Lecture 7

CS 117, S26 © Katy Craig, 2026

Homework 4 due Thursday, April 30th at 11:59pm

Midterm 1 on Wednesday, April 22nd

Practice Midterm 1 posted

Recall:

Useful notation: We will say that a sequence s_n "has a limit" or "the limit exists" if either:

① s_n converges $\lim_{n\to\infty} s_n \in \mathbb{R}$

② s_n diverges to $\pm\infty$ $\lim_{n\to\infty} s_n \in \{\pm\infty\}$

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$. Then $\lim_{n\to\infty} s_n t_n = +\infty$.

Thm: Suppose s_n is a sequence of positive numbers. Then $\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$.

Def: A sequence s_n is...

increasing if $s_n \leq s_{n+1}$ $\forall n$

decreasing if $s_n \geq s_{n+1}$ $\forall n$.

Finally, s_n is **monotone** if it is either increasing or decreasing.

Rmk: s_n is increasing $\iff$ $n \le m$ ensures $s_n \le s_m$.

Thm: All bounded monotone sequences converge.

Thm: If s_n is an unbounded increasing sequence, $\lim_{n\to\infty} s_n = +\infty$. If s_n is an unbounded decreasing sequence, $\lim_{n\to\infty} s_n = -\infty$.

Def (limsup / liminf) For any sequence s_n,

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n > N\}}^{a_N}$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\inf(\{s_n : n > N\})}^{b_N}$$

Last time we showed:

- either $a_N = +\infty \ \forall N \in \mathbb{N}$ or $a_N \in \mathbb{R} \ \forall N \in \mathbb{N}$
- a_N is decreasing
- either $b_N = -\infty \ \forall N \in \mathbb{N}$ or $b_N \in \mathbb{R} \ \forall N \in \mathbb{N}$
- b_N is increasing

Ex: $s_n = -n$

$N \in \mathbb{N}$

$$a_N = \sup(\{s_n : n > N\}) = s_{N+1} = -(N+1)$$

Thus $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N = -\infty$.

Likewise, $\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\inf\{-n : n > N\}}^{=-\infty} = -\infty$.

These coincide with $\lim_{n\to} s_n = -\infty$.

Ex:

$$\limsup_{n\to\infty}(-1)^n = \lim_{N\to\infty} \overbrace{\sup\{(-1)^n : n>N\}}^{=1} = 1$$

$$\liminf_{n\to\infty}(-1)^n = -1$$

<u>Thm</u>: Given a sequence s_n,

$\lim_{n\to\infty} s_n$ exists $\iff \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$

Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

Facts we will use in proof:

① Given $S \subseteq \mathbb{R}$, $m \in \mathbb{R}$ s.t. $s \leq m \ \forall s \in S$, $\sup(S) \leq m$.

② $b_N = \inf\{s_n : n > N\} \leq s_n \leq a_N$ {for $n > N$}

③ Claim: If r_n and t_n are sequences whose limits exist and $r_n \leq t_n \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} r_n \leq \lim_{n\to\infty} t_n$.

Pf: HW 4.

Note: combining 2+3, we see,

$$\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$$

④ Claim: If either limit exists, $\lim_{n\to\infty} -t_n = -\lim_{n\to\infty} t_n$.

Pf: HW 4.

Note, by (4),

$$\limsup_{n\to\infty} -s_n = \lim_{N\to\infty} \sup\{-s_n : n>N\}$$

$$= \lim_{N\to\infty} -\overbrace{\inf\{s_n : n>N\}}^{b_N}$$

$$= -\liminf_{n\to\infty} s_n$$

Pf: Suppose $\lim_{n\to\infty} s_n$ exists.

WTS

$$\lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

$\boxed{\text{Case 1}}$: $\lim_{n\to\infty} s_n = -\infty$. By $-\star$, it suffices to show $\limsup_{n\to\infty} s_n = -\infty$.

Fix $M<0$. Since $\lim_{n\to\infty} s_n = -\infty$, $\exists N_0$ s.t. $n>N_0$ ensures $s_n < M$. Then M is an upper bound for $\{s_n : n > N_0\}$. Thus,

$$a_{N_0} = \sup\{s_n : n > N_0\} \leq m$$

Since a_N is decreasing,
$a_N \leq m$ for all $N \geq N_0$.
Thus, $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N = -\infty$.

$\boxed{\text{Case 2:}}$ $\lim_{n\to\infty} s_n = +\infty$.

By (4), $\lim_{n\to\infty} -s_n = -\infty$. By Case 1,
$\limsup_{n\to\infty} -s_n = \liminf_{n\to\infty} -s_n = -\infty$.

By (☆), $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = +\infty$,
which gives the result.

$\boxed{\text{Case 3}}$ $\lim_{n\to\infty} s_n = s$ for $s \in \mathbb{R}$.

We must show a_N and b_N converge to s. Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, $\exists N_0 \in \mathbb{N}$ s.t. $n > N_0$ ensures

$$s - \varepsilon < s_n < s + \varepsilon.$$

Thus $s+\varepsilon$ is an upper bound for $\{s_n : n > N_0\}$ and $s-\varepsilon$ is a lower bound for this set. Thus $a_{N_0} \leq s+\varepsilon$ and $b_{N_0} \geq s-\varepsilon$. Since the sequences are decreasing/increasing, $\forall N \geq N_0$

$$s - \varepsilon \leq b_N \leq a_N \leq s + \varepsilon$$