Lecture 12

CS 117, S26 © Katy Craig, 2026

Homework 6 due Thursday, May 14th at 11:59pm

<u>Thm</u>: If f and g are real-valued continuous functions, $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then

(i) $f+g$ is cts at x_0
(ii) fg is cts at x_0
(iii) $\frac{f}{g}$ is cts at x_0, as long as $g(x_0) \neq 0$.

<u>Thm</u>: Suppose g is cts at x_0 and f is cts at $g(x_0)$. Then $f \circ g$ is cts at x_0.

Def: f is <u>bounded on $S \subseteq dom(f)$</u> if there exists $M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in S$.

We say f is <u>bounded</u> if it is bounded on $dom(f)$.

Rmk:

- s_n is bounded

  $\Updownarrow$

  $\{s_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded

- f is bounded on S

  $\Updownarrow$

  $\{f(x) : x \in S\} \subseteq \mathbb{R}$ is bounded

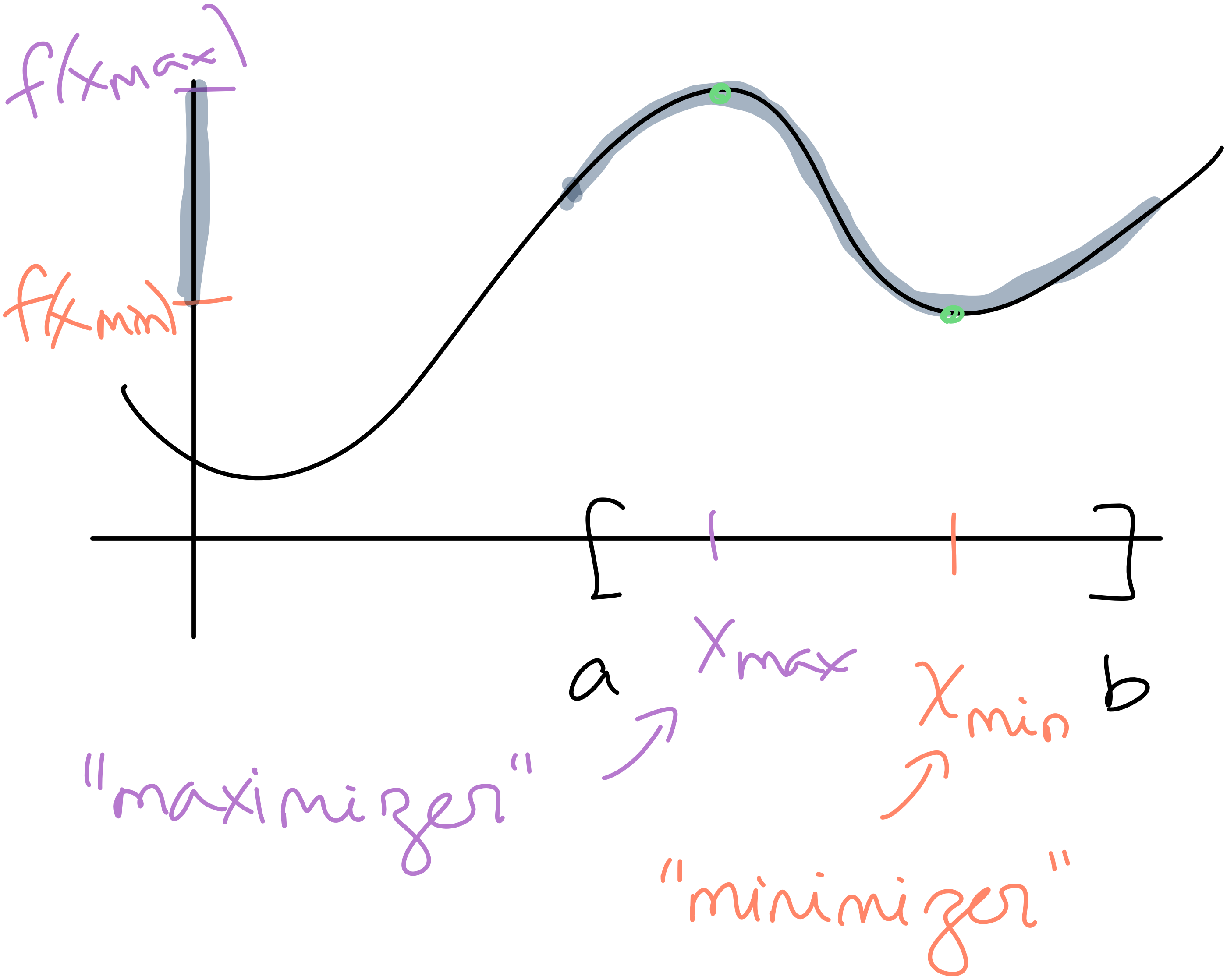
<u>Thm</u>: A real-valued continuous function f "attains its maximum and minimum" on any $[a,b] \subseteq dom(f)$.

