Lecture 11

CS 117, S26 © Katy Craig, 2026

Homework 5 due Thursday, May 7th at 11:59pm

The domain of a function $f(x)$ is the values x at which it is defined.

Abbreviate the domain dom(f).

In this class: $\text{dom}(f) \subseteq \mathbb{R}$
$\text{range}(f) \subseteq \mathbb{R} \cup \{\pm\infty\}$

Def: A function $f : \text{dom}(f) \to \mathbb{R}$ is continuous at a point $x_0 \in \text{dom}(f)$ if, for every sequence x_n in dom(f) satisfying $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Furthermore, <u>f is continuous on a set $S \subseteq dom(f)$</u> if it is continuous at every point $x_0 \in S$.

Finally, <u>f is continuous</u> if it is continuous on $dom(f)$.

<u>Thm</u>: Given $f: dom(f) \to \mathbb{R}$ and $x_0 \in dom(f)$, then

f is cts at x_0

if and only if

$$\boxed{\begin{array}{l}\forall \varepsilon > 0, \exists \delta \text{ s.t. } x \in dom(f) \\ \text{and } |x - x_0| < \delta \text{ ensures} \\ |f(x) - f(x_0)| < \varepsilon.\end{array}}$$

In analogy with sequences, we want to show fns are cts if they are "built" from simpler cts fns.

How can "simple" fns be combined to obtain more "complicated" ones?

Given f and g real valued,

$(f+g)(x) = f(x)+g(x)$ $\quad$ $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$(fg)(x) = f(x)g(x)$ $\quad$ $\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \mathrm{dom}\left(\frac{f}{g}\right) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \{x : g(x) \neq 0\}$$

$$(f \circ g)(x) = f(g(x)) \qquad \mathrm{dom}(f \circ g) = \mathrm{dom}(g) \cap \{x : g(x) \in \mathrm{dom}(f)\}$$

<u>Thm</u>: If f and g are real-valued continuous functions, $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then

(i) $\underbrace{f+g}_{=:h_1}$ is cts at x_0

(ii) $\underbrace{fg}_{=:h_2}$ is cts at x_0

(iii) $\underbrace{\frac{f}{g}}_{=:h_3}$ is cts at x_0, as long as $g(x_0) \neq 0$.

Pf: Define h_1, h_2, h_3 as in thm.

Fix $i = 1, 2,$ or 3.
Fix x_n arbitrary in $\mathrm{dom}(h_i)$
s.t. $x_n \to x_0$.

We must show $\lim_{n\to\infty} h_i(x_n) = h_i(x_0)$.

Note, since $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$,

limit of sum is sum of limit

$$h_1(x_n) = f(x_n) + g(x_n) \to h_1(x_0)$$

$$h_2(x_n) = f(x_n)g(x_n) \xrightarrow{\text{products}} h_2(x_0)$$

$$h_3(x_n) = \frac{f(x_n)}{g(x_n)} \xrightarrow{\text{quotients}} h_3(x_0)$$

$\square$

Thm: Suppose g is cts at x_0 and f is cts at $g(x_0)$. Then $f \circ g$ is cts at x_0.

Pf: Fix arbitrary x_n in $\mathrm{dom}(f\circ g)$ s.t. $x_n \to x_0$

Since g cts, $g(x_n) \to g(x_0)$

Since f cts, $f(g(x_n)) \to f(g(x_0))$. $\square$

Bounded functions also have special properties.

<u>Def</u>: f is <u>bounded on $S \subseteq \mathrm{dom}(f)$</u> if there exists $M>0$ s.t. $|f(x)| \leq M \quad \forall x \in S$.

We say f is <u>bounded</u> if it is bounded on $\mathrm{dom}(f)$.

