Lecture 10

CS 117, S26 © Katy Craig, 2026

Homework 5 due Thursday, April 30th at 11:59pm

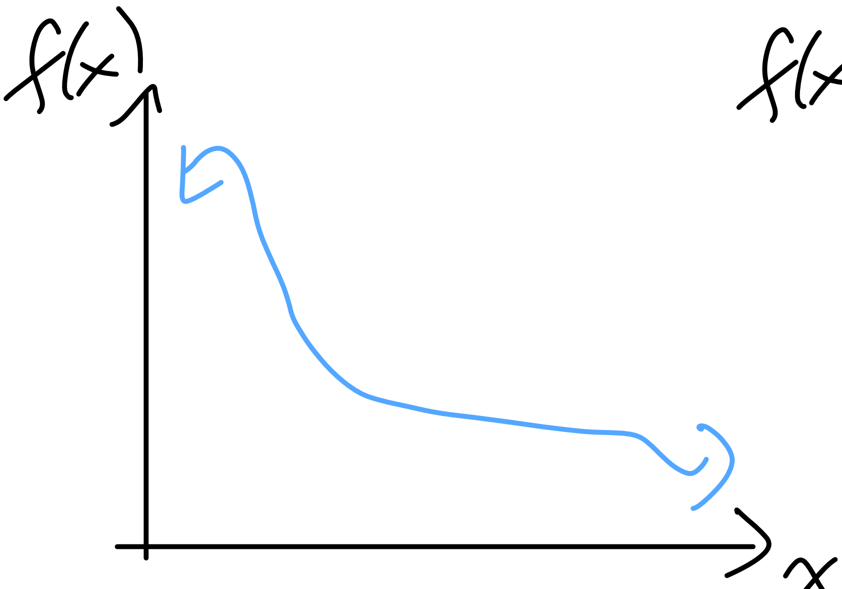
Thm: Every sequence has a monotone subsequence.

Thm (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

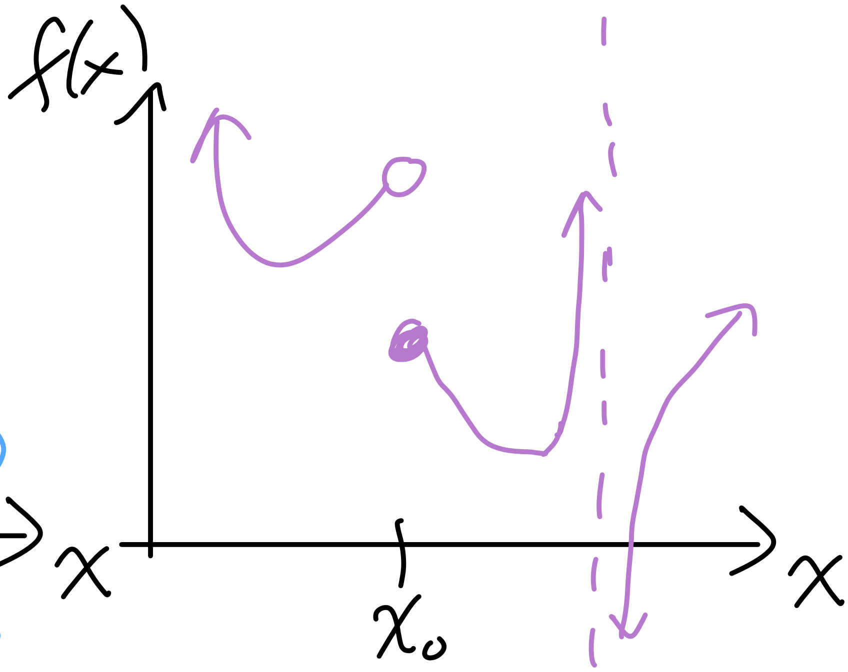
Thm: For any sequence s_n, $\limsup s_n$ and $\liminf s_n$ are the largest and smallest subsequential limits.

Now, apply theory of sequences of real numbers to the study of continuous fns (and generalizations...)

Intuitively, a continuous fn is a fn that "unbroken" or "has no holes".



CONTINUOUS



DISCONTINUOUS

The domain of a function $f(x)$ is the values x at which it is defined.

Abbreviate the domain dom(f).

