Lecture 3

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Office Hours: Tuesdays and Wednesdays, 11:15am-12:15pm, SH 6507

Homework 1 due **Monday**, April 6th at 11:59pm

Homework 2 due Thursday, April 9th at 11:59pm

Def (maximum/minimum)
Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ s.t. $s_0 \geq s$ $\forall s \in S$, then s_0 is the maximum of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in S"
- If there exists $s_0 \in S$ s.t. $s_0 \leq s$ $\forall s \in S$, then s_0 is the minimum of S and write $s_0 = \min(S)$.

"s_0 is the smallest elt in S"

Def (bounded above/below): Suppose $S \subseteq F$, for an ordered field F,

- If there exists $M \in F$ s.t. $s \le M$ $\forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.
- If there exists $m \in F$ s.t. $m \le s$ $\forall s \in S$, then S is <u>bounded below</u> and m is a <u>lower bound</u> of S.

If S is bounded above and below, we say S is <u>bounded</u>.

<u>Def</u>: (supremum / infimum):
Consider an ordered field F.

- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying

(a) M_0 is an upper bound of S
(b) if M is an upper bound of S, then $M_0 \le M$,

we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.
"M_0 is the least upper bound of S"

- If $S \subseteq F$ is bounded below and there exists m_0 satisfying
(a) m_0 is a lower bound of S
(b) if m is a lower bound of S, $m \leq m_0$,

we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.
"m_0 is the greatest lower bound"

<u>Thm</u>: Given $S \subseteq F$, F ordered field,
- if $\max(S)$ exists, $\sup(S) = \max(S)$
- if $\min(S)$ exists, $\inf(S) = \min(S)$.

Ex: $F = \mathbb{Q}$, $S = (0,1) \cap \mathbb{Q}$.

Claim: $\sup(S) = 1$

Pf of Claim: By defn of S, $s \leq 1$ $\forall s \in S$, so 1 is an upper bound of S.

Assume, for the sake of contradiction, $\exists M < 1$ which is an upper bound of S. Note that $M > 0$. By prev thm, $\exists r \in \mathbb{Q}$ s.t. $0 < M < r < 1$. Thus $r \in S$, contradicting that M is an upper bound. $\square$

← Last time: proof by contradiction.

Alternative "direct" proof:

Suppose $x \in \overbrace{\mathbb{Q} \cap (-\infty, 1)}^{A_1}$. Then $\exists\ y \in \mathbb{Q}$ s.t. $\max(\{x, 0\}) < y < 1$, so $y \in S$ and x cannot be an upper bound. Suppose $x \in \underbrace{\mathbb{Q} \cap [1, +\infty)}_{A_2}$. Then x is an upper bound of S.

Since A_1 and A_2 partition $\mathbb{Q}$, A_2 is the set of all upper bounds of S, and its minimum element is $\sup(S) = 1$.

"the least upper bound property of $\mathbb{R}$"

Def (real numbers): The real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above (in $\mathbb{R}$) has a supremum.

Thm: The real numbers exist and are unique.

Quick Review: Induction

Inductive characterization of $\mathbb{N}$

If a subset $S \subseteq \mathbb{N}$ satisfies
(i) $1 \in S$
(ii) if $n \in S$, then $n+1 \in S$
then $S = \mathbb{N}$.

Suppose $\{P_1, P_2, P_3, \dots\}$
$= \{P_k : k \in \mathbb{N}\}$
is a list of statements.

Suppose you can prove that
(a) P_1 is true] base case
(b) For all $n \in \mathbb{N}$, if P_n is true, then P_{n+1} is true] inductive step

inductive hypothesis

Then $S = \{k \in \mathbb{N} : P_k \text{ is true}\}$ satisfies (i) and (ii), so $S = \mathbb{N}$ and P_k is true for all $k \in \mathbb{N}$.

Thm (Archimedean Property)
If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, $\exists\, n \in \mathbb{N}$ s.t. $na > b$. ← bathtub

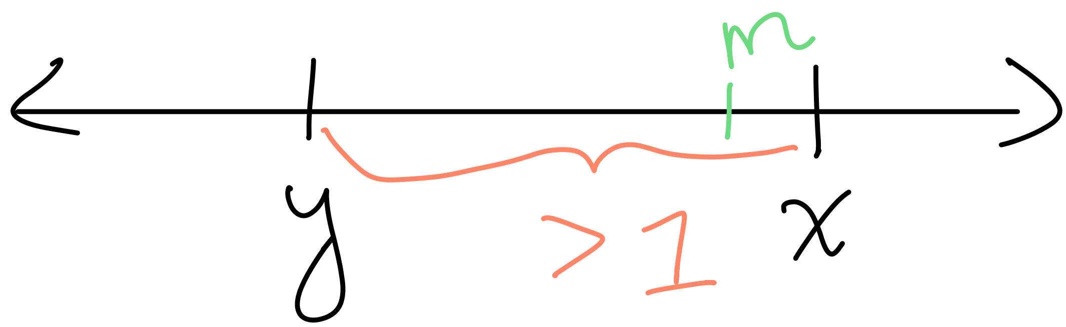
spoon

"even with a very small spoon you can fill a large bathtub"

Lemma: For any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $x < n$.