Rmk:

- s_n is bounded

  $\Updownarrow$

  $\{s_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded

- f is bounded on S

  $\Updownarrow$

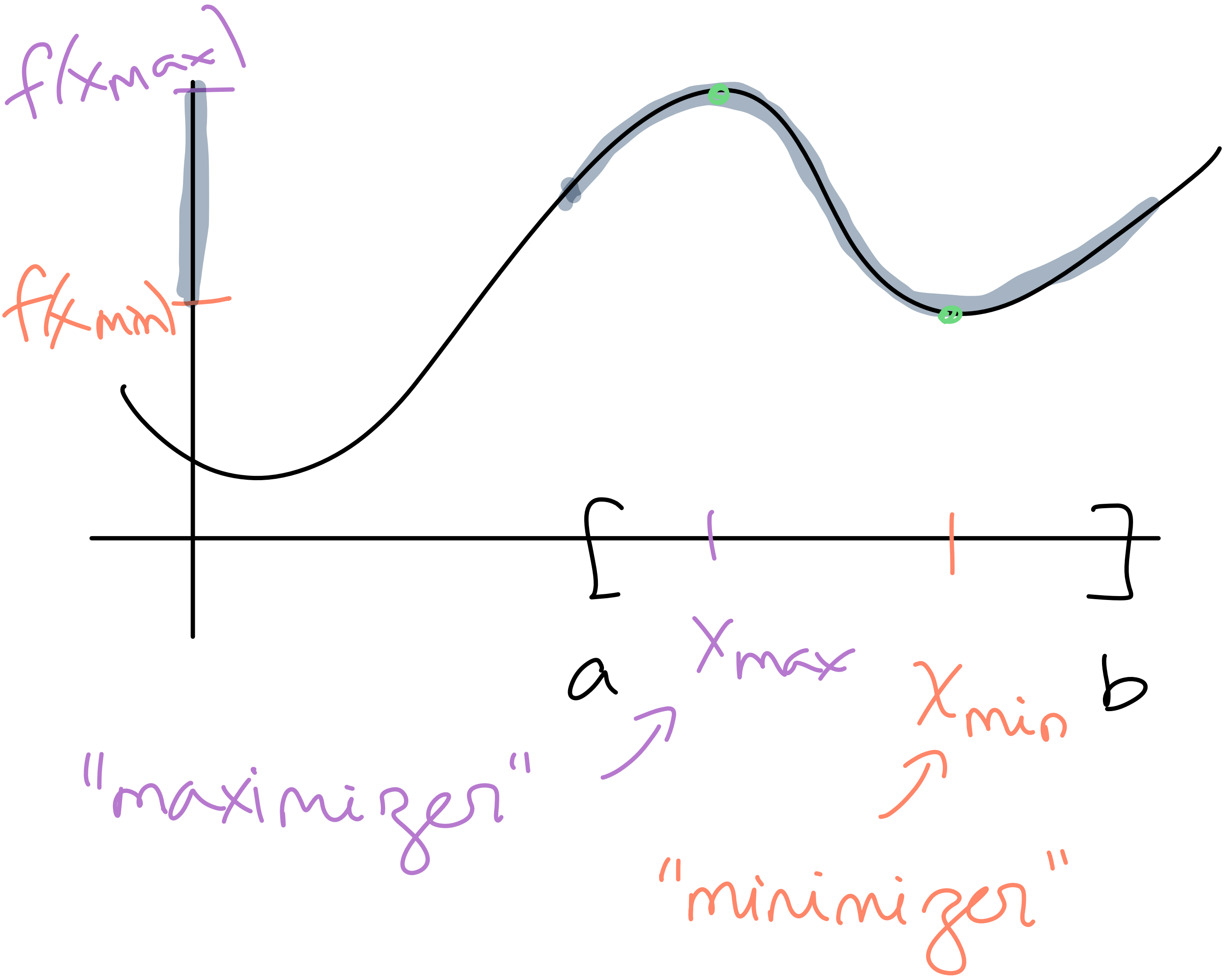
  $\{f(x) : x \in S\} \subseteq \mathbb{R}$ is bounded

Thm: A real-valued continuous function f "attains its maximum and minimum" on any $[a,b] \subseteq \mathrm{dom}(f)$.

