Lecture 6

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Homework 3 due Thursday, April 16th at 11:59pm

Midterm 1 on Wednesday, April 22nd

Practice Midterm 1 posted tomorrow

Recall:

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, then

$$\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences

$$\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequence and $\lim_{n\to\infty} s_n \neq 0$,

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}$$

The hypothesis $\lim_{n\to\infty} s_n \neq 0$ ensures that, up to ignoring finitely many elements of $\frac{t_n}{s_n}$, we never divide by zero.

Rmk: The limiting behavior of the sequence is unchanged if finitely many elements of the original sequence are changed.

Rmk: If s_n is a convergent sequence and $\lim_{n\to\infty} s_n \neq 0$, $\exists N$ s.t. $n > N$ ensures $s_n \neq 0$.

Def: A sequence s_n diverges to $+\infty$ if $\forall M > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $s_n > M$. We write $\lim_{n\to\infty} s_n = +\infty$.

OTOH, a sequence s_n diverges to $-\infty$ if $\forall M < 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $s_n < M$. We write $\lim_{n\to\infty} s_n = -\infty$.

What types of limit theorems hold for sequences that diverge to $\pm\infty$?

Ex: We can't hope for the limit of sum to equal sum of limits, since no canonical defn of $+\infty + (-\infty)$.

	$\lim_{n\to\infty} s_n + t_n$	$\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$
$s_n = n$, $t_n = -n$	0	$+\infty + (-\infty)$
$s_n = 2n$, $t_n = -n$	$+\infty$	$+\infty + (-\infty)$

Ex: Similar with products of limits

	$\lim_{n\to\infty} s_n t_n$	$(\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n)$
$s_n = n$, $t_n = \frac{1}{n}$	1	$(+\infty)(0)$
$s_n = n^2$, $t_n = \frac{1}{n}$	$+\infty$	$(+\infty)(0)$

Under sufficient assumptions, can still prove similar results for sequences that diverge to $\pm\infty$!

Useful notation: We will say that a sequence s_n "has a limit" or "the limit exists" if either:

① s_n converges $\lim_{n\to\infty} s_n \in \mathbb{R}$

② s_n diverges to $\pm\infty$ $\lim_{n\to\infty} s_n \in \{\pm\infty\}$

If neither ① nor ② holds, we say the limit does not exist.

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$. Then $\lim_{n\to\infty} s_n t_n = +\infty$.

Pf: HW on Practice Midterm.

either t_n converges to a positive number or diverges to $+\infty$

Thm: Suppose s_n is a sequence of positive numbers. Then $\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$.

Pf: First, show "$\Rightarrow$" Fix $\varepsilon > 0$.
Since $\lim_{n\to\infty} s_n$, $\exists$ N s.t. $n > N$ ensures
$\frac{1}{\varepsilon} < s_n \iff \frac{1}{s_n} < \varepsilon \iff \left|\frac{1}{s_n} - 0\right| < \varepsilon$
Thus $\lim_{n\to\infty} s_n = 0$.

Now, show "$\Leftarrow$:" Fix $M>0$. Since $\frac{1}{s_n} \to 0$, $\exists N$ s.t. $n>N$ ensures

$$\left|\frac{1}{s_n} - 0\right| < \frac{1}{M} \iff \frac{1}{s_n} < \frac{1}{M} \iff s_n > M.$$

Thus $\lim_{n\to\infty} s_n = +\infty$. $\square$

Q: If $\lim_{n\to\infty} s_n = +\infty$, (and s_n is real valued) is s_n bounded below?

A: Yes! $\exists N$ s.t. $n>N$ ensures $s_n > \pi$. Let $m = \min\{s_n : n \leq N\}$. Then $s_n \geq \min\{\pi, m\}$ $\forall n \in \mathbb{N}$.

Last important type of sequence.

Def: A sequence s_n is...

strictly increasing $s_n < s_{n+1}$

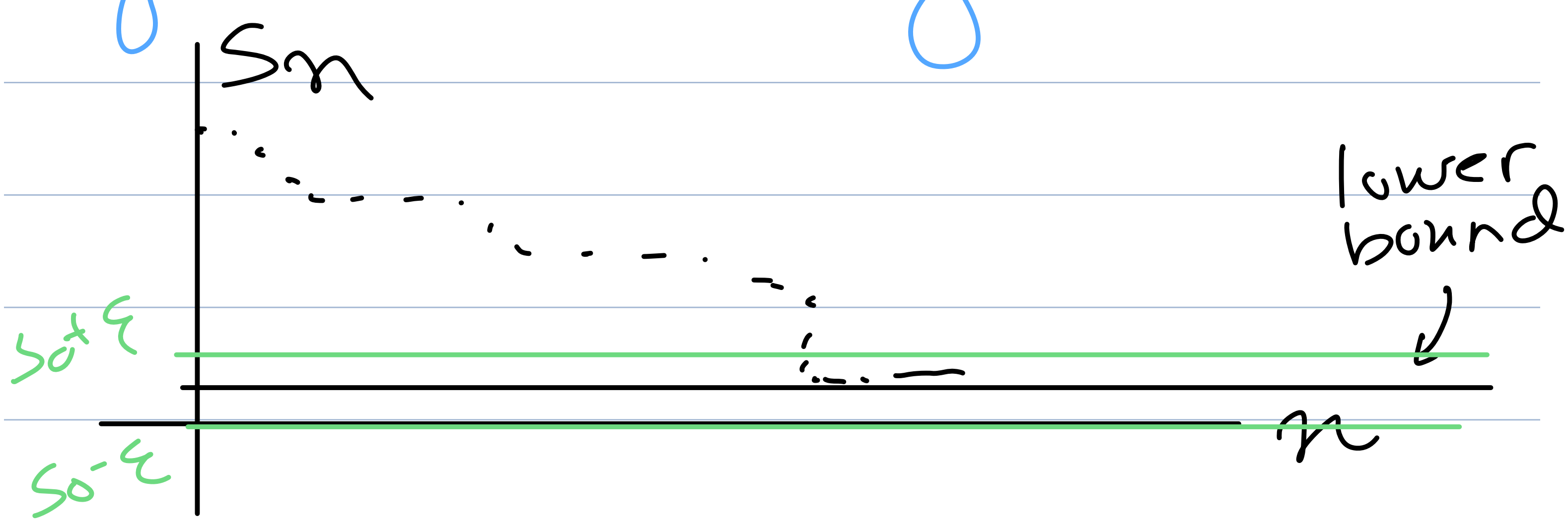
increasing if $s_n \le s_{n+1}$ $\forall n$

