What do these theorems about continuous functions on $\mathbb{R}$ translate to for continuous functions on a topological space (X,τ)?

<u>Def</u>: $f: X \to \mathbb{R}$ is continuous $\forall\, \mathcal{U} \subseteq \mathbb{R}$ open, $f^{-1}(\mathcal{U})$ is open.

<u>Rmk</u>: First, assume f is cts via ε-δ defn. ($f:\mathbb{R}\to\mathbb{R}$) Given $\mathcal{U} \subseteq \mathbb{R}$ open, WTS $f^{-1}(\mathcal{U})$ open, that is, $\forall\, x \in f^{-1}(\mathcal{U})\ \exists\, \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(\mathcal{U})$

Fix $x \in f^{-1}(\mathcal{U}) = \{z : f(z) \in \mathcal{U}\}$. Since $\mathcal{U}$ is open, $\exists\, \varepsilon > 0$ s.t. $|y - f(z)| < \varepsilon \Rightarrow y \in \mathcal{U}$.

By our defn of cty, $\exists\, \delta > 0$ s.t. $|z-x| < \delta$, $|f(z) - f(x)| < \varepsilon$.

Therefore, $B_\delta(x) \subseteq f^{-1}(\mathcal{U})$.

Other direction also holds :)

$[a,b]$

Thm: $f: X \to \mathbb{R}$ is continuous $K \subseteq X$ is compact, then f attains its max and min on K.

any interval

Thm (Intermediate Value): $f: X \to \mathbb{R}$ is continuous, $C \subseteq X$ is path connected, then $\forall\, a, b \in C$, if y is between $f(a)$ and $f(b)$, there exists $\gamma: [0,1] \to C$, continuous, $\gamma(0) = a$, $\gamma(1) = b$ s.t.

$f(\gamma(t)) = y$ for some $t \in [0,1]$.

Lecture 13

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Homework 6 due Thursday, May 14th at 11:59pm

<u>Thm</u>: If f is a cts real valued function on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$, then f is uniformly cts on $[a,b]$.

<u>Thm</u>: If f is a uniformly continuous real-valued fn on $S \subseteq \mathrm{dom}(f)$ and $s_n : \mathbb{N} \to S$ a convergent sequence, then $f(s_n)$ is also convergent.

The precise statement for continuous functions is:--

"Cts fns send convergent sequences with limits in their domain to convergent sequences."

Convex Functions

Across sciences, engineering, machine learning, we have problems of the form

$$\min_{x \in C} f(x)$$

We have seen that continuity of f and $C = [a,b] \subseteq \mathrm{dom}(f)$ is sufficient for existence of minimize.

But how do we "find" a minimizer?