In other words,
$\max\{f(x) : x \in [a,b]\}$
$\min\{f(x) : x \in [a,b]\}$
exist, so $\exists\ x_{max}, x_{min} \in [a,b]$

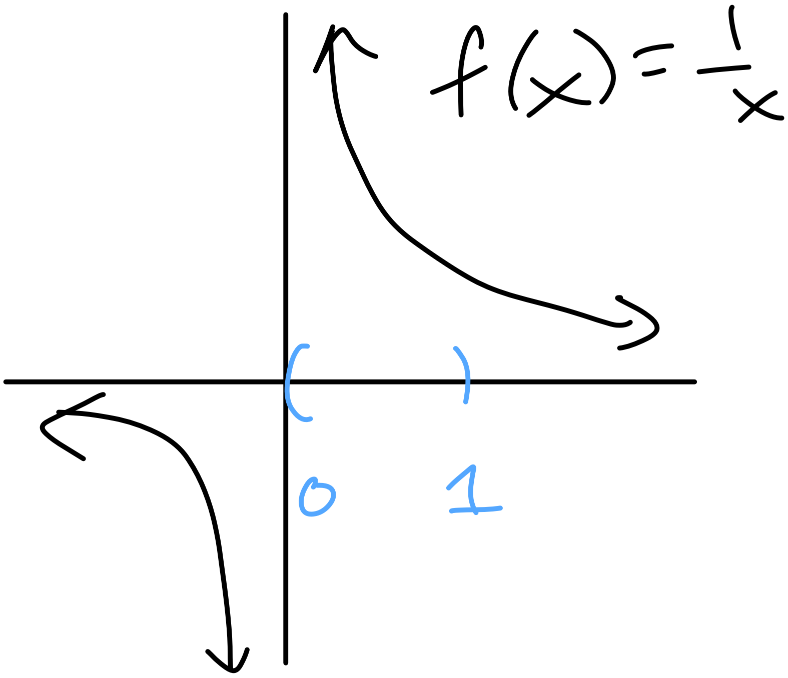
so that

$$f(x_{max}) = \max\{f(x) : x \in [a,b]\}$$
$$f(x_{min}) = \min\{f(x) : x \in [a,b]\}.$$



Rmk: An immediate consequence is that all cts function f are bounded on closed intervals in their domain.

Ex:



$(0,1) \subseteq dom(f) = \mathbb{R} \setminus \{0\}$

$\max\{f(x) : x \in (0,1)\}$ D.N.E.

Pf: We will first show $\max\{f(x) : x \in [a,b]\}$ exists.

Strategy: "Take a maximizing sequence"

Define $m := \sup\{f(x) : x \in [a,b]\}$

Recall: for any $A \subseteq \mathbb{R}$, $\exists$ $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ s.t. $\lim_{n\to\infty} a_n = \sup(A)$.

Thus $\exists$ $x_n \in [a,b]$ s.t.
$$\lim_{n\to\infty} f(x_n) = m. \qquad (*)$$

Since x_n is bounded, by Bolzano-Weierstrass, it has a subsequence $x_{n_k} \in [a,b]$ that converges to some $x_0 \in [a,b]$.

Recall: If the limit of r_n, s_n, t_n exist and $r_n \leq s_n \leq t_n$ then $\lim r_n \leq \lim s_n \leq \lim t_n$

Since f is continuous,
$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0).$$

Combining with $(*)$, we see
$$m = f(x_0).$$

Since $f(x_0) \in \{f(x) : x \in [a,b]\}$,
$$f(x_0) = \max\{f(x) : x \in [a,b]\}.$$

Finally, to show $\min\{f(x) : x \in [a,b]\}$ exists, note that

$$\min\{f(x) : x \in [a,b]\} = -\underbrace{\max\{-f(x) : x \in [a,b]\}}_{\text{this exists, by what we have just shown}}$$

An immediate consequence of this result is...

