Lecture 2

CS 117, S26 © Katy Craig, 2026

Office Hours: Tuesdays and Wednesdays, 11:15am-12:15pm, SH 6507

Homework typically due Thursdays at 11:59pm

Homework 1 due **Monday**, April 6th at 11:59pm

This week: extra office hours **tomorrow** (Thursday) 11:15am-12:15pm

Numbers!

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, \dots\}$

Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

Real numbers $\mathbb{R} = ?$

Def: A binary operation on a set X is a function from $X \times X$ to X.

<u>Def</u>: A set F is a <u>field</u> if it has two binary operations (addition and multiplication) that satisfy the following properties (for all $a,b,c \in F$):

(A1) $a+(b+c) = (a+b)+c$

(A2) $a+b = b+a$

(A3) $\exists!$ element $0 \in F$ s.t. $\forall a \in F$, $a+0 = a$

(A4) for each $a \in F$, $\exists!$ $b \in F$ s.t. $a+b = 0$; denote $-a := b$

(M1) $a(bc) = (ab)c$

(M2) $ab = ba$

(M3) $\exists!$ element $1 \in F \setminus \{0\}$ s.t. $\forall a \in F$, $a \cdot 1 = a$

(M4) for each $a \in F \setminus \{0\}$, $\exists !$ $b \in F$ s.t. $ab = 1$;
denote $\frac{1}{a} = a^{-1} := b$.

(D2) $a(b+c) = ab + ac$

<u>Thm</u>: $\mathbb{Q}$ is a field

<u>Thm</u>: If F is a field, then $\forall a, b, c \in F$,
(i) if $a + c = b + c$, then $a = b$
(ii) $a \cdot 0 = 0$

<u>Def</u>: A field F is an <u>ordered field</u> if it has an ordering relation $\leq$ so that, $\forall a, b, c \in F$

(O1) either $a \le b$ or $b \le a$ totality

(O2) if $a \le b$ and $b \le a$, then $a = b$
antisymmetry

(O3) if $a \le b$ and $b \le c$, then $a \le c$
transitivity

(O4) if $a \le b$, then $a + c \le b + c$

(O5) if $a \le b$ and $c \ge 0$, then $ac \le bc$

<u>Def</u>: Given an ordered field F and $a, b \in F$, if $a \le b$ and $a \ne b$, then write $a < b$.

Thm: Suppose F is an ordered field. Then $\forall\, a,b,c \in F$,

(i) $a \le b \Rightarrow -b \le -a$

(ii) $a \le b$ and $c \le 0 \Rightarrow ac \ge bc$

(iii) $a \ge 0,\ b \ge 0 \Rightarrow ab \ge 0$

(iv) $0 \le a^2$, where $a^2 = a \cdot a$

(v) $0 < a \Rightarrow 0 < \frac{1}{a}$

Rmk: For any ordered field, $0 < 1$.

Thm: $\mathbb{Q}$ is an ordered field

Thm: Suppose F is an ordered field. Then $\forall\, p,q \in F$ with $p < q$, $\exists\, r \in F$ s.t. $p < r < q$.

Def: For any $a \in F$, $|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$.

Thm: (basic properties of $|\cdot|$)

For all $a, b \in F$,

(i) $|a| \geq 0$

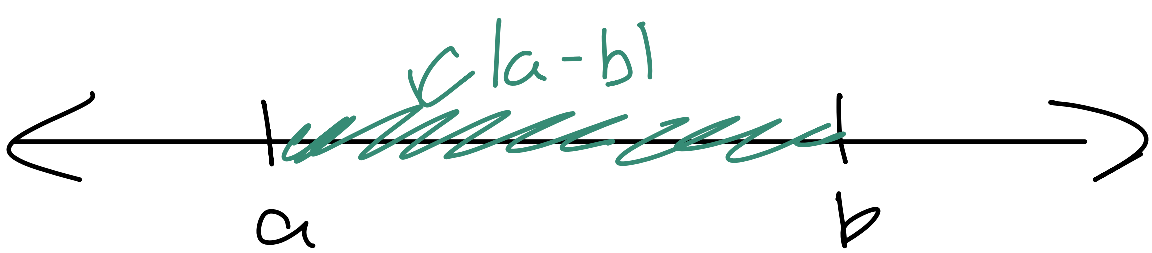
(ii) $|ab| = |a||b|$

(iii) $|a| \geq a$ and $|a| \geq -a$

(iv) $|a+b| \leq |a| + |b|$ triangle inequality

Def: (distance) For any $a, b \in F$,

$$\text{dist}(a,b) = |a-b|.$$



On any ordered field, we can define a notion of maximum and minimum.