A classical approach

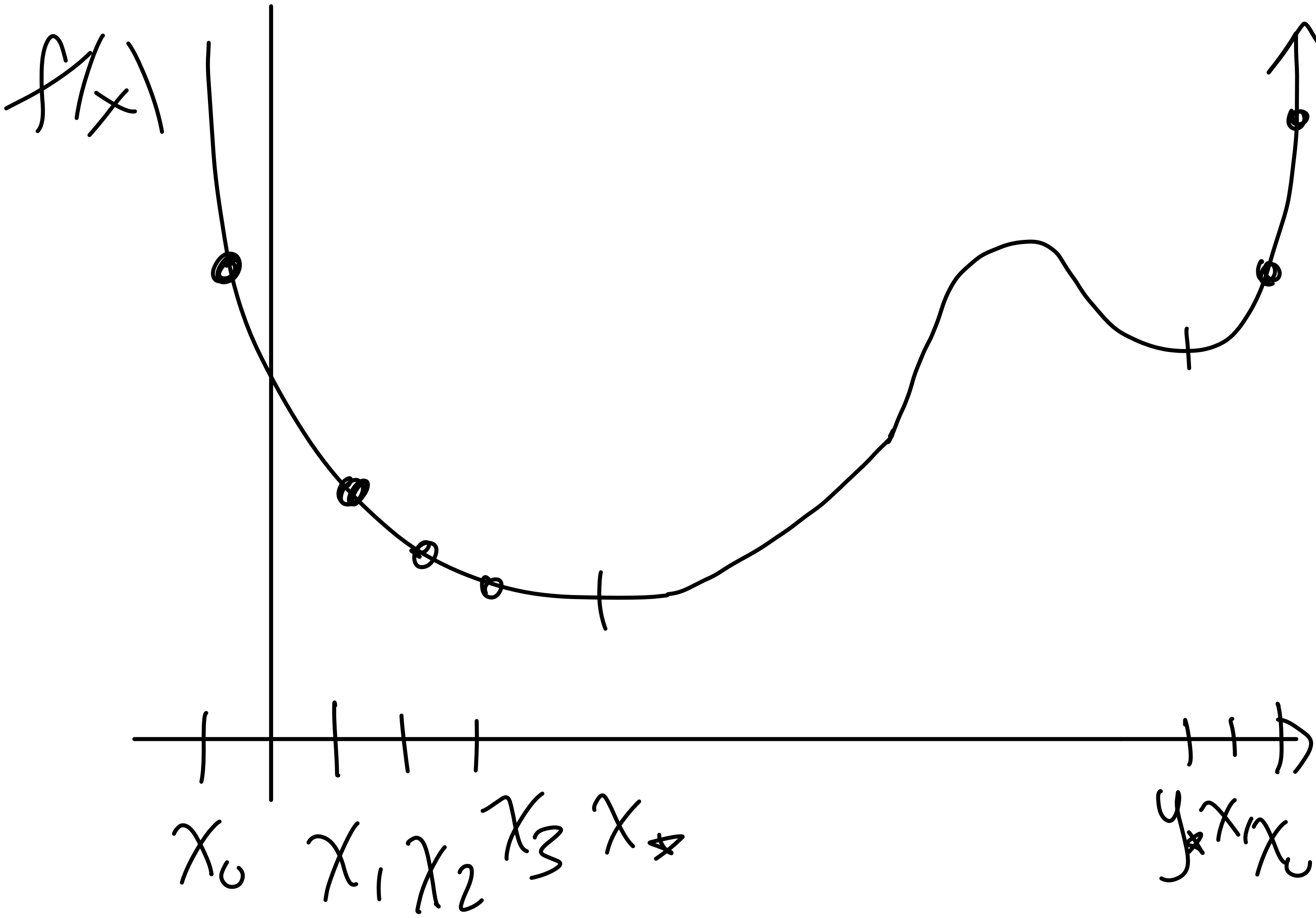
Proximal Point Method
Minimizing Movements
JKO :)

Given $x_0 \in C$, $\forall\, n \in \mathbb{N}$, define x_n to be the minimizer of

$$\min_{x \in C} f(x) + \underbrace{|x - x_{n-1}|^2}_{}$$

in many cases, it's easier to find values of x that make $*$ small and are "close" to x_{n-1}.

Notation: If x_* is a minimizer of f over C, write $x_* = \operatorname{argmin}_{x \in C} f(x)$.



A key hypothesis that allows us to prove convergence of prox point method if convexity.

Ex: Prox point will <u>not</u> be well-defined for arb f.