Thm (Intermediate Value Thm): If f is cts on an interval $I \subseteq \mathrm{dom}(f)$, then for all $a, b \in I$,

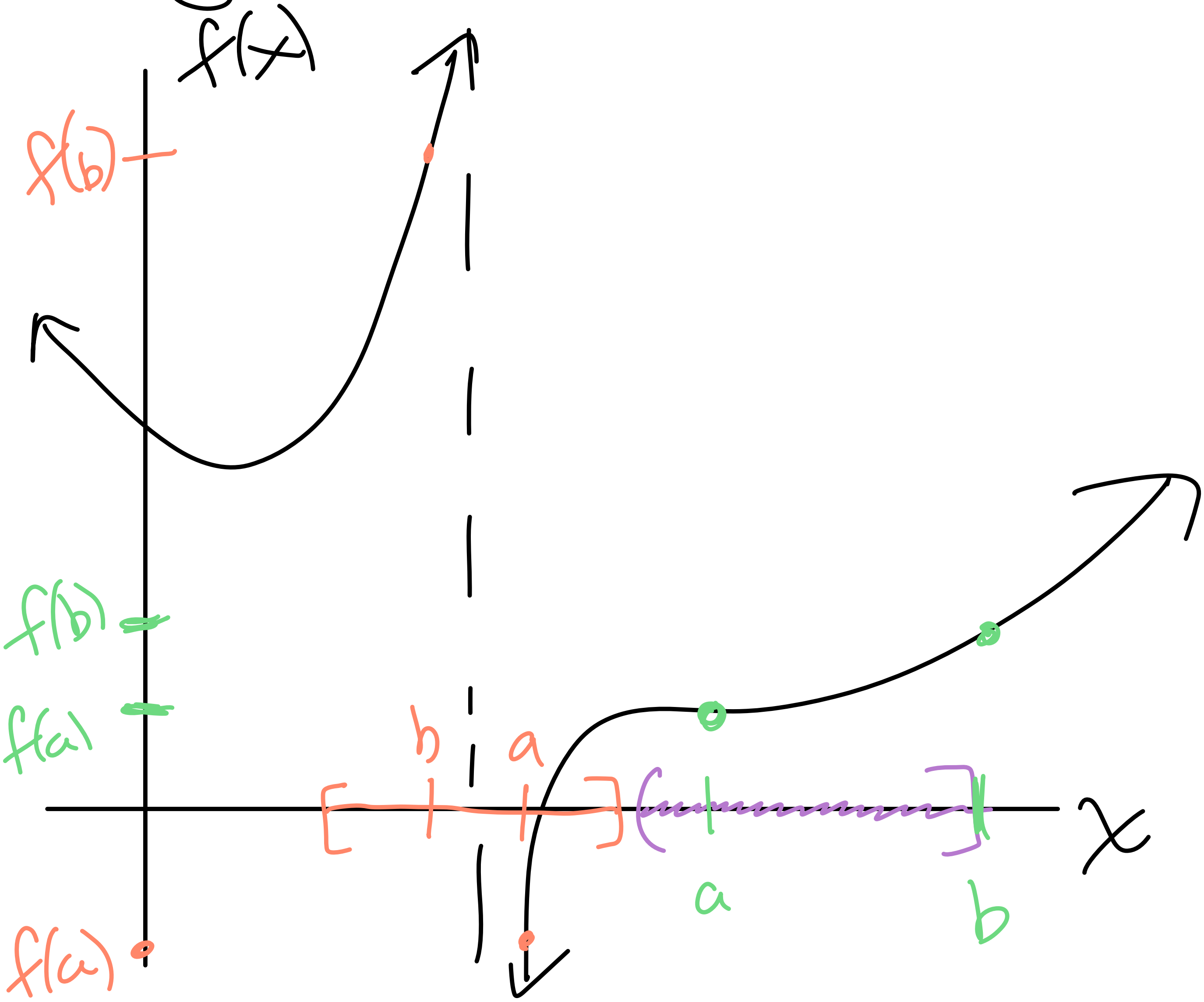
if $\overbrace{y \text{ lies between } f(a) \text{ and } f(b)}^{f(a) \le y \le f(b) \text{ or } f(b) \le y \le f(b)}$

then $\exists$ $\underbrace{x \text{ between } a \text{ and } b}_{a \le x \le b \text{ or } b \le x \le a}$

satisfying $f(x) = y$.