In this class: $\mathrm{dom}(f) \subseteq \mathbb{R}$
$\mathrm{range}(f) \subseteq \mathbb{R} \cup \{\pm\infty\}$

Ex: $f(x) = \frac{1}{x}$, $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$

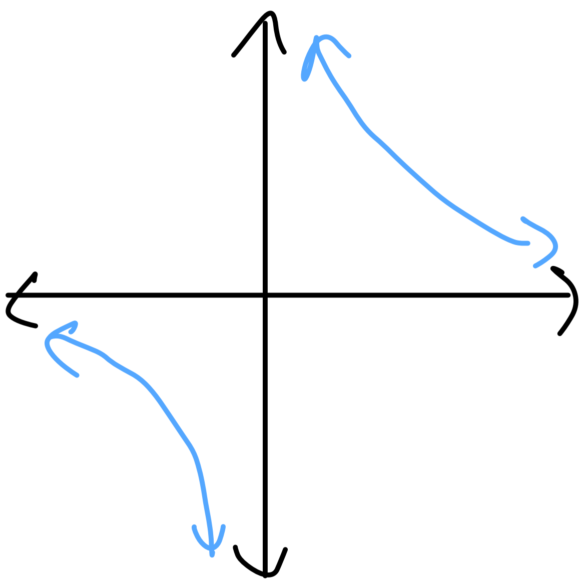
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}, \quad \mathrm{dom}(f) = \mathbb{R}$$

<u>Def</u>: A function $f : \mathrm{dom}(f) \to \mathbb{R}$ is <u>continuous at a point $x_0 \in \mathrm{dom}(f)$</u> if, for every sequence x_n in $\mathrm{dom}(f)$ satisfying $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Furthermore, <u>f is continuous on a set $S \subseteq \text{dom}(f)$</u> if it is continuous at every point $x_0 \in S$.

Finally, <u>f is continuous</u> if it is continuous on $\text{dom}(f)$.

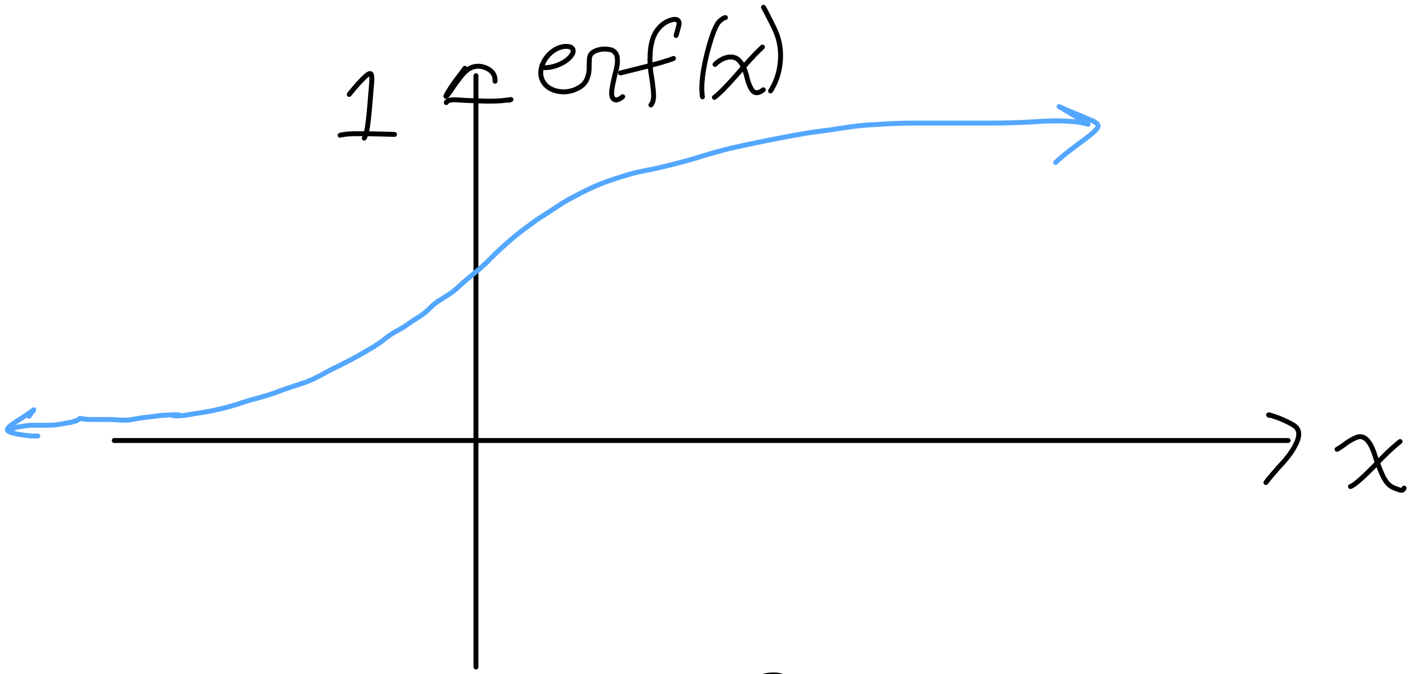
Ex: $f(x) = \frac{1}{x}$, $\text{dom}(f) = \mathbb{R} \setminus \{0\}$ is continuous



