Lecture 17

CS 117, S26 © Katy Craig, 2026

Reminders:

- Fill out course evaluations
- No HW8, Practice Final posted
- Monday, June 8th, 11:15am-12:15pm: extra office hours
- Will email when midterm 2 is ready to be picked up

Recall:

we seek $x_* \in D(f)$ s.t.

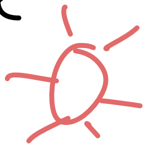
$$f(x_*) = \min_{x \in \mathbb{R}} f(x)$$

$$\Updownarrow$$

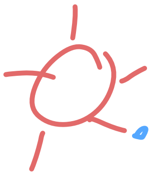
$$x_* \in \operatorname{argmin}_{x \in \mathbb{R}} f(x)$$

Proximal Point Method

Given $x_0 \in D(f)$, for $n \in \mathbb{N}$, define



$$x_n = \operatorname*{argmin}_{x \in \mathbb{R}} \underbrace{f(x) + \overbrace{|x - x_{n-1}|^2}^{g_n(x)}}_{h_n(x)}.$$

Thm: Given $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc, then for any $x_0 \in D(f)$ and $n \in \mathbb{N}$, there exists a unique x_n satisfying .

Lemma: Given f as in thm, $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$f(x) \geq \alpha x + \beta \quad \forall x \in \mathbb{R}.$$

Thm: Given $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc. Suppose a minimizer of f exists. Then $\forall\, x_0 \in D(f)$, the proximal point sequence x_n satisfies $x_n \to x_*$, where x_* is a minimizer of f.

Pf: Since x_n minimizes h_n

$h_n(y) \geq h_n(x_n) + 0$

$$0 \in \partial h_n(x_n)$$

Furthermore, $\forall\, n \in \mathbb{N}$,

$$f(x_n) \leq h_n(x_n) \leq h_n(x_{n-1}) = f(x_{n-1})$$

Thus $f(x_n) \leq f(x_0) \quad \forall\, n \in \mathbb{N}$.

In particular, $x_n \in D(f) \cap D(q_n)$, so

$$0 \in \partial h_n(x_n) = \partial f(x_n) + \partial q_n(x_n)$$

Fact: $q_n'(x) = 2(x - x_{n-1})$

Thus $\partial q_n(x_n) = \{2(x_n - x_{n-1})\}$.

Hence

$$2(x_{n-1} - x_n) \in \partial f(x_n).$$

Fix $x_* \in \underset{x \in \mathbb{R}}{\operatorname{argmin}} f(x)$.

Then $0 \in \partial f(x_*)$.

By monotonicity of ∂f,

$2(x_{n-1}-x_n) \in \partial f(x_n)$, $0 \in \partial f(x_*)$,

$$\left(2(x_{n-1}-x_n)\cancel{-0}\right)(x_n-x_*) \geq 0$$

Thus,

$$|x_{n-1}-x_*|^2$$

$$= |x_{n-1}-x_n + x_n - x_*|^2$$

$$\geq |x_n-x_{n-1}|^2 + |x_n-x_*|^2$$

We see...

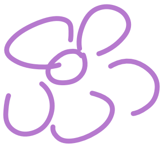
$(*)$ $n \mapsto |x_n-x_*|^2$ is decreasing

Furthermore,

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}|^2 \leq \sum_{n=1}^{\infty} |x_{n-1} - x_*|^2 - (x_n - x_*)^2$$
$$\leq |x_0 - x_*|^2$$

Since the series of nonneg terms is bounded, it converges, so the sequence of terms must go to zero, i.e.,

$$\lim_{n \to 0} |x_n - x_{n-1}|^2 = 0.$$



Since ⊛ ensures
$|x_n| - |x_*| \leq |x_n - x_*| \leq |x_0 - x_*|$,

so x_n is a bounded sequence.

Thus, by Bolzano-Weierstrass, $\exists\, \bar{x} \in \mathbb{R}$ s.t. $\lim_{k\to\infty} x_{n_k} = \bar{x}$.

By ✿, we also have $\lim_{k\to\infty} x_{n_k+1} = \bar{x}$

By definition of $2(x_{n-1} - x_n) \in \partial f(x_n)$, we have

$$f(y) \geq f(x_{n_k}) + 2(x_{n_k-1} - x_{n_k})(y - x_{n_k}), \; \forall y \in \mathbb{R}$$

$$\downarrow y = x_*$$

$$f(x_*) \geq f(x_{n_k}) + 2(x_{n_k-1} - x_{n_k})(x_* - x_n)$$

$$f(x_*) \geq \liminf_{k\to\infty} f(x_{n_k}) + 2(x_{n_k-1} - x_{n_k})(x_* - x_n)$$

lsc

$$\geq f(\bar{x}) + 0$$

Thus $\bar{x} \in \operatorname{argmin}_{x \in \mathbb{R}} f(x)$

Applying (★) for the minimizer $x_* = \bar{x}$, we have

$$n \mapsto |x_n - \bar{x}|^2 \text{ is decreasing.}$$

Thus $\exists\, L \in \mathbb{R}$ s.t.

$$\lim_{n\to\infty} |x_n - \bar{x}|^2 = L.$$

Furthermore, any subsequence must have the same limit. Thus,

$$\lim_{k\to\infty} |x_{n_k} - \bar{x}|^2 = L$$

Since $x_{n_k} \to \bar{x}$, we have $L = 0$.

Thus $x_n \to \bar{x}$. $\square$

Midterm 2

Question 4- Extra Credit

Choose $\tilde{p},\tilde{q} \in D(f)$, $(\tilde{p},\tilde{q}) \subseteq D(f)$.

Fix $a,b \in (\tilde{p},\tilde{q})$, $a<b$. We must find $L_{a,b} \in \mathbb{R}$ so that

$$|f(x)-f(y)| \le L_{a,b}|x-y|, \forall x,y \in (a,b).$$

Assume WLOG $x<y$.

Then $\tilde{p}<a<x<y<b<\tilde{q}$, so...

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(y)}{b-y} \overset{y<b<\tilde{q}}{\le} \frac{f(\tilde{q})-f(b)}{b-\tilde{q}}$$

Analogously, $\exists C_{a,b} \in \mathbb{R}$

$$\frac{f(y)-f(x)}{y-x} \ge C_{a,b}$$

Define

$$L_{ab} := \max\left\{ \left|\frac{f(\tilde{q})-f(b)}{b-\tilde{q}}\right|, |C_{a,b}| \right\}$$

Then,

$$-L_{ab} \le \frac{f(y)-f(x)}{y-x} \le L_{ab}$$

$$\Updownarrow$$

$$\left|\frac{f(y)-f(x)}{y-x}\right| \le L_{ab}$$

$$\Updownarrow$$

$$|f(y)-f(x)| \le L_{ab}|x-y|$$