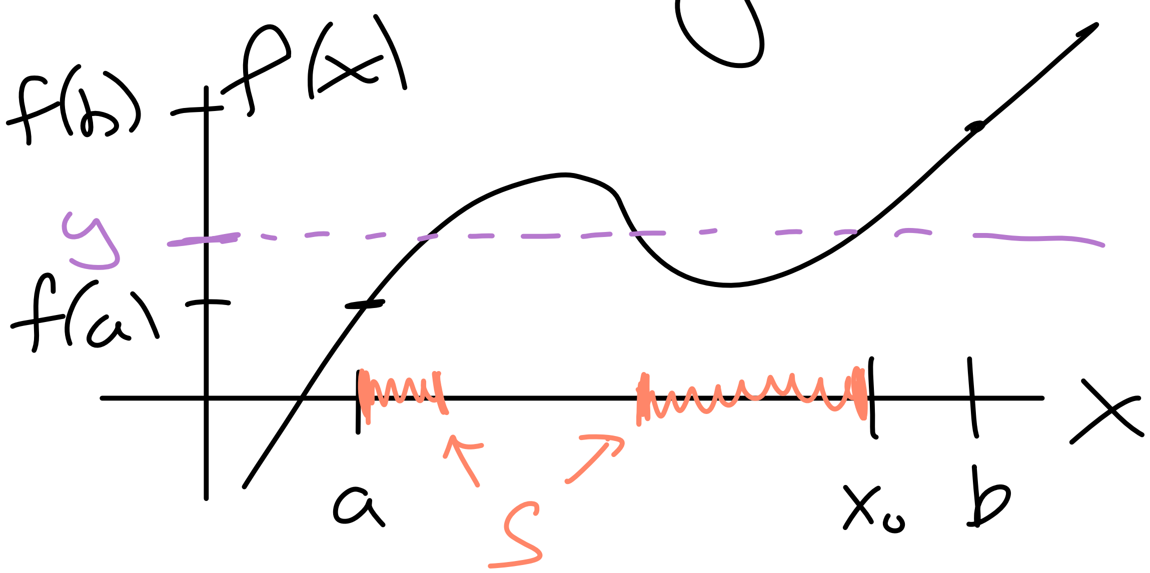
Mental Image:

Cty $\Rightarrow$ no gaps



Pf: Fix $a, b \in I$. Assume WLOG $a \leq b$. Suppose y lies between $f(a)$ and $f(b)$. WTS $\exists\, x \in [a,b]$ s.t. $f(x) = y$.

Case 1: $f(a) \le y \le f(b)$.



Define $S = \{x \in [a,b] : f(x) \le y\}$

Let $x_0 = \sup(S)$. Then $x_0 \in [a,b]$.
We will show $f(x_0) = y$.

Let x_n be a sequence of elements of S satisfying $\lim_{n\to\infty} x_n = x_0$.

Since f is cts, $\lim_{n\to\infty} f(x_n) = f(x_0)$

Since $x_n \in S \Rightarrow f(x_n) \leq y \ \forall n \in \mathbb{N}$,
$f(x_0) \leq y$.

It remains to show
$f(x_0) \geq y$.

If $x_0 = b$, we are done
since in Case 1, $f(b) \geq y$.

We may assume $x_0 < b$.

Define $t_n = \min\{x_0 + \frac{1}{n}, b\}$.
By defn, $\lim_{n\to\infty} t_n = x_0$.

Since f cts on $[a,b] \subseteq I$,
$\lim_{n\to\infty} f(t_n) = f(x_0)$.

