Lecture 16

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Reminders:

- Fill out course evaluations
- No HW8, Practice Final posted Tuesday
- Last Lecture: Review and Q&A (email me the questions you want to discuss)
- Monday, June 8th, 11:15am-12:15pm: extra office hours

Lemma: Given a function $f: \mathbb{R} \to \overline{\mathbb{R}}$, $a \in \mathbb{R}$, $L \in \overline{\mathbb{R}}$,

$$\lim_{x \to a^+} f(x) = L$$

iff $\forall$ sequences $x_n \in (a, +\infty)$ s.t. $x_n \downarrow a$, we have $\lim_{n \to \infty} f(x_n) = L$.

i.e. x_n is decreasing and $x_n \to a$

Analogous result for $\lim_{x \to a^-} f(x)$ and $x_n \uparrow a$.

Def: A function $f:\mathbb{R}\to\overline{\mathbb{R}}$ is

increasing if $x_1\le x_2 \Rightarrow f(x_1)\le f(x_2)$

decreasing if $x_1\le x_2 \Rightarrow f(x_1)\ge f(x_2)$.

Prop: If $f:\mathbb{R}\to\overline{\mathbb{R}}$, $a\in\mathbb{R}$

f increasing $\Rightarrow$
$$\lim_{x\to a^+} f(x) \overset{\text{I}}{=} \inf_{x>a} f(x)$$
$$\lim_{x\to a^-} f(x) \overset{\text{II}}{=} \sup_{x<a} f(x)$$

f decreasing $\Rightarrow$
$$\lim_{x\to a^+} f(x) \overset{\text{III}}{=} \sup_{x>a} f(x)$$
$$\lim_{x\to a^-} f(x) \overset{\text{IV}}{=} \inf_{x<a} f(x)$$

<u>Def</u>: Given $f: \mathbb{R} \to \mathbb{R}$, x s.t. $f(x) \in \mathbb{R}$, the <u>right derivative of f at x</u> is

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

<u>left derivative</u> $= \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$

$$f'_-(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$

Finally, the <u>derivative of f at x</u> is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

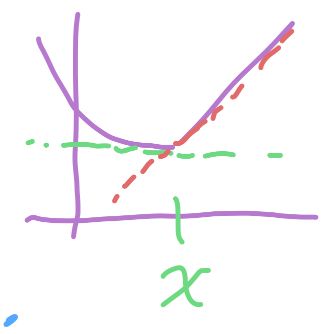
<u>Rmk</u>: For many f, these do not exist. If $f'(x)$ exists, we say f is <u>differentiable</u> at x.

<u>Cor</u>: Given $f:\mathbb{R}\to\overline{\mathbb{R}}$, x s.t. $f(x)\in\mathbb{R}$,

$f_+'(x)$, $f_-'(x)$ exist, $\iff$ $f'(x)$ exists
$f_+'(x)=f_-'(x)$.

If either equivalent condition holds,

$$f_+'(x)=f_-'(x)=f'(x).$$



<u>Thm</u>: Given $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ convex, then for all $x\in D(f)$,

$$\sup_{h>0}\frac{f(x)-f(x-h)}{h} \overset{\mathrm{I}}{=} f_-'(x) \overset{\mathrm{II}}{\leq} f_+'(x) \overset{\mathrm{III}}{=} \inf_{h>0}\frac{f(x+h)-f(x)}{h}$$

We can now characterize ∂f.

<u>Thm</u>: Given $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex and $x \in D(f)$

$$\partial f(x) = [f'_-(x), f'_+(x)] \cap \mathbb{R}$$

<u>Pf</u>: By def of $\partial f(x)$, for any $v \in \mathbb{R}$,

$$v \in \partial f(x)$$
$$\Updownarrow$$
$$f(y) \geq f(x) + v(y-x) \quad \forall y \in \mathbb{R}$$
$$\Updownarrow$$
$$\frac{f(y)-f(x)}{y-x} \geq v \qquad \forall y > x$$
$$\frac{f(y)-f(x)}{y-x} \leq v \qquad \forall y < x$$

