Lecture 4

CS 117, S26 © Katy Craig, 2026

Homework 2 due Thursday, April 9th at 11:59pm

CLAIM:
$S \subseteq \mathbb{Z}$, S bdd above, $S \neq \emptyset$
$\Rightarrow \max(S)$ exists

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$):
If $a, b \in \mathbb{R}$ and $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Def: (Unbounded above/below)
For any nonempty set $S \subseteq \mathbb{R}$,

- if S is not bounded above, write $\sup(S) = +\infty$.
- if S is not bounded below, write $\inf(S) = -\infty$.

Given $S \subseteq \mathbb{R}$ nonempty,

S has a supremum $\overset{\text{by defn of supremum}}{\Longrightarrow}$ S is bounded above

S has a supremum $\overset{\text{by defn of } \mathbb{R}}{\Longleftarrow}$ S is bounded above

$\Updownarrow$

$\sup(S) \in \mathbb{R}$

The supremum of S does not exist $\Longleftrightarrow$ S is not bounded above

$\Updownarrow$

$\sup(S) = +\infty$

note that we do not write $\sup(S)$ D.N.E.

In this way,

$$\sup : \mathcal{P}(\mathbb{R}) \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

$$\inf : \mathcal{P}(\mathbb{R}) \to \underbrace{\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}}_{\text{extended real numbers}}$$

One last convention...