$f(x) = -2x^2$, $x_0 = 0$

$$x_1 := \underbrace{\operatorname{argmin} \overbrace{-2x^2 + x^2}^{-x^2}}$$

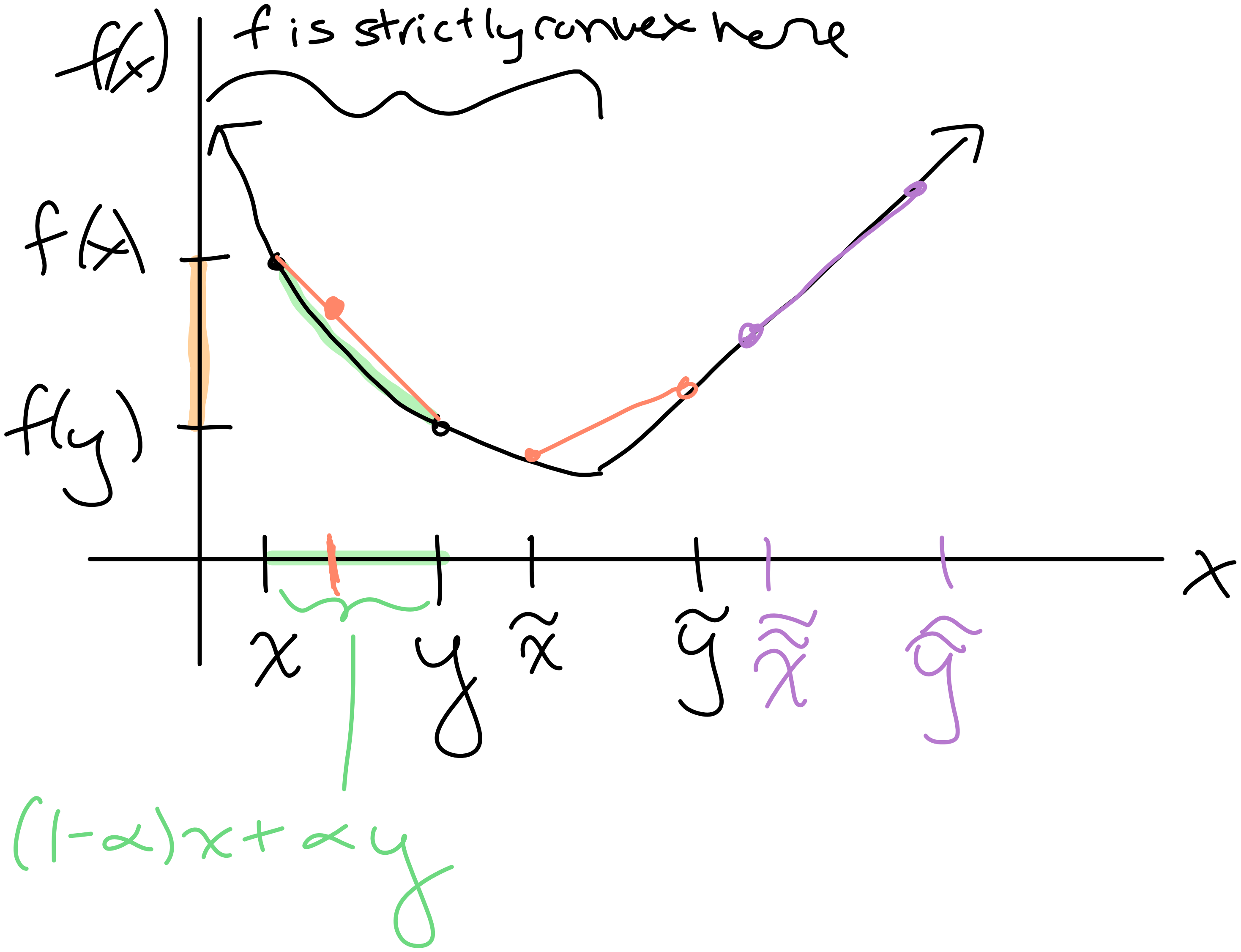
minimizer does not exist!

(−∞,+∞) allowed

Def: Given an interval $I \subseteq \mathbb{R}$, a function $f: I \to \mathbb{R} \cup \{+\infty\}$ is <u>convex</u> if, $\forall x, y \in I, \alpha \in [0,1]$,

(*) $$f((1-\alpha)x + \alpha y) \leq (1-\alpha) f(x) + \alpha f(y).$$

We say f is <u>strictly</u> <u>convex</u> if $\forall x \neq y, \alpha \in (0,1)$ (*) holds with $<$.



Rmk: If $I \subseteq \mathbb{R}$ is an interval and $f: I \to \mathbb{R} \cup \{+\infty\}$ is convex, we may extend f to be a convex function on all of $\mathbb{R}$ by defining

$$\tilde{f}(x) = \begin{cases} f(x) & x \in I \\ +\infty & x \notin I \end{cases}$$

(⋆) always holds if either x or y is not in I.

One way that we can see convexity is important when study minimization problems...

$$\min_{x\in C} f(x) \iff \min_{x\in\mathbb{R}} \tilde{f}(x)$$

where $\tilde{f}(x) = \begin{cases} f(x) & x\in C \\ +\infty & x\notin C \end{cases}$

<u>Thm</u>: If $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is strictly convex, minimizers of f are unique.