Aside...

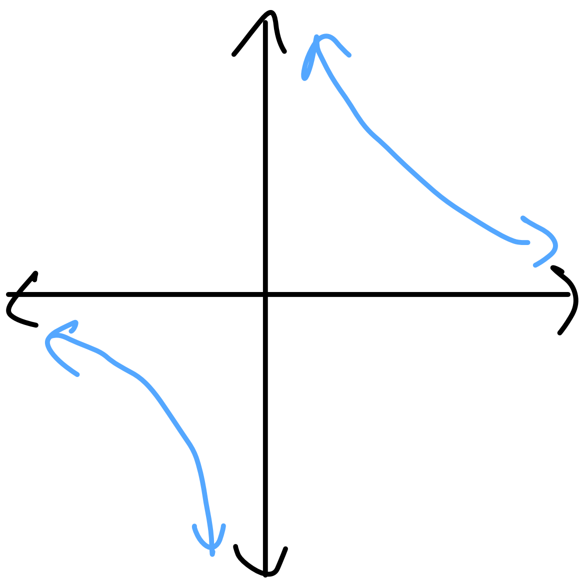
We will also consider extended real valued fns...

There is a nice topology on $\overline{\mathbb{R}}$, which you can think of like this...



$\text{erf}: \overline{\mathbb{R}} \longrightarrow [0,1]$

Ex: $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$, $\text{dom}(f) = \mathbb{R}$



Does there exist $a \in \mathbb{R}$ so that f is continuous?

<u>Rmk</u>: A function $f: \text{dom}(f) \to \mathbb{R}$ is continuous iff for every convergent sequence x_n in $\text{dom}(f)$,

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right).$$

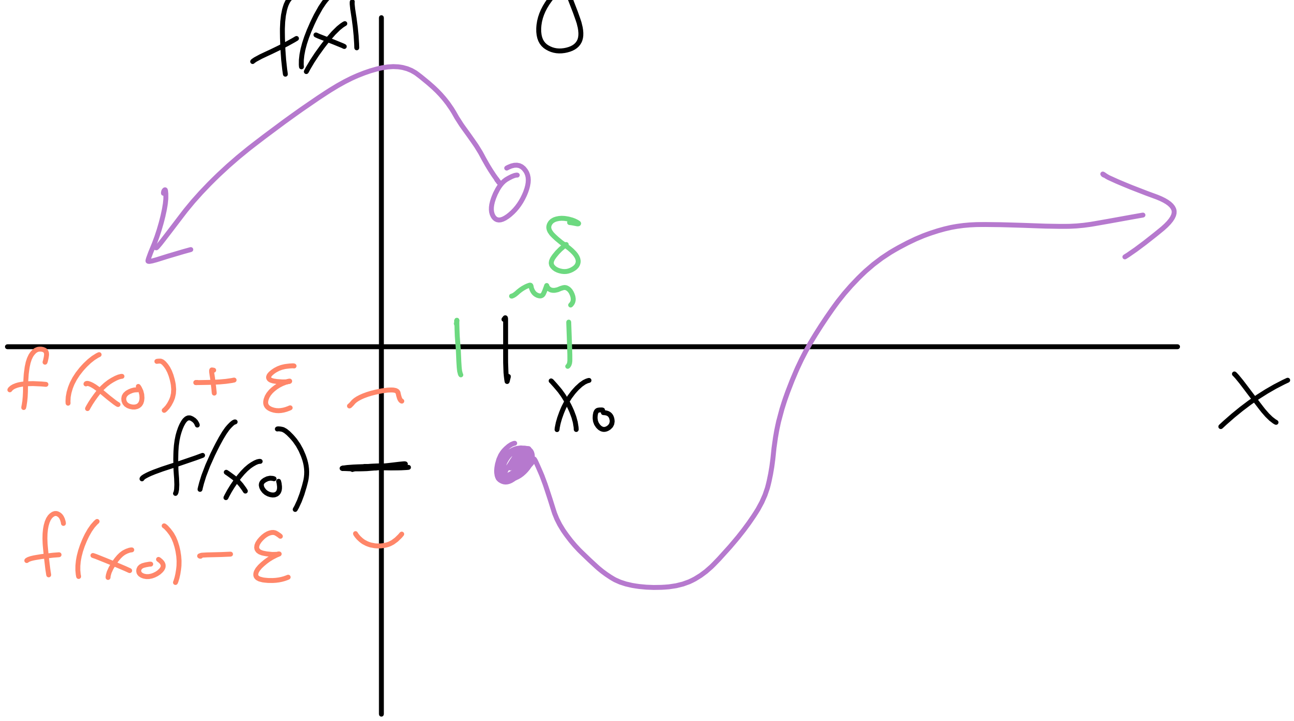
Thm: Given $f : dom(f) \to \mathbb{R}$ and $x_0 \in dom(f)$,