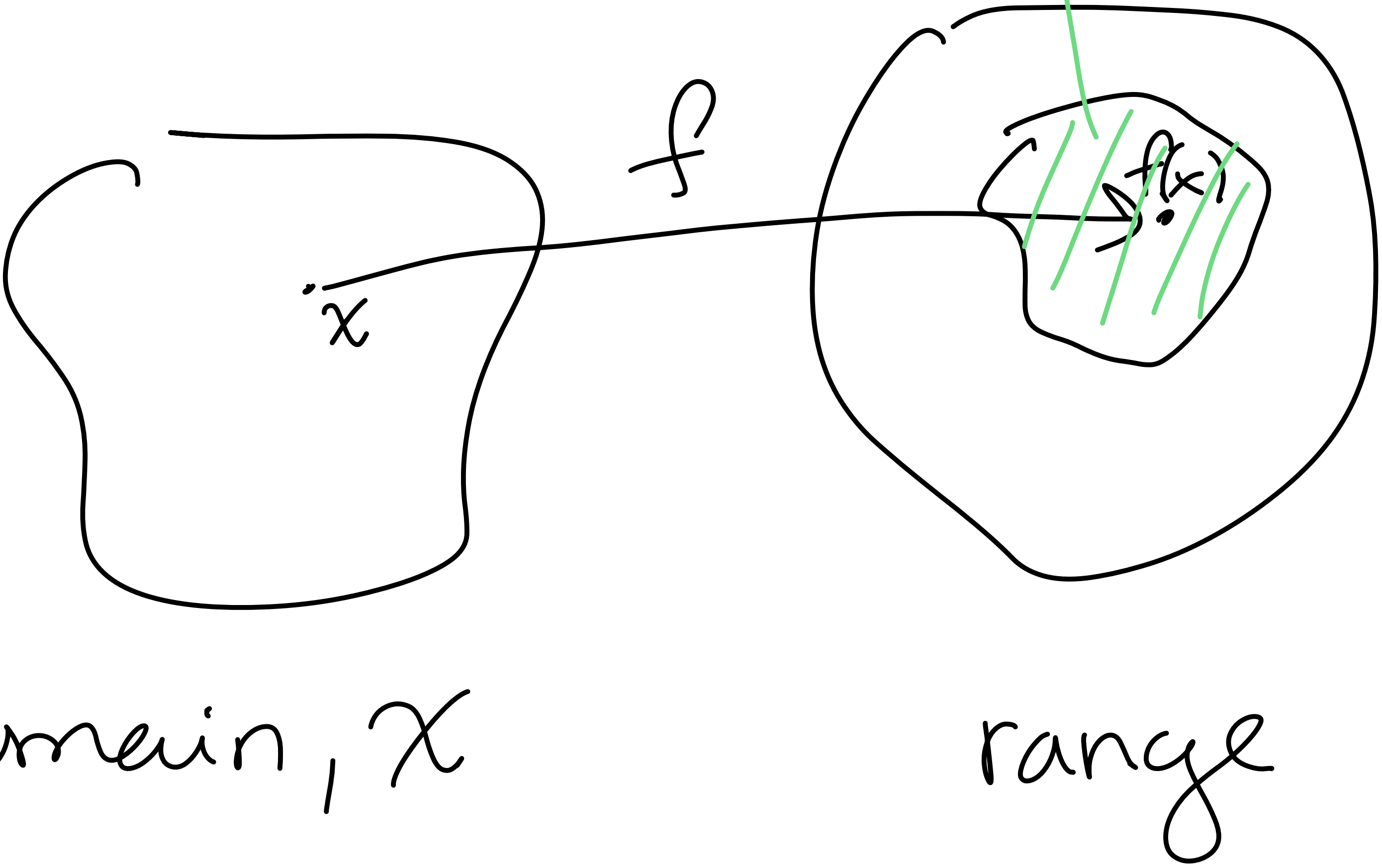
$\sup(\emptyset) = -\infty$

$\inf(\emptyset) = +\infty$

Ch. 2 Sequences

image $= \{f(x) : x \in X\}$

Recall: functions



<u>Def</u> (sequence): A <u>sequence</u> is a function whose domain is a set of the form

$$\{m, m+1, m+2, \ldots\} = \{m+k : k \in \mathbb{N} \cup \{0\}\}$$

for some $m \in \mathbb{Z}$.

In this course, we will study sequences whose range is $\mathbb{R}$.

Typically, the domain will be either $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$.

To empacize that a sequence is a special type of function, instead of

$$f(n)$$

we write

$$s_n.$$

Def (convergence): A sequence s_n converges to $s \in \mathbb{R}$ provided that $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

We call $s \in \mathbb{R}$ the limit of s_n and write

$$\lim_{n \to \infty} s_n = s$$

or

$$s_n \to s.$$

A sequence that does not converge to any $s \in \mathbb{R}$ is said to diverge.

Remark

- Recall: $|b| < a \iff -a < b < a$
- Similarly,

$$|s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon$$

$$\iff \underbrace{s - \varepsilon < s_n < s + \varepsilon}$$

Unless specified, domain always $\mathbb{N}$.

Ex: Consider the sequence $s_n = \frac{1}{n^2}$.

We expect $\lim_{n \to \infty} s_n = 0$. Let's prove it!

Scratchwork:

$$\left|\frac{1}{n^2} - 0\right| < \varepsilon \iff \frac{1}{n^2} < \varepsilon \iff \overbrace{\frac{1}{\sqrt{\varepsilon}}}^{N} < n$$

$\Longleftarrow$ just need this direction

Now, we can write the proof. Fix $\varepsilon > 0$ arbitrary. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Then $n > N$ ensures $\frac{1}{n^2} < \varepsilon \Leftrightarrow \left| \frac{1}{n^2} - 0 \right| < \varepsilon$. Therefore, by the definition of convergence, $\lim_{n \to \infty} \frac{1}{n^2} = 0$.

could also have chosen $N = \frac{1}{\sqrt{\varepsilon}} + \pi \ldots$

Now, let's consider a more complex example of a convergent sequence where it's not worth trouble to pick "minimal" N.

Ex: $s_n = \frac{2n-1}{3n+2}$ $= \frac{2-\frac{1}{n}}{3+\frac{2}{n}} \to \frac{2}{3}$

idea from calc.