In other words,

$\max\{f(x) : x \in [a,b]\}$
$\min\{f(x) : x \in [a,b]\}$
exist, so $\exists$ $x_{max}, x_{min} \in [a,b]$

so that

$f(x_{max}) = \max\{f(x) : x \in [a,b]\}$
$f(x_{min}) = \min\{f(x) : x \in [a,b]\}$.



Rmk: An immediate consequence is that all cts function f are bounded on closed intervals in their domain.

An immediate consequence of this result is...

Thm (Intermediate Value Thm): If f is cts on an interval $I \subseteq \mathrm{dom}(f)$, then for all $a, b \in I$,

if $\overbrace{y \text{ lies between } f(a) \text{ and } f(b)}^{f(a) \leq y \leq f(b) \text{ or } f(b) \leq y \leq f(b)}$

then $\exists$ $\underbrace{x \text{ between } a \text{ and } b}_{a \leq x \leq b \text{ or } b \leq x \leq a}$

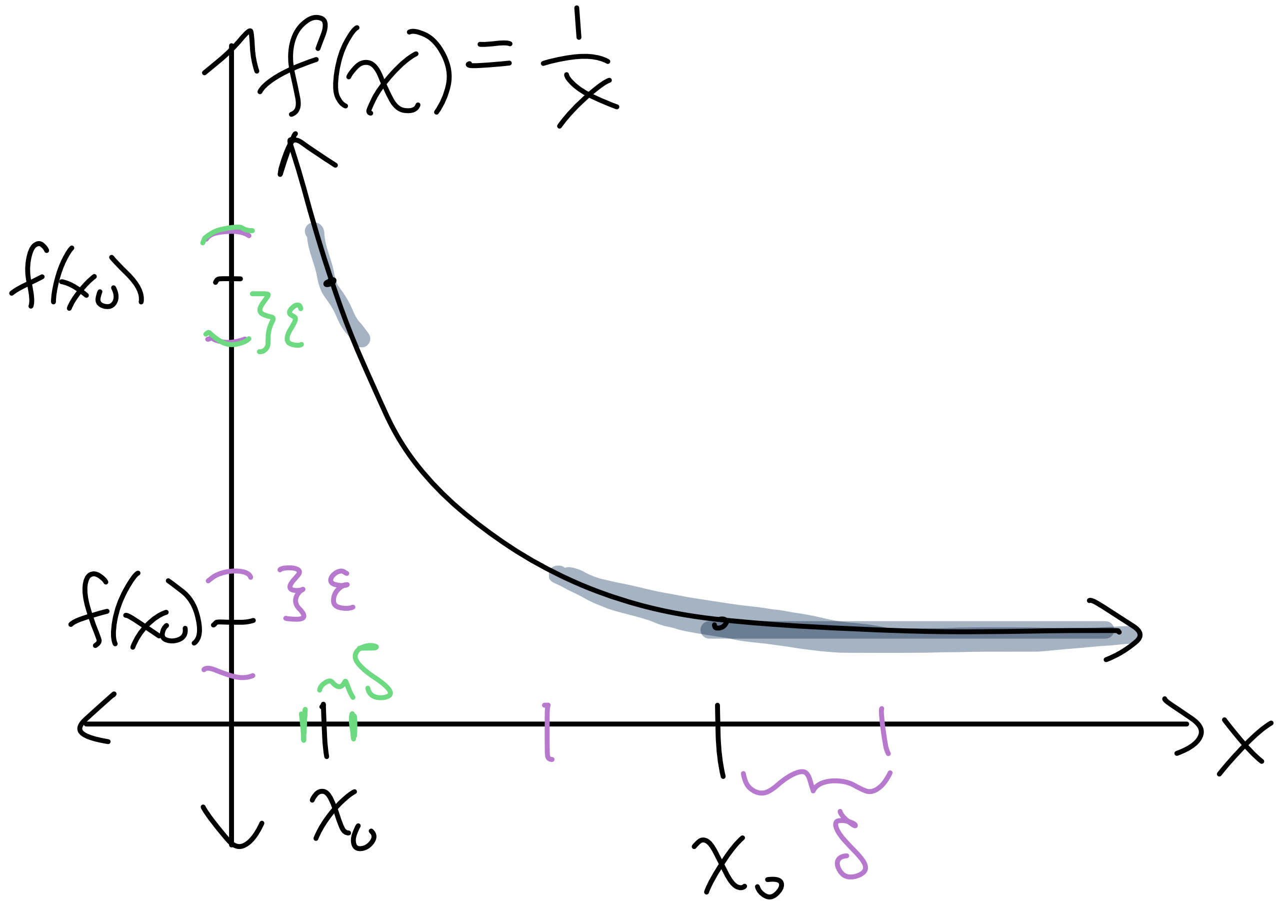
satisfying $f(x) = y$.