Lemma: For $a, b \in \mathbb{R}$, $a < b$, $\exists\, n \in \mathbb{N}$ s.t. $a + \frac{1}{n} < b$.

<u>Lemma</u>: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists\, m \in \mathbb{Z}$ s.t. $y < m < x$.



<u>CLAIM</u>: $S \subseteq \mathbb{Z}$, $S \neq \emptyset$, S bdd above $\Rightarrow \max(S)$ exists

Pf of Claim: Choose $s_0 \in S$. Let $M \in \mathbb{R}$ be an upper bound of S. By lemma, $\exists\, n_M \in \mathbb{N}$ s.t. $n_M > M$ and $\exists\, n_0 \in \mathbb{N}$ s.t. $n_0 > -s_0 \Leftrightarrow s_0 > -n_0$.
Consider
$S' := \{s \in S : s_0 \le s\}$.

Then $|S'| \leq n_M + n_0 + 1$.
Since S' is a finite set, it has a maximum. Since $\max(S) = \max(S')$ by construction, we have that $\max(S)$ exists.

Pf: Consider the set
$S := \{n \in \mathbb{Z} : n \leq y\}$.
By previous claim, its maximum exists.
Let $M_0 := \max(S)$. Define $m := M_0 + 1$.

To see that $m < x$, note that
$M_0 \in S \Rightarrow M_0 \leq y \Rightarrow m \leq y + 1 < x$.

Assume, for the sake of contradiction

that $m_0+1 = m \leq y$. Then $m_0+1 \in S$, contradicting that m_0 was the maximum. $\square$

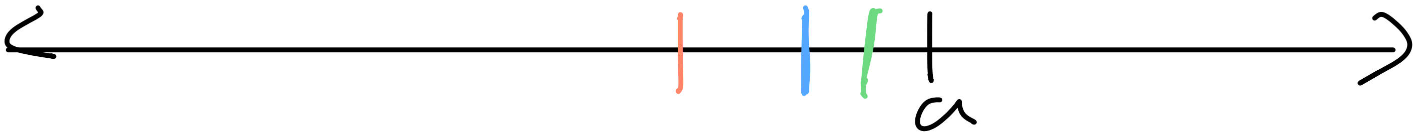
Now, we can apply previous theorems to show:

<u>Thm</u> ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ and $a<b$, $\exists r \in \mathbb{Q}$ s.t. $a<r<b$.

Mental image: "$\mathbb{Q}$ is sprinkled throughout $\mathbb{R}$"

Pf: By lemma, $\exists n \in \mathbb{N}$ s.t. $a+\frac{1}{n}<b \iff 1<bn-an$.
By other lemma, $\exists m \in \mathbb{Z}$ s.t. $an<m<bn \iff a<\frac{m}{n}<b$. $\square$

Ex: Given $d \in \mathbb{R}$, $\exists\, S \subseteq \mathbb{Q}$ s.t. $\sup(S) = d$.



Pf: $S := \{ s \in \mathbb{Q} : s \leq d \}$.
By defn, d is a upper bound of S.

Assume, for the sake of contradiction, that $\exists$ an upper bound c with $c < d$. By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, r \in \mathbb{Q}$ s.t. $c < r < d$.
Then $r \in S$, which contradicts the defn of c as an upper bound of S.

We will use symbols $+\infty$, $-\infty$ to simplify notation for sup & inf.

Def: (Unbounded above/below)
For any nonempty set $S \subseteq \mathbb{R}$,
- if S is not bounded above, write $\sup(S) = +\infty$.
- if S is not bounded below, write $\inf(S) = -\infty$.

Remark: Given a nonempty set $S \subseteq \mathbb{R}$ ...