<u>Pf</u>: Suppose x_0 and x_1 are minimizers of f and $x_0 \neq x_1$.

$$\min_{x\in\mathbb{R}} f(x) = f(x_0) = f(x_1)$$

By strict convexity of f, $\forall \alpha\in(0,1)$,

$$f((1-\alpha)x_0 + \alpha x_1) < \underbrace{(1-\alpha)f(x_0) + \alpha f(x_1)}_{f(x_0) = \min_{x\in\mathbb{R}} f(x)}$$

This contradicts the definition of minimum. $\square$

Convexity alone is not enough to guarantee existence of a minimum...

Ex: $f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}$

$\min_{x \in \mathbb{R}} f(x)$ D.N.E.

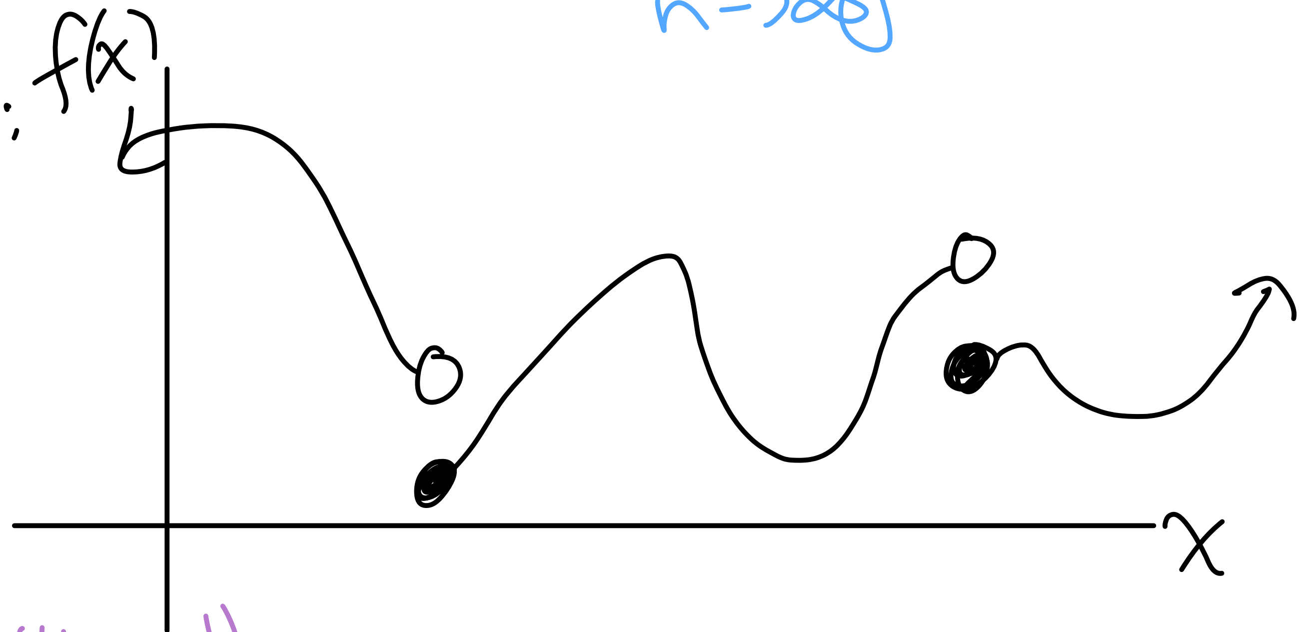
We already saw that cts functions on closed intervals attain their min, cty is too strong for our setting.

A weaker condition...

Def: A function $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous at x_0 if, for all $x_n \to x_0$,

$$f(x_0) \leq \liminf_{n \to \infty} f(x_n)$$

Ex:



"lsc"

Thm: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous, f attains its minimum on any $[a,b]$.

Pf: If $f \equiv +\infty$ on $[a,b]$, the result is trivially true. Suppose $\exists x \in [a,b]$ s.t. $f(x) < +\infty$.