Uniform Continuity

Recall: in usual ε-δ defn of cty, δ depends on x_0.

Uniform cty: same $\delta>0$ works for all x

<u>Def</u>: A real valued function f is <u>uniformly continuous on $S \subseteq dom(f)$</u> iff $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $x, y \in S$ and $|x-y|<\delta$ ensures $|f(x)-f(y)|<\varepsilon$.



f(x) = 1/x
f(x₀)
ε
f(x₀)
ε
δ
x
x₀
x₀
δ

Ex: $f(x)=\frac{1}{x}$, $\mathrm{dom}(f)=\mathbb{R}\setminus\{0\}$.

We will show $f(x)$ is <u>not</u> uniformly on $\mathrm{dom}(f)$.

Assume, for the sake of contradiction that it is.
Fix $\varepsilon=1$ and take $\delta>0$ as in defn of uniform cty.

Then, $\forall\, x,y\in\mathbb{R}\setminus\{0\}$
$|x-y|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<1$.

Suppose $x=\frac{1}{n}$, $y=\frac{1}{n+1}$.
Then for any $\delta>0$, we may choose n large enough so that
$|x-y|\leq\frac{1}{n}<\delta$, but $\left|\frac{1}{x}-\frac{1}{y}\right|=1\not<1$.

On the positive side...

<u>Thm</u>: If f is a cts real valued function on a closed interval $[a,b] \subseteq dom(f)$, then f is uniformly cts on $[a,b]$.

It's necessary that $[a,b]$ is closed...

Ex: $f(x) = \frac{1}{x}$ is <u>not</u> unif cts on $(0,1] \subseteq dom(f)$

Quick review of facts we will use to prove:

- If $s_n \in [a,b]$ $\forall n \in \mathbb{N}$ and s_n converges to s, then $s \in [a,b]$.

(This is false if you replace $[a,b]$ with (a,b).)

Pf:

Assume, for the sake of contradiction, that f is not unif cts on $[a,b]$, that is, $\exists\ \varepsilon > 0$ s.t. $\forall\ \delta > 0$, $\exists\ x, y \in [a,b]$ s.t. $|x-y| < \delta$ and $|f(x)-f(y)| \geq \varepsilon$.

In particular, $\forall\ n \in \mathbb{N}$, $\exists\ x_n, y_n \in [a,b]$ s.t. $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$.

Since x_n is bdd, there exists a subsequence

x_{n_k} that converges to x_0.
Since y_{n_k} is bounded, so it has a convergent subsequence $y_{n_{k_\ell}}$ that converges to $y_0 \in [a,b]$

Since a subsequence of a convergent sequence has the same limit $x_{n_{k_\ell}}$ converges to $x_0 \in [a,b]$

By ☆,

$$y_{n_{k_\ell}} - \frac{1}{n_{k_\ell}} \leq x_{n_{k_\ell}} \leq y_{n_{k_\ell}} + \frac{1}{n_{k_\ell}}$$

Thus, sending $\ell \to \infty$,

$$y_0 \leq x_0 \leq y_0 \Rightarrow x_0 = y_0 \in [a,b]$$

By cty of f,

$$\lim_{\ell \to \infty} f(x_{n_{k_\ell}}) - f(y_{n_{k_\ell}}) = f(x_0) - f(y_0) = 0$$

This contradicts ⊛, since

$$|f(x_{n_{k_\ell}}) - f(y_{n_{k_\ell}})| > \varepsilon. \square$$

Why do we need another notion of continuity?

Thm: If f is a uniformly continuous real-valued fn on $S \subseteq \mathrm{dom}(f)$ and $s_n : \mathbb{N} \to S$ a convergent sequence, then $f(s_n)$ is also convergent.

"uniformly cts fns send convergent sequences to convergent sequences"

Wait... isn't that what cts fn do?

Ex: $f(x)=\frac{1}{x}$, $s_n=\frac{1}{n}$,
$f(s_n)=n$ does not converge

The precise statement for continuous functions is...

"Cts fns send convergent sequences with limits in their domain to convergent sequences."

Pf:

It suffices to show $f(s_n)$ is a Cauchy sequence.

Fix $\varepsilon > 0$. By defn of uniform cty, $\exists\, \delta > 0$ s.t. $\forall\, x, y \in S$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Since s_n is convergent, it is Cauchy and $\exists\, N$ s.t. $n, m > N$ ensures $|s_n - s_m| < \delta$.

Thus, $n, m > N$ ensures $|f(s_n) - f(s_m)| < \varepsilon$. $\square$

Convex Functions

Across sciences, engineering, machine learning, we have problems of the form

$$\min_{x \in C} f(x)$$

We have seen that continuity of f and $C = [a,b] \subseteq \mathrm{dom}(f)$ is sufficient for existence of minimize.

But how do we "find" a minimizer?

A classical approach

Proximal Point Method
Minimizing Movements
JKO :)

Next time