S has a supremum $\overset{\text{by defn of } \mathbb{R}}{\Longleftarrow}$ S is bounded above

S has a supremum $\underset{\text{by defn of supremum}}{\Longrightarrow}$ S is bounded above

S has a supremum

$\Updownarrow$

$\sup(S) \in \mathbb{R}$

The supremum of S does not exist $\Longleftrightarrow$ S is not bounded above

The supremum of S does not exist

$\Updownarrow$

$\sup(S) = +\infty$

note that we do not write $\sup(S)$ D.N.E.

In this way,

$$\sup : \mathcal{P}(\mathbb{R}) \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

$$\inf : \mathcal{P}(\mathbb{R}) \to \underbrace{\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}}_{\text{extended real numbers}}$$

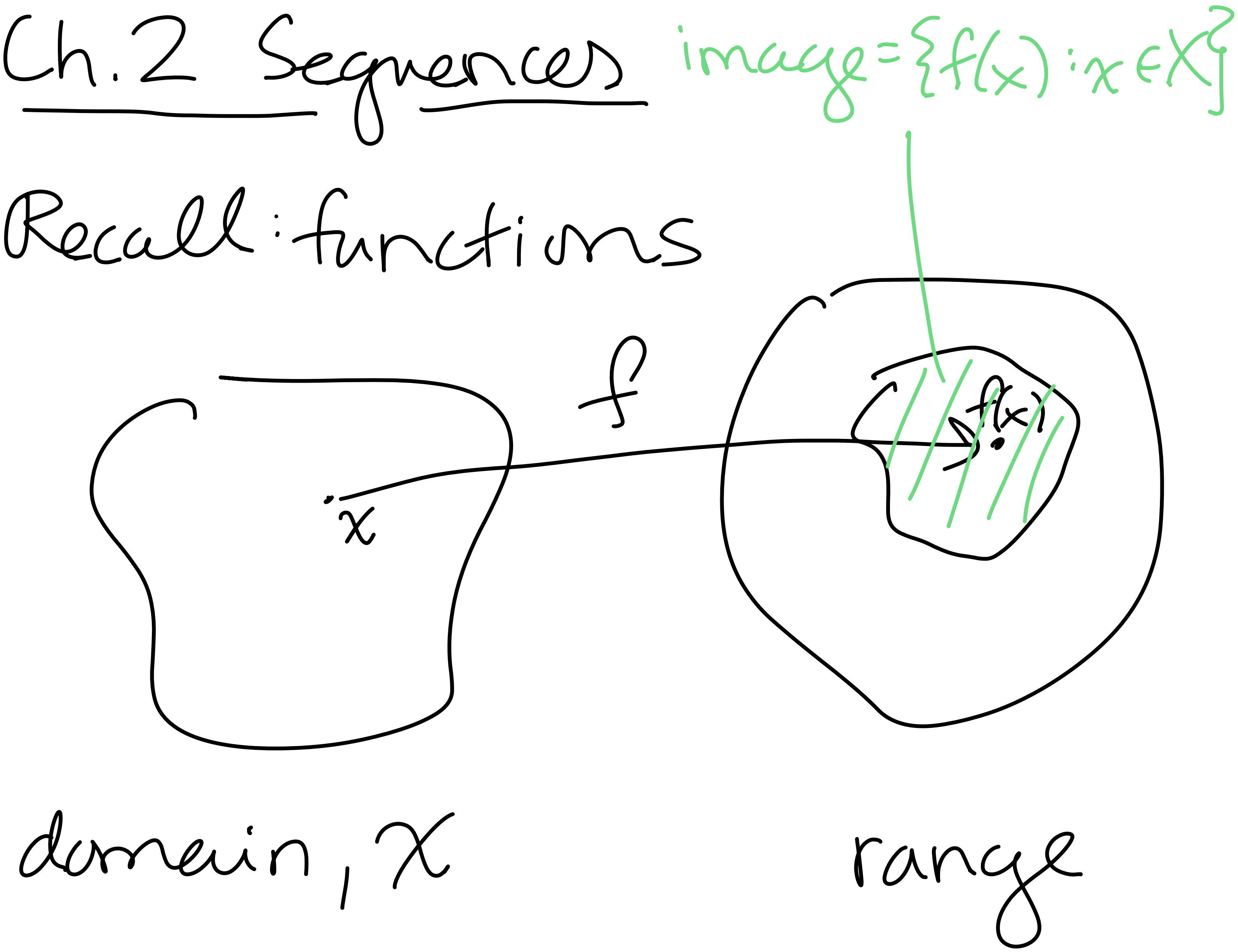
One last convention...