(I) f is continuous at x_0

iff

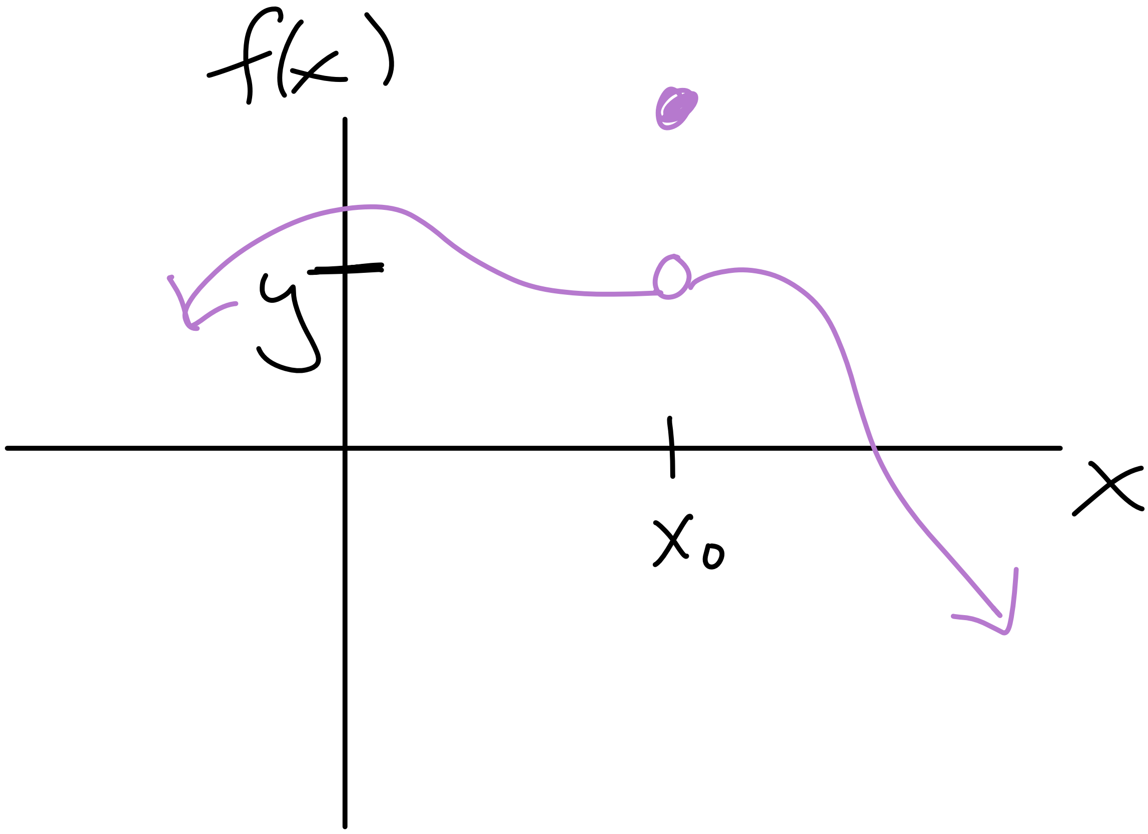
(II) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in dom(f)$ and $|x - x_0| < \delta$ ensures $|f(x) - f(x_0)| < \varepsilon$.