$$\Updownarrow$$

$$\inf_{y>x} \frac{f(y)-f(x)}{y-x} \geq v$$

$$\sup_{y<x} \frac{f(y)-f(x)}{y-x} \leq v$$

$$\Updownarrow$$

$$\inf_{h>0} \frac{f(x+h)-f(x)}{h} \geq v$$

$$\sup_{h>0} \frac{f(x)-f(x-h)}{h} \leq v$$

$$\Updownarrow$$

$$f_+'(x) \geq v$$

$$f_-'(x) \leq v$$

HW: $f+g$ is convex

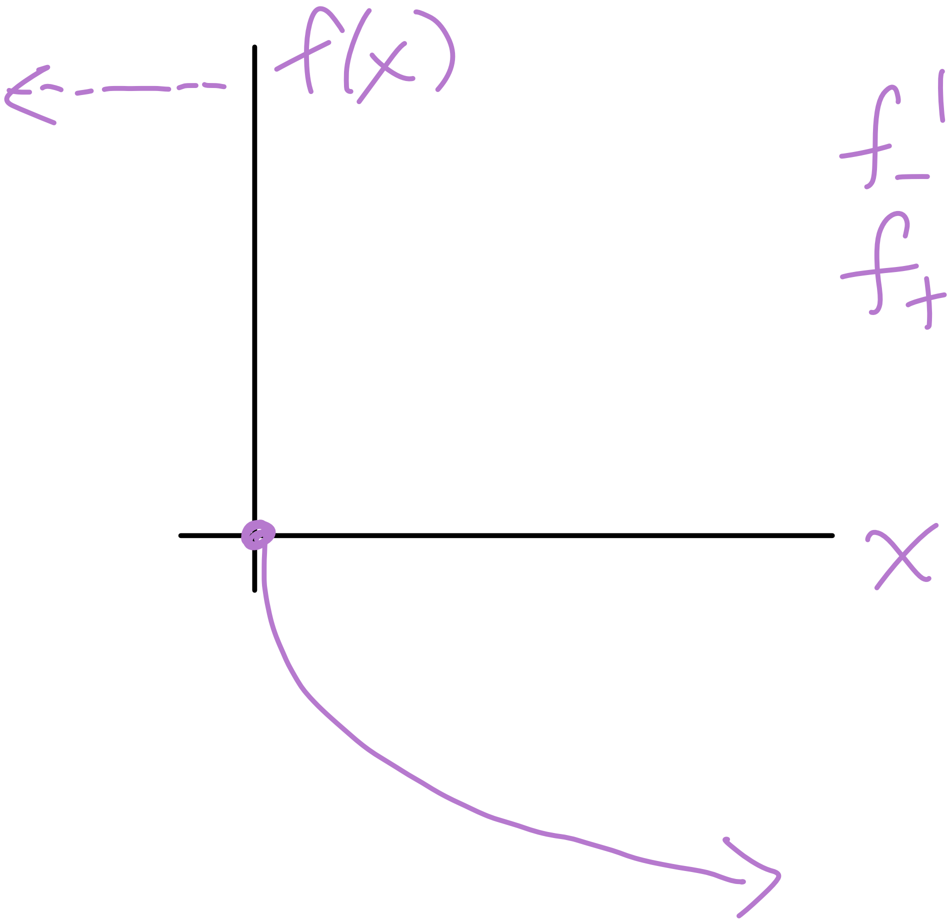
$\square$

Cor: If $f, g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, $x \in D(f) \cap D(g)$, then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)$$

Example for why we need $\bigcap \mathbb{R}$:

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0 \end{cases}$$



$$f_-'(0) = -\infty$$
$$f_+'(0) = -\infty$$

in calculus...

$$f'(x) = \frac{-1}{2}\frac{1}{x} \xrightarrow{x \to 0^+} -\infty$$

Fact: $\nexists\, v \in \mathbb{R}$ s.t.

$$f(y) \geq vy \quad \forall\, y \in \mathbb{R}$$

Thus $\partial f(0) = \emptyset$.

Pf: By prev thm,

$$\partial f(x) = [f_-'(x), f_+'(x)] \cap \mathbb{R}$$
$$\partial g(x) = [g_-'(x), g_+'(x)] \cap \mathbb{R}$$
$$\partial (f+g)(x) = [f_-'(x) + g_-'(x), f_+'(x) + g_+'(x)] \cap \mathbb{R}$$

$\square$

The monotonicity of secant lines extends to the subdifferential.

Thm: Given $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex, $x, y \in D(f)$ if $u \in \partial f(x)$ and $v \in \partial f(y)$, then

$$(u - v)(x - y) \geq 0.$$

i.e. $x \geq y \Rightarrow u \geq v$

Recall: $\min_{x \in C} g(x)$

Suppose $C = [a,b]$, $a<b$

Suppose $g: C \to \mathbb{R} \cup \{+\infty\}$ proper, lsc, convex.

Rmk: Convexity is not necessary for existence of a minimizer, but it does help us find a minimizer...

Rmk: If g is strictly convex, minimizers are unique.