$\sup(\emptyset) = -\infty$

$\inf(\emptyset) = +\infty$

Ch. 2 Sequences

Recall: functions



<u>Def</u> (sequence): A <u>sequence</u> is a function whose domain is a set of the form

$\{m, m+1, m+2, \ldots\} = \{m+k : k \in \mathbb{N} \cup \{0\}\}$

for some $m \in \mathbb{Z}$.

In this course, we will study sequences whose range is $\mathbb{R}$.

Typically, the domain will be either $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$.

To empacize that a sequence is a special type of function, instead of

$$f(n)$$

we write

$$s_n.$$

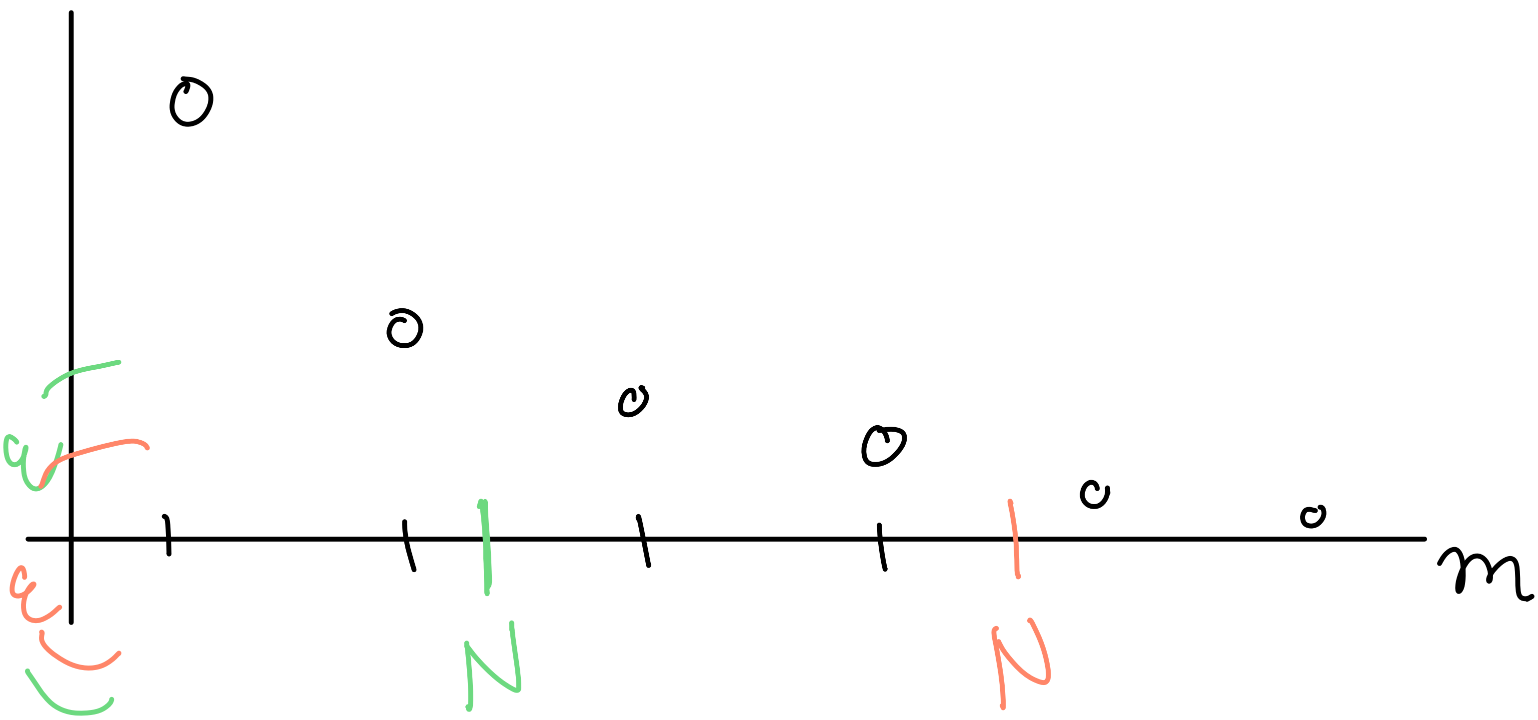
We can either specify a sequence by formula,

$$s_m = \frac{1}{m},$$

or by listing first few elements,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right).$$

$s_m = \frac{1}{m}$



Def (convergence): A sequence s_n converges to $s \in \mathbb{R}$ provided that $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

We call $s \in \mathbb{R}$ the limit of s_n and write

$$\lim_{n \to \infty} s_n = s$$

or

$$s_n \to s.$$

A sequence that does not converge to any $s \in \mathbb{R}$ is said to diverge.