Take a "minimizing sequence", that is, a sequence $x_n : \mathbb{N} \to [a,b]$

$$\lim_{n\to\infty} f(x_n) = \inf_{x\in[a,b]} f(x) = \inf \underbrace{\{f(x) : x \in [a,b]\}}_{\subseteq \mathbb{R} \cup \{+\infty\}} = \inf \underbrace{\{f(x) : x \in [a,b], f(x) < +\infty\}}_{\subseteq \mathbb{R}}$$

$\exists$ convergent subsequence $x_{n_k} \to x_* \in [a,b]$. By lsc,

$$\liminf_{k\to\infty} f(x_{n_k}) \geq f(x_*)$$

But since $f(x_n)$ converges to $\inf\{f(x) : x \in [a,b]\}$,

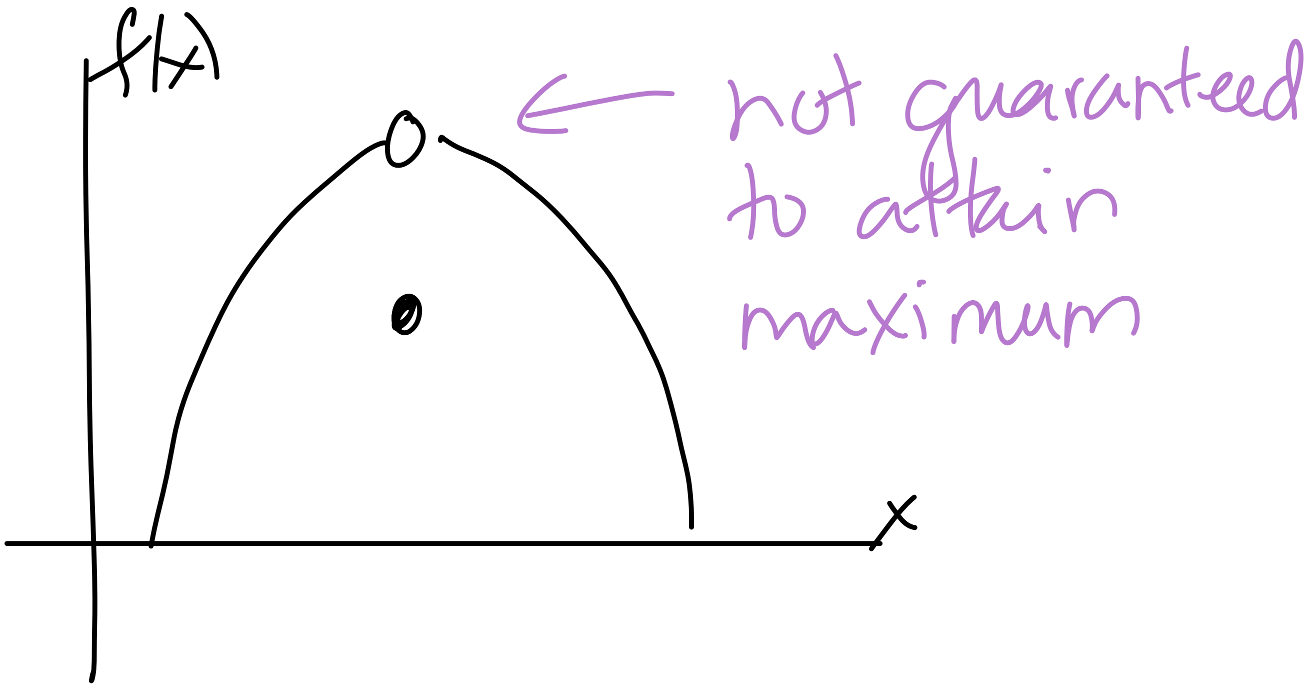
$$\lim_{k\to\infty} f(x_{n_k}) = \inf_{x\in[a,b]} f(x)$$

Thus

$$f(x_*) \le \inf_{x\in[a,b]} f(x),$$

so x_* is a minimizer of f on $[a,b]$. $\square$

Ex:



Goal: combine lsc and convexity to prove convergence of proximal point method.

