Lecture 5

CS 117, S26 © Katy Craig, 2026

Homework 3 due Thursday, April 16th at 11:59pm

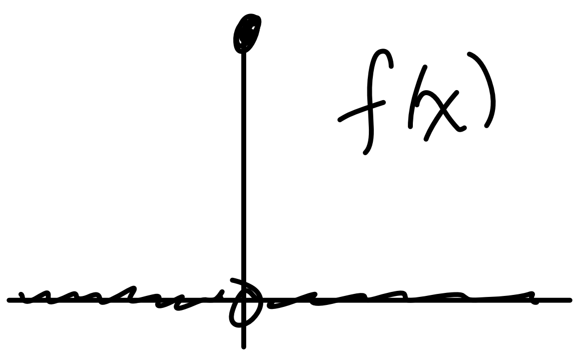
Recall:

Def: A sequence s_n is bounded if there exists $M \in \mathbb{R}$ s.t. $|s_n| \leq M$ for all n.

Thm: (real-valued) Convergent sequences are bounded.

extended real numbers

$\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$



"Delta function"

$$f(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

Ex: $\lim_{n\to\infty} \frac{5}{n} - \frac{100}{n^2} = 0$

from Calculus we expect...

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, then

$$\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

Ex: The theorem fails if s_n and t_n are not convergent

$s_n = (-1)^n$, $t_n = (-1)^{n+1}$

$s_n + t_n = 0$

Pf: Let $s := \lim_{n\to\infty} s_n$, $t := \lim_{n\to\infty} t_n$.
Fix $\varepsilon > 0$. Since $s_n \to s$, $t_n \to t$, $\exists$ $N_s, N_t \in \mathbb{R}$ s.t.

$$n > N_s \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$$
$$n > N_t \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$$

Thus for $N := \max\{N_s, N_t\}$, $n > N$ ensures

"$\frac{\varepsilon}{2}$ proof"

$$|s_n - s| < \frac{\varepsilon}{2}, \quad |t_n - t| < \frac{\varepsilon}{2}.$$

Thus, $n > N$ implies
$$|s_n - s| + |t_n - t| < \varepsilon.$$

By the triangle inequality, $n > N$ implies
$$|(s_n - s) + (t_n - t)| < \varepsilon$$
$$\|$$
$$|(s_n + t_n) - (s+t)|$$

$\square$

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences

$$\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Pf: Let $s := \lim_{n\to\infty} s_n$, $t := \lim_{n\to\infty} t_n$.
Since s_n convergent, it is bdd, and $\exists M \in \mathbb{R}$ s.t. $|s_n| \le M$ $\forall n \in \mathbb{N}$.
WLOG, we may assume $M > 0$.
Thus, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + t(s_n - s)| \\
&\overset{\Delta \text{ineq}}{\le} |s_n(t_n - t)| + |t(s_n - s)| \\
&= |s_n||t_n - t| + |t||s_n - s| \\
&\le M|t_n - t| + |t||s_n - s|
\end{aligned}$$

Fix $\varepsilon > 0$. Since $s_n \to s$, $\exists\, N_s$ s.t. $n > N_s$,

$$|s_n - s| < \begin{cases} \dfrac{\varepsilon}{2|t|} & \text{if } t \neq 0 \\ 250 & \text{if } t = 0 \end{cases}$$

Since $t_n \to t$, $\exists\, N_t$ s.t. $n > N_t$ ensures $|t_n - t| < \dfrac{\varepsilon}{2m}$.

Therefore $n > \max\{N_s, N_t\}$ ensures

$$|s_n t_n - st| < m \frac{\varepsilon}{2m} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Aside: A (real valued) sequence s_n converges to $s \in \mathbb{R}$ if $\forall$

$\varepsilon > 0$ $\cancel{\varepsilon \geq 0}$

$\exists$

$N \in \mathbb{R}$ $N \in \mathbb{N}$

s.t.

$\underline{n > N}$ $\qquad$ $\underline{n \geq N}$

ensures

$\underline{|s_n - s| < \varepsilon}$ $\qquad$ $\underline{|s_n - s| \leq \varepsilon}$

<u>Claim</u>

$s_n \to s \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{R}$
s.t. $n > N$ ensures
$|s_n - s| \leq \varepsilon$

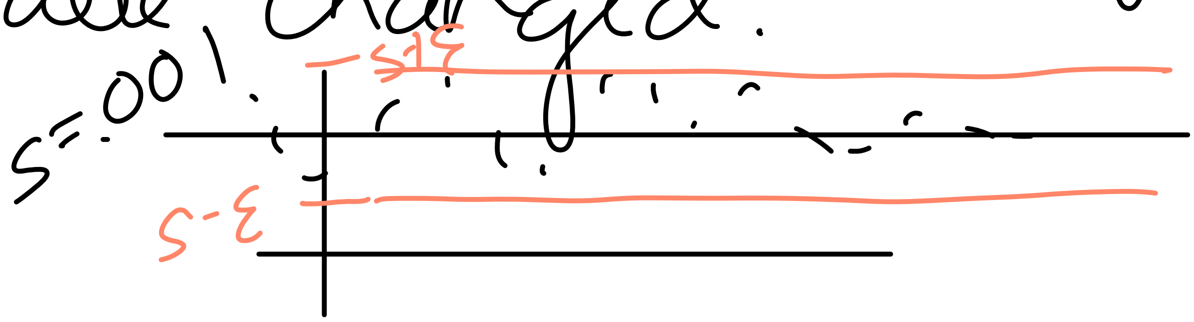
<u>Pf of "$\Leftarrow$"</u>: Fix $\varepsilon > 0$. Then $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| \leq \frac{\varepsilon}{2} < \varepsilon$.

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequence and $\lim_{n\to\infty} s_n \neq 0$,

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}$$

The hypothesis $\lim_{n\to\infty} s_n \neq 0$ ensures that, up to ignoring finitely many elements of $\frac{t_n}{s_n}$, we never divide by zero.

Rmk: The limiting behavior of the sequence is unchanged if finitely many elements of the original sequence are changed.



Rmk: If s_n is a convergent sequence and $\lim_{n\to\infty} s_n \neq 0$, $\exists N$ s.t. $n>N$ ensures $s_n \neq 0$.

Let $s := \lim_{n\to\infty} s_n$, $\varepsilon := |s| > 0$.
By defn of convergence, $\exists N$ s.t. $n>N$ ensures

reverse triangle ineq $|a|-|b| \leq |a-b|$

$$|s|-|s_n| \leq |s_n - s| < |s| \Rightarrow 0 < |s_n|.$$

Pf of Thm: HW :)

Ex: $\lim_{n\to\infty} n^2 = +\infty$

__Def__ (diverges $+\infty$ or $-\infty$): A sequence s_n __diverges to $+\infty$__ if $\forall M>0$, $\exists N\in\mathbb{R}$ s.t. $n>N$ ensures $s_n>M$. We write $\lim_{n\to\infty} s_n = +\infty$.

OTOH, a sequence s_n __diverges to $-\infty$__ if $\forall M<0$, $\exists N\in\mathbb{R}$ s.t. $n>N$ ensures $s_n<M$. We write $\lim_{n\to\infty} s_n = -\infty$.