decreasing if $s_n \ge s_{n+1}$ $\forall n$.

Finally, s_n is monotone if it is either increasing or decreasing.

Rmk: s_n is increasing $\iff$ $n \le m$ ensures $s_n \le s_m$.

Thm: All bounded monotone sequences converge.



Pf: Case I: Suppose s_n is a bounded decreasing sequence. Let $S = \{s_n : n \in \mathbb{N}\}$. Then S is a bounded set. Define $s_0 := \inf(S)$.

Fix $\varepsilon > 0$. Note that $s_0 - \varepsilon < s_0 \leq s_n$ $\forall n \in \mathbb{N}$. Since s_0 is the greatest lower bound, $s_0 + \varepsilon$ is not a lower bound of S. There $\exists N \in \mathbb{N}$ s.t. $s_N < s_0 + \varepsilon$. Since s_n is decreasing, $\forall n > N$, $s_n < s_0 + \varepsilon$. Thus $\forall n > N$, we have,

$$s_0 - \varepsilon < s_n < s_0 + \varepsilon \iff |s_n - s_0| < \varepsilon.$$

Thus $s_n \to s_0$.

Case II: Suppose s_n is a bounded increasing sequence.

Then $t_n := -s_n$ is a bounded decreasing sequence. By Case I, it converges to some t_0.

Then s_n is the product of t_n and $(-1, -1, -1, \dots)$. Thus, $\lim_{n \to \infty} s_n = t_0 \cdot (-1)$, so s_n converges. $\square$

Even unbounded monotone sequences must have a limit.

Thm: If s_n is an unbounded increasing sequence, $\lim_{n\to\infty} s_n = +\infty$.
If s_n is an unbounded decreasing sequence, $\lim_{n\to\infty} s_n = -$

Rmk: s_n bdd $\Leftrightarrow \exists\, M > 0$ s.t.
$|s_n| \leq M \;\; \forall n \in \mathbb{N}$

s_n unbdd $\Leftrightarrow \forall\, M > 0 \;\exists\, n \in \mathbb{N}$
s.t. $|s_n| > M$.

Pf:
Case I: Suppose s_n is an unbounded increasing sequence.
Fix $M > 0$. Since s_n is an increasing sequence it is bounded below by s_1.
Therefore s_n must unbounded above. Hence $\exists\, N \in \mathbb{N}$ s.t.

$s_N > M$. Since s_n is increasing $n > N$ ensures $s_n \geq s_N > M$.

Thus $\lim_{n\to\infty} s_n = +\infty$.

Case II: decreasing

Exercise ☺ □

In summary, if s_n is monotone,

$$\lim_{n\to\infty} s_n = \begin{cases} +\infty & \text{if } s_n \text{ unbdd above} \\ s \text{ for } s \in \mathbb{R} & \text{if } s_n \text{ bdd} \\ -\infty & \text{if } s_n \text{ unbdd below} \end{cases}$$

Thus, the limit always exists.

We will now show how we can study the limiting behavior of arbitrary real-valued sequences by reducing to the case of monotone sequence.

Just like sup generalized max, we can also generalize $\lim_{n\to\infty}$.

Def (limsup / liminf) For any sequence s_n,

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\sup\{s_n : n > N\}}_{a_N}$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf(\{s_n : n > N\})}_{b_N}$$

Note: a_N is $\mathbb{R} \cup \{+\infty\}$ valued

① $a_N = +\infty$ for all N

② $a_{N_0} \in \mathbb{R}$ for some N_0

↳ if $\{s_n : n > N_0\}$ is bounded above, since $\{s_n : n \leq N_0\}$ is finite, it is also bounded above, so $\{s_n : n \in \mathbb{N}\} = \{s_n : n > N_0\} \cup \{s_n : n \leq N_0\}$ is bounded above.

↳ Thus $a_N \in \mathbb{R}$ for all N.

Likewise, either $b_N = -\infty$ $\forall N$ or $b_N \in \mathbb{R}$ $\forall N$.

Rmk: For a_N as in definition, $\lim_{N\to\infty} a_N$ always exists.

① Either $a_N \equiv +\infty$, $\lim_{N\to\infty} a_N = +\infty$

② $a_N \in \mathbb{R}$ and

$$a_N = \sup(\underbrace{\{s_n : n > N\}}_{S}) \geq \sup(\underbrace{\{s_n : n > N+1\}}_{T}) = a_{N+1}$$

$T \subseteq S$

So a_N is decreasing, so its limit exists.

Similarly, for liminf, since b_N is increasing!

== End of material for Midterm 1 ==