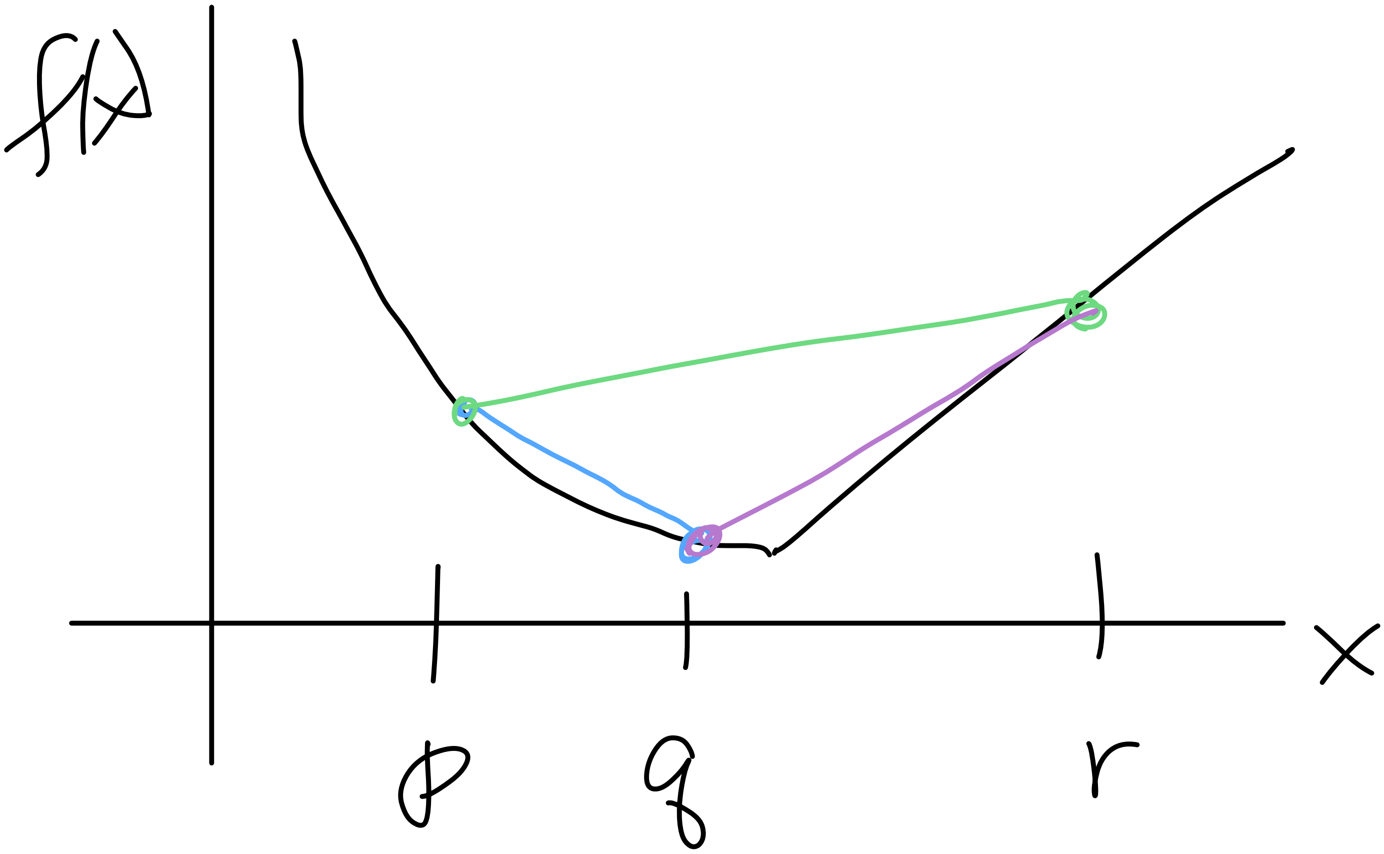
We will rely on a useful equivalent characterization of convexity:

Def: The (convex) domain of $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is

$$D(f) := \{x : f(x) < +\infty\}$$

Prop: $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex iff for all $p, q, r \in D(f)$, $p < q < r$, we have

$$\frac{f(q)-f(p)}{q-p} \leq \frac{f(r)-f(p)}{r-p} \leq \frac{f(r)-f(q)}{r-q}$$



Slopes of lines are increasing.

Prove next time :)