Mental image:



Aside: Consider this variant of the definition...

∃ $y \in \mathbb{R}$ s.t. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in dom(f)$ and $0 < |x - x_0| < \delta$ ensures $|f(x) - $ ~~$f(x_0)$~~ $y| < \varepsilon$.



"The discontinuity at x_0 is removable."

Rmk:

(¬II) $\iff \exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$
there exists $x \in dom(f)$
$|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$

Pf of Thm:

First show (II) $\Rightarrow$ (I). Fix arbitrary x_n in dom(f) s.t. $x_n \to x_0$.
We must show $f(x_n) \to f(x_0)$.
$(x_1, x_2, x_3, \ldots)$, $(f(x_1), f(x_2), f(x_3), \ldots)$
Fix $\varepsilon > 0$. By (II), $\exists\, \delta > 0$ s.t.
$x \in dom(f)$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Since $x_n \to x_0$, $\exists\, N$ s.t.
$n > N$ ensures $|x_n - x_0| < \delta$. Thus,
$n > N$ ensures $|f(x_n) - f(x_0)| < \varepsilon$.

Now, show (¬II) $\Rightarrow$ (¬I).

We must show there exists a sequence x_n in dom(f) s.t. $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$.

By (¬II), $\exists\, \varepsilon > 0$ s.t $\forall n \in \mathbb{N}$, $\exists\, x_n \in \text{dom}(f)$ s.t. $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$.

Since

$$x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$$

by Squeeze Lemma $x_n \to x_0$

OTOH, $f(x_n) \not\to f(x_0)$. $\square$

Some examples...

Ex: $f(x) = 3x^2 - 2$, $\text{dom}(f) = \mathbb{R}$

CLAIM: f is continuous

PF OF CLAIM: Fix $x_0 \in \mathbb{R}$.
Fix x_n arbitrary s.t. $x_n \to x_0$.

By our limit theorems,

$$3x_n^2 - 2 \to 3x_0^2 - 2.$$

ALT PF OF CLAIM:
(an example of when one characterization of cty is much easier to work with than other)

Fix $x_0 \in \mathbb{R}$ arbitrary. Fix $\varepsilon > 0$.

Scratchwork:

$$|f(x) - f(x_0)|$$
$$=$$
$$|(3x^2 - 2) - (3x_0^2 - 2)|$$
$$=$$
$$3|x^2 - x_0^2|$$
$$=$$
$$3|x - x_0||x + x_0|$$
$\wedge |$ Δ ineq
$$3|x - x_0|(|x| + |x_0|) < \varepsilon$$

want

$|x - x_0| < \delta$, $\quad (|x| + |x_0|) < 2|x_0| + 1$

Let's make sure choose $\delta < 1$. Then $|x - x_0| < 1$ ensures, by reverse Δ ineq

$$|x| - |x_0| < 1$$
$$\Updownarrow$$
$$|x| < 1 + |x_0|$$

Thus it suffices to have

$$|x-x_0| < \frac{\varepsilon}{3(2|x_0|+1)}$$

Choose $\delta < \min\left\{1, \frac{\varepsilon}{3(2|x_0|+1)}\right\}$

Then $|x-x_0| < \delta \leq 1$ ensures, by reverse Δ, $|x| \leq |x_0| + 1$. ☆

therefore,

$$|f(x) - f(x_0)|$$
$$=$$
$$|(3x^2-2) - (3x_0^2-2)|$$
$$=$$
$$3|x^2 - x_0^2|$$
$$=$$
$$3|x-x_0||x+x_0| \overset{\Delta \text{ ineq}}{\leq} 3|x-x_0|(|x|+|x_0|)$$
$$\overset{☆}{\leq} 3|x-x_0|(2|x_0|+1)$$

$$\ldots \leq 3\delta(2|x_0|+1) \quad \text{;} \quad \delta < \frac{\varepsilon}{3(2|x_0|+1)}$$

$$< \varepsilon$$

□