Scratchwork:

$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \varepsilon$

$\iff \left| \frac{3(2n-1) - 2(3n+2)}{3(3n+2)} \right| < \varepsilon$

$\iff \frac{7}{3(3n+2)} < \varepsilon$

$\Leftarrow \frac{7}{9n} < \varepsilon$

$\iff \underbrace{\frac{7}{9\varepsilon}}_{N} < n$

We will prove s_n converges to $\frac{2}{3}$.
Fix $\varepsilon > 0$ arbitrary. Let $N = \frac{7}{9\varepsilon}$.
Then $n > N$ ensures
$\varepsilon > \frac{7}{9n} > \frac{7}{3(3n+2)}$

Therefore

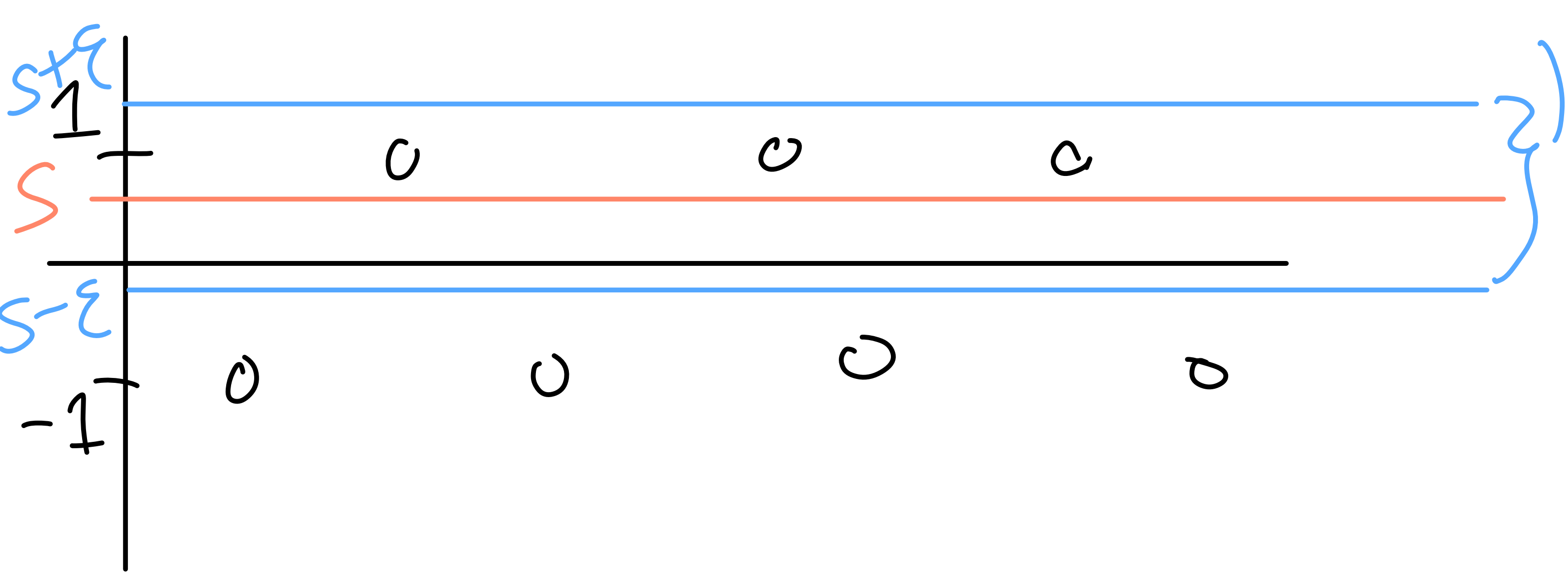
$$\left|\frac{3(2n-1)-2(3n+2)}{3(3n+2)}\right| < \varepsilon$$

$$\Updownarrow$$

$$\left|\frac{2n-1}{3n+2}-\frac{2}{3}\right| < \varepsilon$$

Therefore, $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$.

Ex: Let's prove $(-1)^n$ diverges.
Assume, for the sake of contradiction, $\exists\, s \in \mathbb{R}$ s.t. $(-1)^n$ converges to s.



By definition of convergence, for $\varepsilon = 1$, $\exists N$ s.t. $n > N$ ensures $|(-1)^n - s| < 1$.

Claim: there exist even and odd numbers n s.t. $n > N$.

