Lecture 1

CS 117, S26 © Katy Craig, 2026

Course goal: transition to upper division

"What is a proof?" ⇒ "Let's prove interesting things!"

This is a mathematical writing course.

↳ You must back up your claims using clear, logical arguments.

↳ You must be able to precisely state important definitions and theorems.

Pro tips:

① Buy and read the textbook.

② Come to office hours

③ Ask for help, early and often! (Often it's a tiny missing piece that throws you off.)

Q: Why analysis? What is analysis?

A: It takes everything you learned in Calculus and puts it on rigorous mathematical footing.

A': Analysis is the mathematics of approximation and perturbation, allowing us to understand and formulate mathematical models in terms of how they behave in the limit or when "jostled" or "nudged".

- ▷ more data, more computational power...
- ▷ measurement inaccuracies, robustness to adversary...

Numbers!

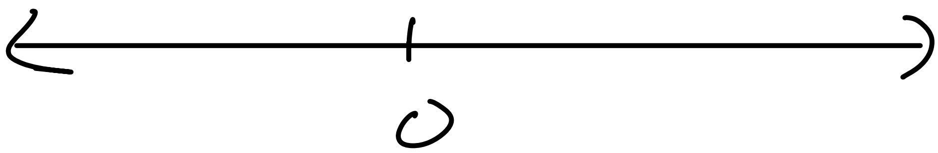
Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, \ldots\}$

Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

Real numbers $\mathbb{R} = ?$

Intuitively, $\mathbb{R}$ is all the numbers on the number line...



Goal: define $\mathbb{R}$.

<u>Def</u>: A binary operation on a set X is a function from $X \times X$ to X.

Def: A set F is a field if it has two binary operations (addition and multiplication) that satisfy the following properties (for all $a,b,c \in F$):

(A1) $a+(b+c)=(a+b)+c$
(A2) $a+b=b+a$
(A3) $\exists!$ element $0 \in F$ s.t. $\forall a \in F$, $a+0=a$
(A4) for each $a \in F$, $\exists!\ b \in F$ s.t. $a+b=0$; denote $-a:=b$

(M1) $a(bc)=(ab)c$
(M2) $ab=ba$
(M3) $\exists!$ element $1 \in F \setminus \{0\}$ s.t. $\forall a \in F$, $a \cdot 1=a$

(M4) for each $a \in F \setminus \{0\}$, $\exists!$ $b \in F$ s.t. $ab = 1$; denote $\frac{1}{a} = a^{-1} := b$.

(D2) $a(b+c) = ab + ac$

<u>Remark</u>: $\mathbb{Z}$ is not a field

<u>Thm</u>: $\mathbb{Q}$ is a field

Using the definition of a field, you can rigorously prove familiar algebraic properties.

<u>Thm</u>: If F is a field, then $\forall a, b, c \in F$,

(i) if $a+c=b+c$, then $a=b$
(ii) $a\cdot 0=0$

Pf: First, we show (i). By (A4), $\exists -c$ s.t. $c+(-c)=0$. Thus,

$$a+c=b+c \implies a+c+(-c)=b+c+(-c)$$
$$\overset{(A1)}{\implies} a+(c+(-c))=b+(c+(-c))$$
$$\overset{(A4)}{\implies} a+0=b+0$$
$$\overset{(A3)}{\implies} a=b$$

We now show (ii). By (A3), $0=0+0$, so $\forall a\in F$,

$$a\cdot 0=a\cdot(0+0) \overset{(D2)}{\implies} a\cdot 0=a\cdot 0+a\cdot 0$$
$$\overset{(A3)}{\implies} 0+a\cdot 0=a\cdot 0+a\cdot 0$$
$$\overset{(i)}{\implies} 0=a\cdot 0. \quad \square$$

Along with the behavior of addition and multiplication, another equally important attribute of $\mathbb{R}$ is that its elements possess an "order", from left to right on the real line.

Def: A field F is an <u>ordered field</u> if it has an ordering relation $\leq$ so that, $\forall a,b,c \in F$

(O1) either $a \leq b$ or $b \leq a$ totality

(O2) if $a \leq b$ and $b \leq a$, then $a = b$

antisymmetry

(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$

transitivity

(O4) if $a \leq b$, then $a + c \leq b + c$

(O5) if $a \le b$ and $c \ge 0$, then $ac \le bc$

Def: Given an ordered field F and $a, b \in F$, if $a \le b$ and $a \ne b$, then write $a < b$.

Using defn of an ordered field, we obtain familiar rules about inequalities.

Thm: Suppose F is an ordered field. Then $\forall\, a, b, c \in F$,

(i) $a \le b \Rightarrow -b \le -a$

(ii) $a \le b$ and $c \le 0 \Rightarrow ac \ge bc$

(iii) $a \ge 0,\ b \ge 0 \Rightarrow ab \ge 0$

(iv) $0 \le a^2$, where $a^2 = a \cdot a$

(v) $0 < a \Rightarrow 0 < \frac{1}{a}$

Pf: We will show (i) and (iii).

To see (i), suppose $a \leq b$.
By (O4), $a + (-a) + (-b) \leq b + (-a) + (-b)$.
By (A1–A4), $-b \leq -a$.

To see (iii), suppose $a \geq 0$, $b \geq 0$.
By (O5), $0 \cdot b \leq a \cdot b$. By previous theorem, $0 \cdot b = 0$, which gives the result.

Rmk: For any ordered field, $0 < 1$.

Thm: $\mathbb{Q}$ is an ordered field

Ex: $\left\{ \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} : q \in \mathbb{Q} \right\}$ is an ordered field.

<u>Rmk</u>: We will show on the homework that any ordered field F has a subfield that is isomorphic to $\mathbb{Q}$.

An important property of an ordered field is...

<u>Thm</u>: Suppose F is an ordered field. Then $\forall p, q \in F$ with $p < q$, $\exists r \in F$ s.t. $p < r < q$.

Pf: Let $2 := 1+1$. Since $0<1$,
$\underbrace{0+1}_{1} < \underbrace{1+1}_{2}$, so $0<2$.

By previous thm (v), $0 < \frac{1}{2}$.

$\boxed{\text{Idea: } r = \frac{p+q}{2}}$

By (D2), $p+p = (1+1)p = 2p$.
By (M3) $\frac{p+p}{2} = p$.

Therefore,

$$p = \frac{p+p}{2} \overset{(D2)}{=} \frac{p}{2} + \frac{p}{2} \overset{(O4)}{<} \overbrace{\frac{p}{2} + \frac{q}{2}}^{r:=} \overset{(O4)}{<} \frac{q}{2} + \frac{q}{2} = q$$

On any ordered field F, we may define a notion of absolute value and distance.

Def: For any $a \in F$, $|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$

Thm: (basic properties of $|\cdot|$)
For all $a, b \in F$,

(i) $|a| \geq 0$

(ii) $|ab| = |a||b|$

(iii) $|a| \geq a$ and $|a| \geq -a$

(iv) $|a+b| \leq |a| + |b|$

triangle inequality

Pf: Homework

Def: (distance) For any $a, b \in F$,

$$\text{dist}(a,b) = |a-b|.$$