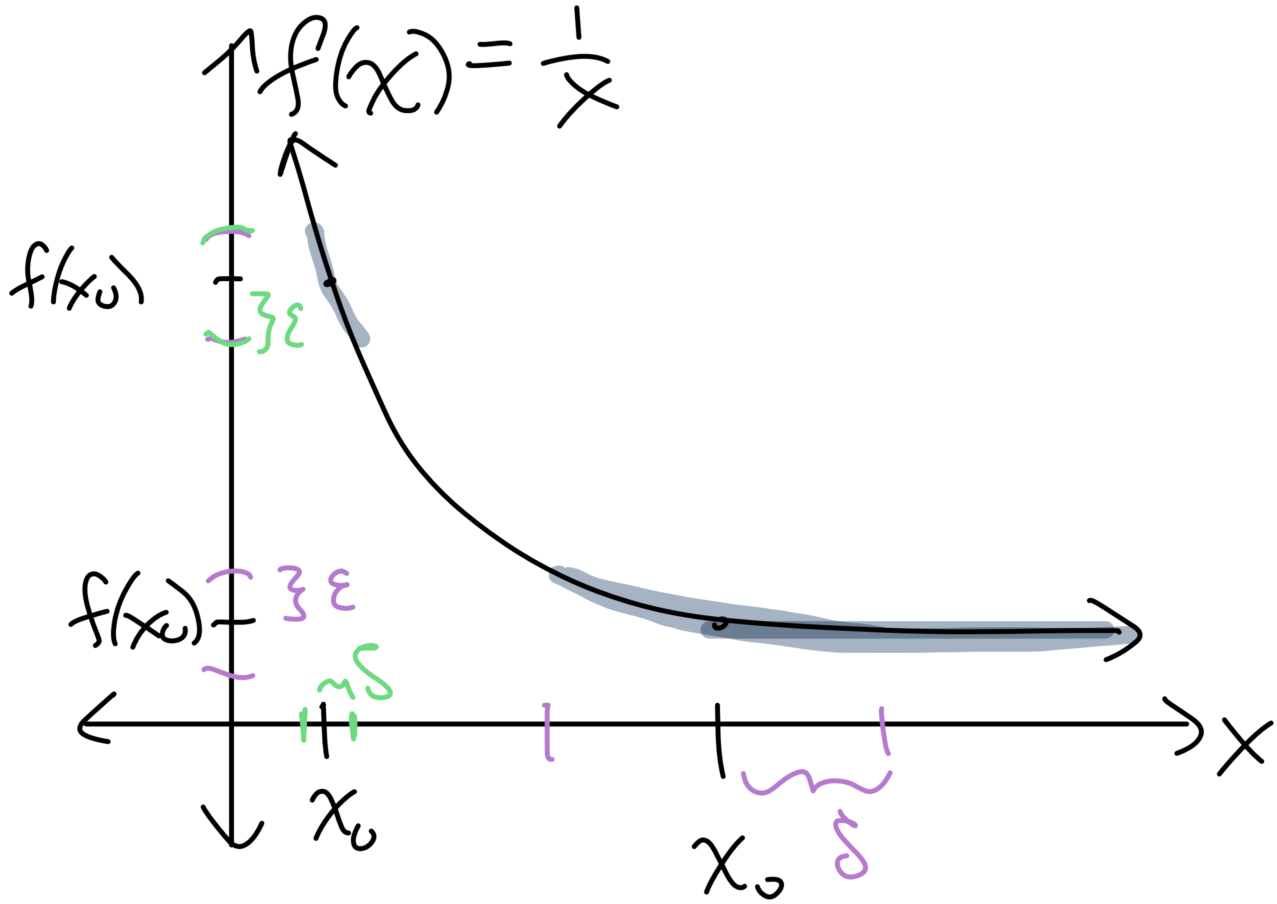
Since $t_n > x_0$, $t_n \notin S$,
so $f(t_n) > y$ $\forall n \in \mathbb{N}$.
Thus $f(x_0) = \lim_{n \to \infty} f(t_n) \geq y$.

Case 2: $f(b) \leq y \leq f(a)$.

Then, $-f(a) \leq -y \leq -f(b)$.
Applying Case 1 to $-f$,
we find $x_0 \in [a,b]$ s.t.
$-f(x_0) = -y \Rightarrow f(x_0) = y$. $\square$

Uniform Continuity

Recall: in usual ε-δ defn
of cty, δ depends on x_0.



Uniform cty: same $\delta > 0$ works for all x

<u>Def</u>: A real valued function f is <u>uniformly continuous on $S \subseteq dom(f)$</u> iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $x, y \in S$ and $|x-y| < \delta$ ensures $|f(x) - f(y)| < \varepsilon$.