If n is even, $|1 - s| < 1$
$\iff s \in (0, 2)$.
If n is odd, $|(-1) - s| < 1$
$\iff s \in (-2, 0)$

There is no such s. This is a contradiction. $\square$

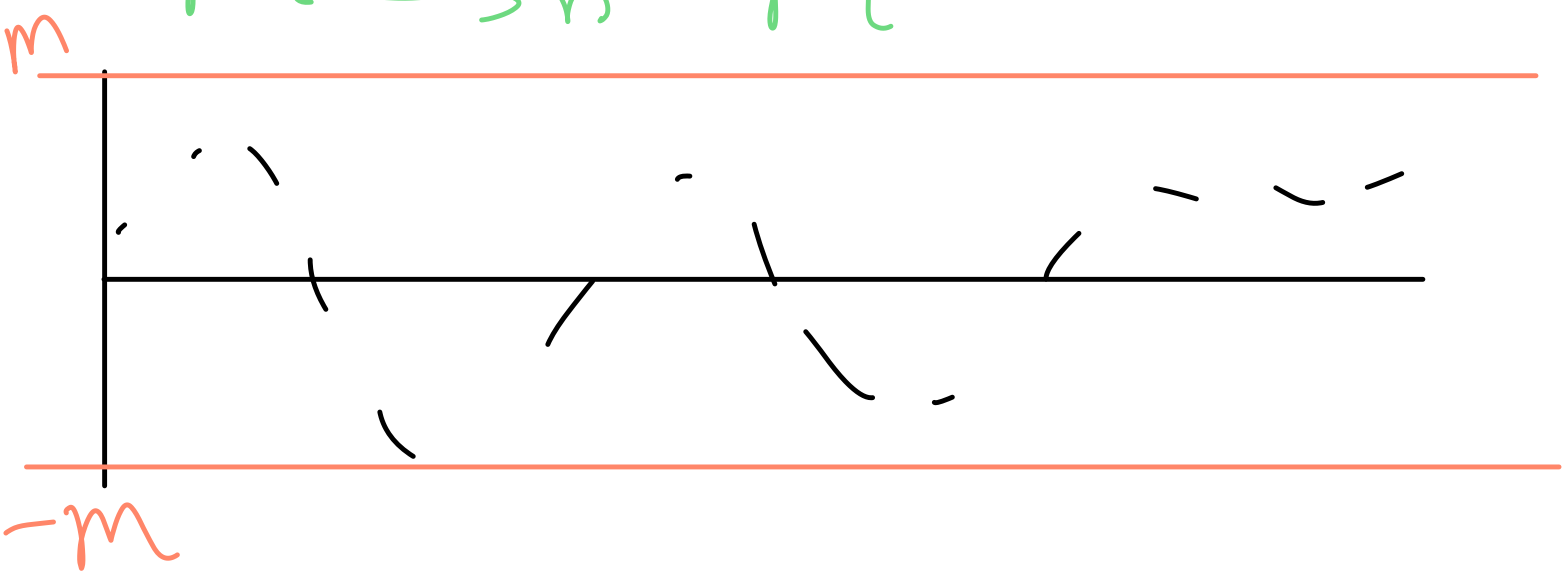
Another important type of sequence is...

Def: A sequence s_n is **bounded** if there exists $M \in \mathbb{R}$ s.t. $|s_n| \le M$ for all n.

