Lecture 15

CS 117, S26 © Katy Craig, 2026

Homework 7 due Thursday, May 21st at 11:59pm

Midterm 2 on Wednesday, May 27th

Prop: $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex iff $D(f)$ is an interval and for all $p, q, r \in D(f)$, $p < q < r$, we have

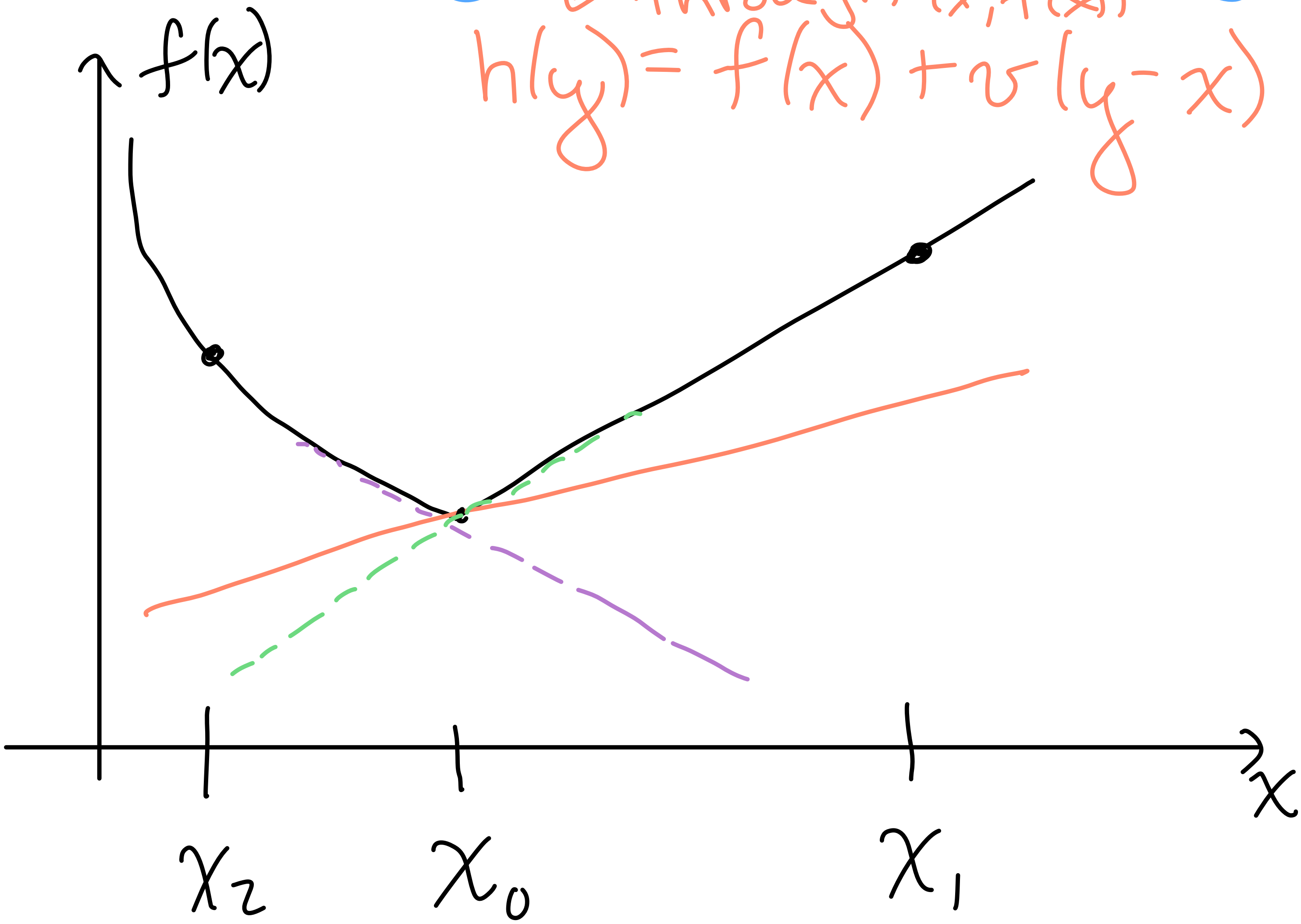
$$\frac{f(q)-f(p)}{q-p} \underset{☆}{\leq} \frac{f(r)-f(p)}{r-p} \underset{☆}{\leq} \frac{f(r)-f(q)}{r-q}$$

Def: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, the <u>subdifferential</u> of f at $x \in \mathbb{R}$ is

$$\partial f(x) := \{ v \in \mathbb{R} : f(y) \geq f(x) + v(y-x) \ \forall y \in \mathbb{R} \}$$

line with slope v through $(x, f(x))$

$$h(y) = f(x) + v(y-x)$$



<u>Def</u>: $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is <u>proper</u> if $D(f) \neq \emptyset$.