Def (maximum/minimum)
Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ s.t. $s_0 \geq s \; \forall s \in S$, then s_0 is the <u>maximum</u> of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in S"
- If there exists $s_0 \in S$ s.t. $s_0 \leq s \; \forall s \in S$, then s_0 is the minimum of S and write $s_0 = \min(S)$.
  "s_0 is the smallest elt in S"

Ex: Given an ordered field F, then any finite set

$$\{s_1, s_2, \ldots, s_n\} =: S \subseteq F,$$

has a maximum and a minimum.

Ex: $F = \mathbb{Q}$,
$S = (0,1) \cap \mathbb{Q}$
$= \{x \in \mathbb{Q} : 0 < x < 1\}$

"does not exist"

Claim: $\max(S)$ D.N.E.

Pf of Claim: Assume, for the sake of contradiction that $s_0 = \max(S)$ exists. Since $s_0 \in S$, $0 < s_0 < 1$, so by *previous theorem

$\exists\, r \in \mathbb{Q}\quad 0 < s_0 < r < 1.$

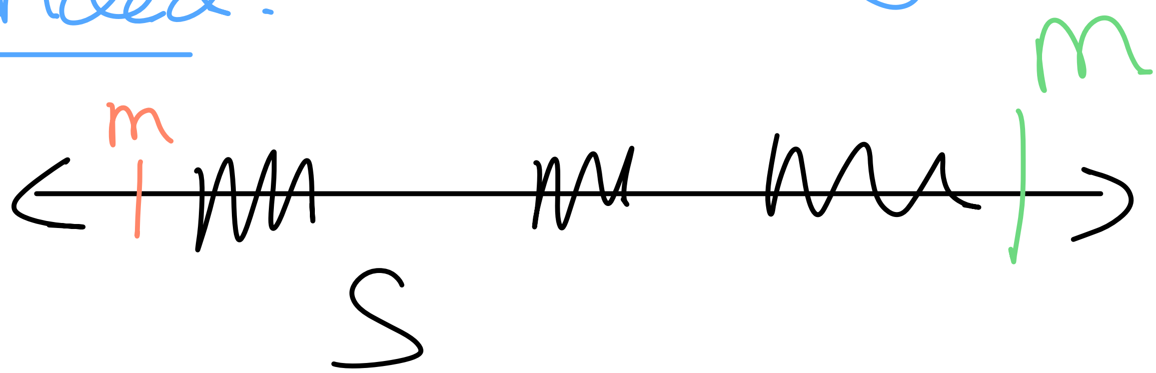
Thus $r \in S$, but $r > \max(S)$, which is a contradiction.

<u>Def</u> (bounded above/below): Suppose $S \subseteq F$, for an ordered field F,

- If there exists $M \in F$ s.t. $s \leq M \;\; \forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.

- If there exists $m \in F$ s.t. $m \leq s \;\; \forall s \in S$, then S is <u>bounded below</u> and m is a <u>lower bound</u> of S.

If S is bounded above and below, we say S is <u>bounded</u>.



Ex: $(0,1) \cap \mathbb{Q}$ is bounded
$\mathbb{N}$ is not bounded

What about when a set "almost" has a maximum?

<u>Def</u>: (supremum / infimum):
Consider an ordered field F.
- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying

(a) M_0 is an upper bound of S
(b) if M is an upper bound of S, then $M_0 \le M$,

we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.
"M_0 is the least upper bound of S"

- <u>If</u> $S \subseteq F$ is bounded below and there exists m_0 satisfying

(a) m_0 is a lower bound of S
(b) if m is a lower bound of S, $m \le m_0$,

we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.
"m_0 is the greatest lower bound"

Thm: Given $S \subseteq F$, F ordered field,

- if $\max(S)$ exists, $\sup(S) = \max(S)$
- if $\min(S)$ exists, $\inf(S) = \min(S)$

Pf: HW

Ex: $F = \mathbb{Q}$, $S = (0,1) \cap \mathbb{Q}$.

Claim: $\sup(S) = 1$

Pf of Claim: By defn of S, $s \leq 1$ $\forall s \in S$, so 1 is an upper bound of S.

Assume, for the sake of contradiction, $\exists m < 1$ which is an upper bound. Note that $m > 0$.

of S. By prev thm, $\exists\, r \in \mathbb{Q}$ s.t. $0 < M < r < 1$. Thus $r \in S$, contradicting that M is an upper bound. $\square$

"the least upper bound property of $\mathbb{R}$"

Def (real numbers): The real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above (in $\mathbb{R}$) has a supremum.

Thm: The real numbers exist and are unique.

Pf: Spivak, Calculus, last chapter.

Ex: $S := \{q \in \mathbb{Q} : q^2 < 2\} \subseteq \mathbb{Q}$

On the homework you will show that S is bounded above but does <u>not</u> have a supremum (in $\mathbb{Q}$).