Define $f(x) = \begin{cases} g(x) & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}$

Then, we seek $x_* \in D(f)$ s.t.

$$f(x_*) = \min_{x \in \mathbb{R}} f(x)$$

$$\Updownarrow$$

$$x_* \in \operatorname{argmin}_{x \in \mathbb{R}} f(x)$$

<u>Proximal Point Method</u>

Given $x_0 \in D(f)$, for $n \in \mathbb{N}$, define

$$x_n = \operatorname{argmin}_{x \in \mathbb{R}} \underbrace{f(x) + \overbrace{|x - x_{n-1}|^2}^{g_n(x)}}_{h_n(x)}.$$

Is this well-defined, in the sense that a unique minimizer always exists?

Thm: Given $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc, then for any $x_0 \in D(f)$ and $n \in \mathbb{N}$, there exists a unique x_n satisfying

The proof relies on a lemma...

Lemma: Given f as in thm, $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$f(x) \geq \alpha x + \beta \quad \forall x \in \mathbb{R}.$$

Pf: Recall $D(f)$ is an interval. If $D(f) = \{x_0\}$ for some $x_0 \in \mathbb{R}$, then $f(x) \geq f(x_0)$ $\forall\, x \in \mathbb{R}$.

Otherwise, $\exists\ a, b \in \mathbb{R},\ a<b,$ s.t. $[a,b] \subseteq D(f)$.

Define $m := \dfrac{f(b)-f(a)}{b-a}$.

By monotonicity of the slope,

$$x>b \Rightarrow m \leq \frac{f(x)-f(b)}{x-b}$$

$$f(x) \geq \underbrace{f(b)+m(x-b)}_{(*)}$$

these are equal

$$x<a \Rightarrow \frac{f(a)-f(x)}{a-x} \leq m$$

$$f(x) \geq \underbrace{f(a)+m(x-a)}_{(*)}$$

Thus $(*)$ is an affine function

that bounds f from below on $[a,b]^c$.

Since $\exists\ c \in [a,b]$ s.t.

$$f(c) \leq f(x) \quad \forall\ x \in [a,b].$$

Thus, there exists $\alpha, \beta \in \mathbb{R}$ s.t. $f(x) \geq \alpha x + \beta \quad \forall\ x \in \mathbb{R}$. $\square$

<u>Pf of Thm</u>:
~~Note~~ that $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper, strictly convex, lsc fn.
First, we will show a minimizer x_n exists.
<u>Claim</u>: There exists $m \in \mathbb{N}$ s.t.
$$\min_{x \in \mathbb{R}} h(x) = \min_{x \in [-m,m]} h(x).$$

Assume, for the sake of contradiction, that for all $m \in \mathbb{N}$, $\exists\, z_m \in [-m,m]^c$ s.t.

$$h(z_m) \le \min_{x \in [-m,m]} h(x)$$

$|z_m| \ge m$ $\forall m$

Since h is proper, $\exists\, m_0$ s.t. $\displaystyle\min_{x \in [-m_0,m_0]} h(x) = C < +\infty$.

Thus $\forall\, m \ge m_0$,

$$f(z_m) + |z_m - x_{n-1}|^2 = h(z_m) \le C.$$

$$\rotatebox{90}{$\le$} \quad \exists\, \alpha, \beta \in \mathbb{R}$$

$$\alpha z_m + \beta + |z_m - x_{n-1}|^2$$

$$\|$$

$$\underbrace{z_m(\alpha - 2x_{n-1} + z_m) + \beta + x_{n-1}^2}_{\star}$$

By , $\exists$ a subsequence z_{m_k} s.t. either $\lim_k z_{m_k} = +\infty$ or $\lim_k z_{m_k} = -\infty$.

However, taking limit of $\star$ along this subsequence gives $+\infty$, contradicting that it is bdd above by c.

Finally, since h is strictly convex, its minimizers are unique. $\square$

<u>Thm</u>: Given $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc. Suppose a minimizer of f exists. Then $\forall x_0 \in D(f)$, the proximal point sequence x_n satisfies

$x_n \to x_*$, where x_* is a minimizer of f.

Pf: Since x_n minimizes h_n $\{h_n(y) \geq h_n(x_n) + 0\}$

$$0 \in \partial h_n(x_n)$$

Furthermore, $\forall n \in \mathbb{N}$,

$$f(x_n) \leq h_n(x_n) \leq h_n(x_{n-1}) = f(x_{n-1})$$

Thus $f(x_n) \leq f(x_0) \quad \forall n \in \mathbb{N}$.

In particular, $x_n \in D(f) \cap D(q_n)$, so

$$0 \in \partial h_n(x_n) = \partial f(x_n) + \partial q_n(x_n)$$

Fact: $q_n'(x) = 2(x - x_{n-1})$

Thus $\partial q_n(x_n) = \{2(x_n - x_{n-1})\}$.

Hence

$$2(x_{n-1} - x_n) \in \partial f(x_n).$$

Finish next time...