<u>Remark</u>:

f is proper and $x \notin D(f) \Rightarrow \partial f(x) = \emptyset$.

x is a minimizer of $f \Leftrightarrow 0 \in \partial f(x)$

<u>Def</u>: <u>Given a function</u> $f: \mathbb{R} \to \overline{\mathbb{R}}$, $a \in \overline{\mathbb{R}}$, $L \in \overline{\mathbb{R}}$, we say

$$\lim_{x \to a} f(x) = L$$

if, for all sequences $x_n \in \mathbb{R} \setminus \{a\}$ s.t. $x_n \to a$, $\lim_{n\to\infty} f(x_n) = L$.

The previous def is sometimes known as "two sided limit".

The following is "one sided limit":

Def: Given a function $f: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $a \in \mathbb{R}$, $L \in \overline{\mathbb{R}}$, we say

$$\lim_{x \to a^+} f(x) = L$$

"limit as x approaches a from above"

if, for all sequences $x_n \in (a, +\infty)$ s.t. $x_n \to a$, $\lim_{n \to \infty} f(x_n) = L$.

Likewise, we define $\lim_{x \to a^-} f(x) = L$.

Prop: Given a function $f:\mathbb{R}\to\overline{\mathbb{R}}$, $a\in\mathbb{R}$, $L\in\overline{\mathbb{R}}$,

$$\lim_{x\to a} f(x) = L \text{ iff } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Lemma: Given a function $f:\mathbb{R}\to\overline{\mathbb{R}}$, $a\in\mathbb{R}$, $L\in\overline{\mathbb{R}}$,

$$\lim_{x\to a^+} f(x) = L$$

iff $\forall$ sequences $x_n\in(a,+\infty)$ s.t. $\underbrace{x_n \downarrow a}$, we have $\lim_{n\to\infty} f(x_n) = L$.

i.e. x_n is decreasing and $x_n \to a$

Analogous result for $\lim_{x\to a^-} f(x)$ and

$x_n \nearrow a$.

$\left\{ \inf\limits_{x\in C} f(x) = \inf\{f(x) : x \in C\} \right\}$

Pf: Practice Midterm 2.

Def: A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is

increasing if $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$

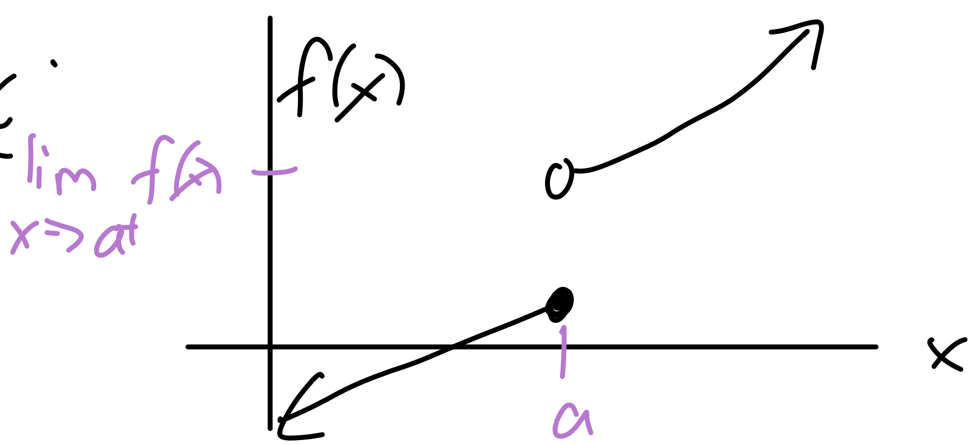
decreasing if $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$.