$$\underbrace{|s_n| \le M}$$

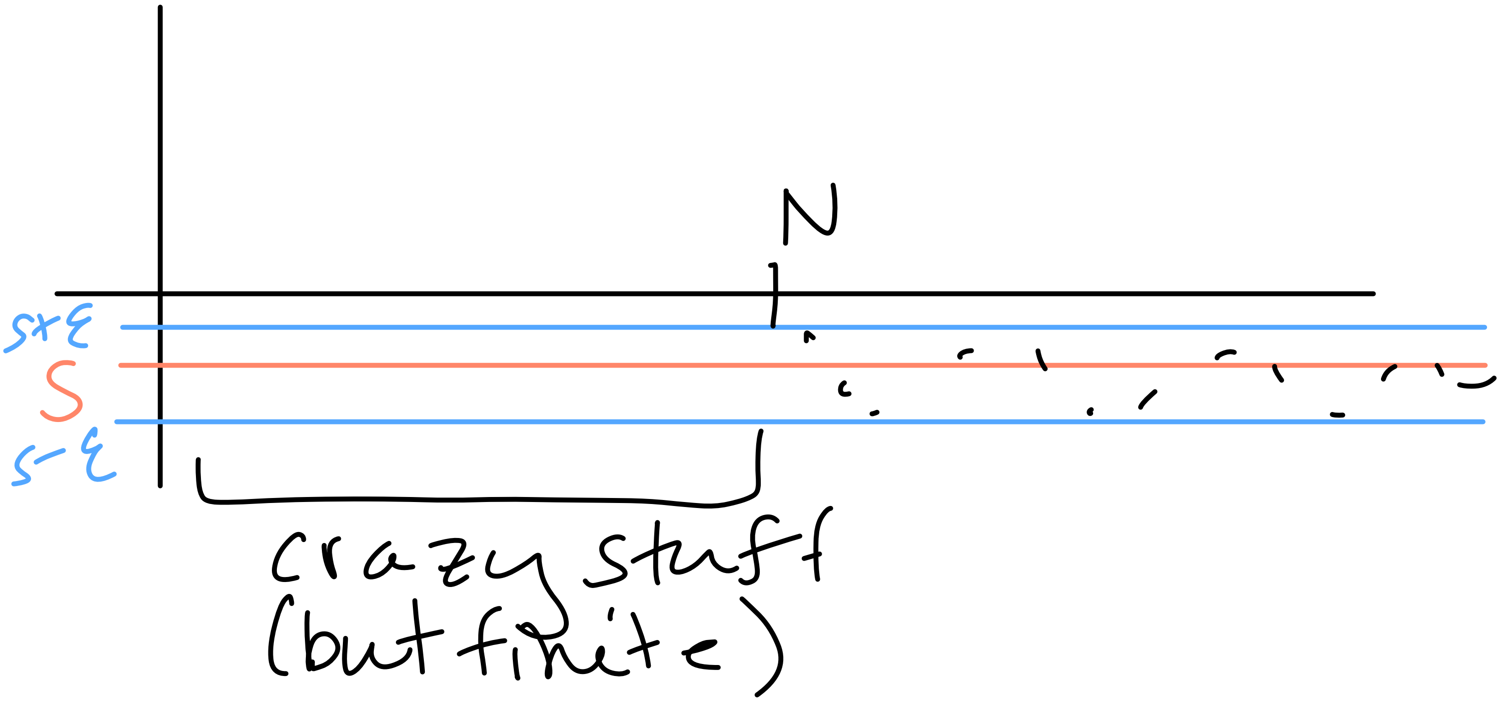
$$\Updownarrow$$

$$-M \le s_n \le M$$



Rmk: On HW, will show that s_n is bounded iff the set $S = \{s_n : n \in \mathbb{N}\}$ is bounded.

Thm: Convergent sequences are bounded.



Recall: Reverse triangle inequality

$\forall a,b \in \mathbb{R}$

$$|a|-|b| \le \big||a|-|b|\big| \le |a-b|$$

$y \le z$ $\quad$ $-y \le z$ $\quad$ $|y| \le z$

$|a|-|b| \le |a-b|$ AND $|b|-|a| \le |a-b| \Rightarrow \big||a|-|b|\big| \le |a-b|$

Pf: Assume s_n converges to $s \in \mathbb{R}$. Then, for $\varepsilon = 7$, $\exists$ N s.t. $n > N$ ensures $|s_n - s| < 7$.

By the reverse triangle ineq.,
$|s_n| - |s| < 7 \iff |s_n| < \underbrace{7 + |s|}_{M}$

Let $S := \{|s_1|, |s_2|, \dots |s_{N}|\} \subseteq \mathbb{R}$.
Since S is finite its maximum M_0 exists. Therefore $\forall n \in \mathbb{N}$,

$$|s_n| \leq \max\{M_0, M\}$$

Thus s_n is bounded. $\square$

<u>Recall</u>: For $a \in \mathbb{R}$
$\lfloor a \rfloor = \max\{z \in \mathbb{Z} : z \leq a\}$ "floor"
$\lceil a \rceil = \min\{z \in \mathbb{Z} : z \geq a\}$ "ceiling"