How does $\mathbb{R}$ relate to other numbers?

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

$A \subsetneq B$
$\Updownarrow$
$A \subseteq B$
$A \neq B$

Previous Course: $\sqrt{2} \notin \mathbb{Q}$

Homework: $\sqrt{2} \in \mathbb{R}$

Quick Review: Induction

Inductive characterization of $\mathbb{N}$

If a subset $S \subseteq \mathbb{N}$ satisfies

(i) $1 \in S$

(ii) if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$.

mental picture:

This is the basis for proof by induction.

Suppose $\{P_1, P_2, P_3, \ldots\}$
$= \{P_k : k \in \mathbb{N}\}$
is a list of statements.

$P_k = k \cdot 3$ is even

$P_k =$ Katy wants k cookies

Suppose you can prove that

(a) P_1 is true] base case

(b) For all $n \in \mathbb{N}$, if P_n is true, then P_{n+1} is true] inductive step

inductive hypothesis

Then $S = \{k \in \mathbb{N} : P_k \text{ is true}\}$
satisfies (i) and (ii),
so $S = \mathbb{N}$ and P_k is
true for all $k \in \mathbb{N}$.

We will now study two major theorems for $\mathbb{R}$:

Thm (Archimedean Property)
If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, $\exists\, n \in \mathbb{N}$ s.t. $na > b$. ← bathtub

↑ spoon

"even with a very small spoon, you can fill a large bathtub"

Pf: Let's assume, for the sake of contradiction, that $na \le b$ $\forall\, n \in \mathbb{N}$.

Define $S := \{na : n \in \mathbb{N}\}$. Then S is bounded above.

Let $s_0 := \sup(S)$. Since $a > 0$, $s_0 - a < s_0$ and $s_0 - a$ is <u>not</u> an upper bound.

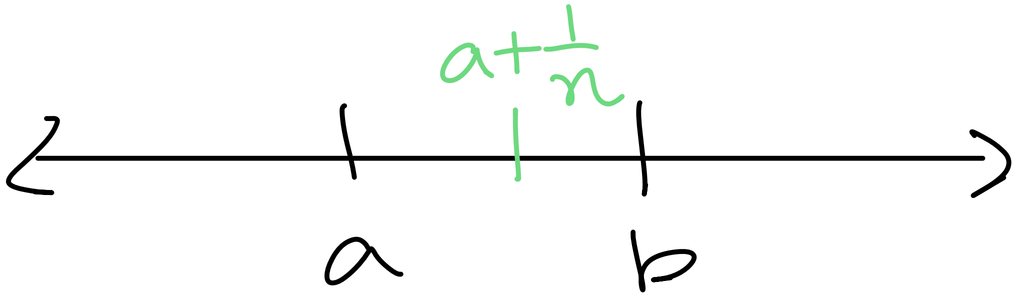
Thus $\exists\, n_0 \in \mathbb{N}$ s.t.
$n_0 a > s_0 - a \Rightarrow (n+1)a > s_0$.
This contradicts that s_0 is an upper bound. $\square$

As a consequence of A.P., we obtain a few useful lemmas...

<u>Lemma</u>: For any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $x < n$.

Pf: :)

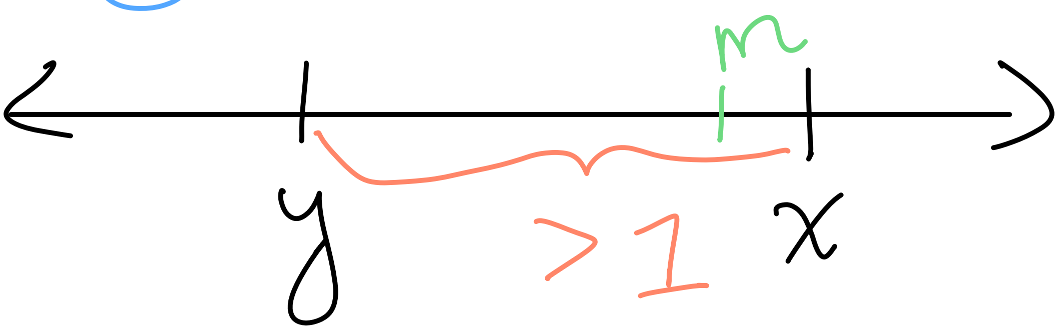
Lemma: For $a, b \in \mathbb{R}$, $a < b$, $\exists\, n \in \mathbb{N}$ s.t. $a + \frac{1}{n} < b$.



Note: $\iff \quad 1 < n(b-a)$

Pf: ☺

Lemma: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists\, m \in \mathbb{Z}$ s.t. $y < m < x$.



Pf: Next time ☺