Prop: If $f : \mathbb{R} \to \overline{\mathbb{R}}$, $a \in \mathbb{R}$,

$$f \text{ increasing} \Rightarrow \begin{cases} \lim\limits_{x\to a^+} f(x) \overset{\mathrm{I}}{=} \inf\limits_{x>a} f(x) \\ \lim\limits_{x\to a^-} f(x) \overset{\mathrm{II}}{=} \sup\limits_{x<a} f(x) \end{cases}$$

$$f \text{ decreasing} \Rightarrow \begin{cases} \lim\limits_{x\to a^+} f(x) \overset{\mathrm{III}}{=} \sup\limits_{x>a} f(x) \\ \lim\limits_{x\to a^-} f(x) \overset{\mathrm{IV}}{=} \inf\limits_{x<a} f(x) \end{cases}$$

Ex:



Pf: We will show (I). Suppose f is increasing and $L := \inf_{x>a} f(x)$

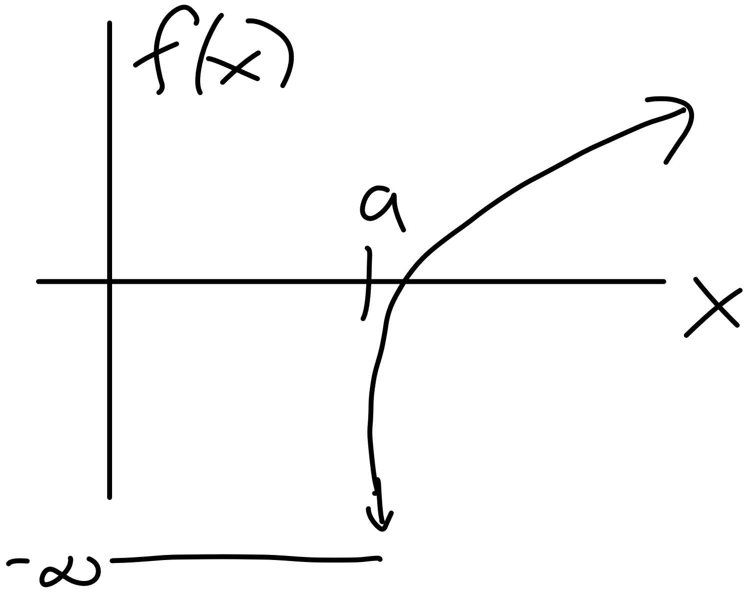
Fix $x_n : \mathbb{N} \to (a, +\infty)$ s.t. $x_n \to a$.
It suffices to show $\lim_{n\to\infty} f(x_n) = L$.

First, suppose $L \in \mathbb{R}$. Then $\forall \varepsilon > 0$, $\exists\, x_0 > a$ s.t. $f(x_0) < L + \varepsilon$.
Since $x_n \to a$, $\exists N$ s.t. $\forall n \geq N$ $x_n < x_0 \Rightarrow$

$$L \leq f(x_n) \leq f(x_0) < L + \varepsilon.$$

Thus $\lim_{n\to\infty} f(x_n) = L$.

Now, suppose $L = -\infty$, so $\{f(x) : x > a\}$ is unbdd below.
Fix $M \in \mathbb{R}$. Then $\exists\, x_0 > a$ s.t.



$f(x_0) < M$. Since $x_n \to a$
$\exists N$ s.t. $\forall n \geq N$, $x_n < x_0 \Rightarrow$

$$f(x_n) \leq f(x_0) < M.$$

Thus $\lim_{n\to\infty} f(x_n) = -\infty$.

Next, want to show III.
Then $g = -f$ is increasing,

$$-\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \inf_{x > a} g(x) = -\sup_{x > a} f(x)$$

Next, show (III). Let $g(x) = f(-x)$.
Then g decreasing,

$$\lim_{y \to a^-} f(y) = \lim_{x \to -a^+} g(x) = \sup_{x > -a} g(x)$$
$$= \sup\{f(-x) : x > -a\}$$
$$= \sup\{f(y) : y < a\}$$

(IV) similar :) $\square$

Def: Given $f : \mathbb{R} \to \overline{\mathbb{R}}$, x s.t. $f(x) \in \mathbb{R}$, the right derivative of f at x is

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

left derivative $= \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$

$$f'_-(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$

Finally, the <u>derivative of</u> <u>f at x</u> is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Rmk</u>: For many f, these do not exist. If $f'(x)$ exists, we say f is <u>differentiable</u> at x.

<u>Rmk</u>: If f and g are differentiable at x, then $(f+g)'(x) = f'(x) + g'(x)$.

