Lecture 8

CS 117, S26 © Katy Craig, 2026

Homework 4 due Thursday, April 30th at 11:59pm

Thm: Given a sequence s_n,

$\lim_{n\to\infty} s_n$ exists $\iff \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$

Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

Def: A (real valued) sequence s_n is **Cauchy** if, $\forall \varepsilon > 0$, $\exists N$ s.t. $n, m > N$ ensures

$$|s_n - s_m| < \varepsilon.$$

Thm: All convergent sequences are Cauchy.

Prop: All Cauchy sequences are bounded.

Thm: All Cauchy sequences converge.

Recall

(a) $s_n \le t_n$ for all but finitely many n and limits exist, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$

(b) $a \le b+\varepsilon \;\; \forall \varepsilon > 0 \Rightarrow a \le b$.

Pf: Let s_n be a Cauchy sequence.

We will show

$$\underbrace{\lim_{N\to\infty} \sup\{s_n : n > N\}}_{a_N} = \limsup s_n = \liminf s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n > N\}}_{b_N} \in \mathbb{R} \quad (*)$$

Fix $\varepsilon > 0$. Since s_n is Cauchy, $\exists N$ s.t. $n, m > N$ ensure

$$|s_n - s_m| < \varepsilon$$
$$\Updownarrow$$
$$s_m - \varepsilon < s_n < s_m + \varepsilon$$

This implies $s_m + \varepsilon$ is an upper bound of $\{s_n : n > N\}$, so

$$a_N \leq s_m + \varepsilon \iff a_N - \varepsilon \leq s_m$$

Thus $a_N - \varepsilon \le b_N$.
By monotonicity, $\forall\ k > N$,

$$a_k - \varepsilon \le b_k.$$

By (a),

$$\limsup_{n\to\infty} s_n - \varepsilon \le \liminf_{n\to\infty} s_n$$

Since $\varepsilon > 0$ was arbitrary, by (b),

$$\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n.$$

Since Cauchy sequences are bounded and $\liminf s_n \le \limsup s_n$ always holds, we get $(*)$.

Types of Sequences

	MONOTONE	NOT MONOTONE	
BOUNDED	$\frac{1}{n}$	$\frac{(-1)^n}{n}$	$(-1)^n$
UNBOUNDED	n	$(1,0,1,2,3,\ldots)$	$n(-1)^n$

CONVERGENT

DIVERGES TO $\pm\infty$

the limit exists

Final topic in sequences:
bounded sequences

Our most important theorem will rely on notion of subsequence.

Recall:
<u>Def</u>: A <u>sequence</u> is a function whose domain is a set of the form $\{m, m+1, m+2, \ldots\}$ for some $m \in \mathbb{Z}$.

We write s_n instead of $s(n)$ to emphasize it is a special type of function.

__Def__: Given a sequence, s_n, $n \in \mathbb{N}$, and a strictly increasing sequence of natural numbers n_k, $k \in \mathbb{N}$, a sequence of the form $s_{n_k} = s(n(k))$ is a __subsequence__ of s_n.

$$n_k < n_{k+1}$$

Ex: $s_n = (\overset{n=1}{-1}, \overset{n=2}{2}, \overset{n=3}{-3}, 4, -5, \dots)$

$$n_k = (\overset{k=1}{1}, \overset{k=2}{3}, \overset{k=3}{5}, \overset{k=4}{7}, \dots)$$

$$s_{n_k} = (\overset{k=1}{-1}, \overset{k=2}{-3}, \overset{k=3}{-5}, \overset{k=4}{-7}, \dots)$$

Informally, a subsequence s_{n_k} of s_n is any infinite collection of elements from s_n listed in order.

Limits of Subsequences

Lemma: For any strictly increasing sequence of natural numbers n_k, we have $n_k \geq k \; \forall k \in \mathbb{N}$.

Pf: by induction :)

Def: A subsequential limit of a sequence s_n is a real number or $\pm\infty$ that is the limit of some subsequence of s_n.

Ex: $s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots)$
0 and $+\infty$ are subsequential limits. In fact, they are the only subsequential limits.

Thm: If a sequence s_n converges to $s \in \mathbb{R}$, then every subsequence also converges to s.

Pf: Assume s_n converges to s. Consider a subsequence s_{n_k}.

Fix $\varepsilon > 0$. $\exists N$ s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

By Lemma, $k > N$ ensures $n_k > N$, so

$$|s_{n_k} - s| < \varepsilon. \qquad \square$$

Thm (main subsequences thm):

Let s_n be a sequence.

(a) For any $t \in \mathbb{R}$

t is a subsequential limit of s_n

$\Updownarrow$

$\forall \varepsilon > 0$,

$\varepsilon\}| = +\infty$

(b) $+\infty$ is a subsequential limit

$\Updownarrow$

s_n is unbounded above

(c) $-\infty$ is a subsequential limit

$\Updownarrow$

s_n is unbounded below.

Pf:

First, we show (a). Fix $t \in \mathbb{R}$.

Assume $\forall \varepsilon > 0$,

$$|\{n : |s_n - t| < \varepsilon\}| = +\infty$$
$$\iff |\{n : t - \varepsilon < s_n < t + \varepsilon\}| = +\infty$$

Strategy: show there exists s_{n_k} of s_n satisfying

$(*)$ $t - \frac{1}{k} \leq s_{n_k} \leq t + \frac{1}{k} \quad \forall k \in \mathbb{N}$

By Squeeze Lemma, this will ensure $\lim_{k \to \infty} s_{n_k} = t$.

We will construct the sequence inductively.

Let $k=1$. Taking $\varepsilon=1$,

$$|\{n : |s_n - t| < 1\}| = +\infty,$$

so $\exists\, n_1 \in \mathbb{N}$ s.t.

$$t-1 < s_{n_1} < t+1.$$

Suppose we have defined $s_{n_1}, s_{n_2}, \dots, s_{n_k}$ s.t.

$$s_{n_\ell} \in \{s_n\}_{n\in\mathbb{N}},\quad n_\ell < n_{\ell+1} \quad \forall \ell \in \mathbb{N}$$

Taking $\varepsilon = \frac{1}{k+1}$

$$|\{n : |s_n - t| < \tfrac{1}{k+1}\}| = +\infty$$

Choose n in this set satisfying $n > n_k$. Let $n_{k+1} := n$.

This constructs a subsequence of sn that satisfies (*), so t is a subsequential limit.