<u>Cor</u>: Given $f: \mathbb{R} \to \mathbb{R}$ x s.t. $f(x) \in \mathbb{R}$,

$f_+'(x)$, $f_-'(x)$ exist, $\Longleftrightarrow$ $f'(x)$ exists
$f_+'(x) = f_-'(x)$

If either equivalent condition holds,

$$f_+'(x) = f_-'(x) = f'(x)$$

Pf: This follows from prev proposition.

When f is convex, the left and right derivatives have nice properties and characterize $\partial f(x)$...

Thm: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, then for all $x \in D(f)$,

$$\sup_{h>0} \frac{f(x)-f(x-h)}{h} \overset{\text{I}}{=} f_-'(x) \overset{\text{II}}{\leq} f_+'(x) \overset{\text{III}}{=} \inf_{h>0} \frac{f(x+h)-f(x)}{h}$$

Pf: If $x-h \notin D(f)$ for all $h>0$, then I & II hold since $-\infty = -\infty$.
WLOG, suppose $x - \bar{h}_0 \in D(f)$ for some $\bar{h}_0$; since $D(f)$ interval,

$x-h \in D(f) \quad \forall \ 0 \leq h < h_0.$

Analogously, we may assume $\exists \ h_1$ s.t. $x+h \in D(f) \ \forall 0 \leq h \leq h_1$

Going forward, all h's we consider will satisfy $h < \max\{h_0, h_1\}$

Recall: secant monotonicity

$p, q, r \in D(f)$, $p < q < r$,

$$\frac{f(q)-f(p)}{q-p} \leq \frac{f(r)-f(p)}{r-p} \leq \frac{f(r)-f(q)}{r-q}.$$

If $0 < h_0 < h_1$, $x-h_1 < x-h_0 < x$,

$$\frac{f(x)-f(x-h_1)}{h_1} \leq \frac{f(x)-f(x-h_0)}{h_0}$$

Thus $h \mapsto \frac{f(x)-f(x-h)}{h}$ is decreasing, so I holds.

Again, using the secant inequality, $h \mapsto \frac{f(x+h)-f(x)}{h}$ is increasing, so III holds.

It remains to show II. Since $x-h < x < x+h$ for $h>0$,

$$\frac{f(x)-f(x-h)}{h} \leq \qquad\qquad \leq \frac{f(x+h)-f(x)}{h}$$

Sending $h \to 0^+$, $f_-'(x) \leq f_+'(x)$. $\square$

We can now characterize ∂f.

<u>Thm</u>: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex and $x \in D(f)$

$$\partial f(x) = [f'_-(x), f'_+(x)]$$

<u>Pf</u>: By def of $\partial f(x)$,

$$v \in \partial f(x)$$

$$\Updownarrow$$

$$f(y) \geq f(x) + v(y-x) \quad \forall y \in \mathbb{R}$$

$$\Updownarrow$$

$$\frac{f(y)-f(x)}{y-x} \geq v \qquad \forall y > x$$

$$\frac{f(y)-f(x)}{y-x} \leq v \qquad \forall y < x$$

$\Updownarrow$

$$\inf_{y>x} \frac{f(y)-f(x)}{y-x} \geq v$$

$$\sup_{y<x} \frac{f(y)-f(x)}{y-x} \leq v$$

$\Updownarrow$

$$\inf_{h>0} \frac{f(x+h)-f(x)}{h} \geq v$$

$$\sup_{h>0} \frac{f(x)-f(x-h)}{h} \leq v$$

$\Updownarrow$

$$f_+'(x) \geq v$$
$$f_-'(x) \leq v$$

HW: $f+g$ is convex

$\square$

Cor: If $f, g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, $x \in D(f) \cap D(g)$, then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)$$

Pf: By prev thm,

$$\partial f(x) = [f_-'(x), f_+'(x)]$$
$$\partial g(x) = [g_-'(x), g_+'(x)]$$
$$\partial (f+g)(x) = [f_-'(x) + g_-'(x), f_+'(x) + g_+'(x)]$$

$\square$

The monotonicity of secant lines extends to the subdifferential.

Thm: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, $x, y \in D(f)$, if $u \in \partial f(x)$ and $v \in \partial f(y)$, then

$$(u-v)(x-y) \geq 0.$$

i.e. $x \geq y \Rightarrow u \geq v$

Pf:

By defn,

$$f(y) \geq f(x) + u(y-x)$$
$$f(x) \geq f(y) \quad v(x-y)$$

Adding inequalities gives result. $\square$