Lecture 9

CS 117, S26 © Katy Craig, 2026

Homework 4 due Thursday, April 30th at 11:59pm

Def: Given a sequence, s_n, $n \in \mathbb{N}$, and a strictly increasing sequence of natural numbers n_k, $k \in \mathbb{N}$, a sequence of the form $s_{n_k} = s(n(k))$ is a subsequence of s_n.

Lemma: For any strictly increasing sequence of natural numbers n_k, we have $n_k \geq k$ $\forall k \in \mathbb{N}$.

Def: A subsequential limit of a sequence s_n is a real number or $\pm\infty$ that is the limit of some subsequence of s_n.

Thm: If a sequence s_n converges to $s \in \mathbb{R}$, then every subsequence also converges to s.

Thm (main subsequences thm):

Let s_n be a sequence.

(a) For any $t \in \mathbb{R}$

t is a subsequential limit of s_n

$\Updownarrow$

$\forall \varepsilon > 0, \ |\{n : |s_n - t| < \varepsilon\}| = +\infty$

(b) $+\infty$ is a subsequential limit

$\Updownarrow$

s_n is unbounded above

(c) $-\infty$ is a subsequential limit

$\Updownarrow$

s_n is unbounded below.

Rmk: (a) $\not\Leftrightarrow$ $\forall \varepsilon > 0, \ |\{s_n : |s_n - t| < \varepsilon\}| = +\infty$

e.g. $s_n = (-1)^n, \ t = 1$

Pf:

First, we show (a). Fix $t \in \mathbb{R}$.

Last time, we showed $\Uparrow$.

Now, show $\Downarrow$. Thus, there is a subsequence s_{n_k} that converges to t. Fix $\varepsilon > 0$. There exists N s.t. $k \geq N$ ensures $|s_{n_k} - t| < \varepsilon$.
In other words,

$$\{n_k : k \geq N\} \subseteq \{n : |s_n - t| < \varepsilon\}$$

Thus $|\{n : |s_n - t| < \varepsilon\}| = +\infty$

Now, we show (b).

Suppose s_n is unbounded above. We will define s_{n_k} inductively. Choose n_1 so that $s_{n_1} \geq 1$. Given n_k, note that s_{n_k} and $k+1$ cannot be an upper bounds for

$$\{s_n : n > n_k\},$$

so we may choose $n_{k+1} > n_k$ with $s_{n_{k+1}} \geq \max\{s_{n_k}, k+1\}$.

Then s_{n_k} is an increasing subsequence satisfying $s_{n_k} \geq k$. Its limit exists and is $+\infty$.

OTOH, suppose $+\infty$ is a subsequential limit, and s_{n_k} satisfies $\lim_{k\to\infty} s_{n_k} = +\infty$. For any $M>0$, $\exists\, K$ s.t. $k \geq K$ ensures $s_{n_k} > M$. Thus M is not an upper bound.

Finally (c) follows by considering $-s_n$, $-t$! $\square$

One more observation about subsequences...

<u>Thm</u>: Every sequence has a monotone subsequence.

Pf: Fix an arbitrary sequence s_n

We will say that the n^{th} element of a sequence is dominant if it is greater than everything that follows; that is,

$$s_n > s_m \ \forall m \geq n.$$

$\boxed{\text{Case 1}}$ Suppose s_n has infinitely many dominant elements.

We may define s_{n_k} to be all the dominant elements, in order, and it is a decreasing sequence.

Case 2 Suppose s_n has finitely many dominant elements.

Choose n_1 so that s_{n_1} is beyond all dominant elements. Assume we have chosen s_{n_k}. Since s_{n_k} is not dominant, we may find $n_{k+1} > n_k$ with $s_{n_{k+1}} \geq s_{n_k}$.

This gives an increasing subsequence. $\square$

Immediate consequence:

<u>Thm</u> (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Pf: We may choose a subsequence that is monotone. Since it is bounded and monotone, it converges. □

While a_N and b_N are not (in general) subsequences...

Thm: For any sequence s_n, $\limsup s_n$ and $\liminf s_n$ are the largest and smallest subsequential limits.

Ex: $s_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

$b_N = \inf\{s_n : n > N\} = 0 \quad \forall N$

Pf:
First, we will show that $\limsup s_n$ and $\liminf s_n$ are subsequential limits.

We begin with limsup s_n.

$\boxed{\text{Case 1}}$ $\limsup s_n = -\infty$

Since $\liminf s_n \le \limsup s_n$,

$\lim_{n\to\infty} s_n = -\infty$.

$\boxed{\text{Case 2}}$ $\limsup s_n = +\infty = \lim_{N\to\infty} a_N$

For all $M>0$, $\exists$ N_0 s.t. $N > N_0$

$$\sup\{s_n : n > N\} > M.$$

Assume, for the sake of contradiction, that s_n is bounded above, so $\exists$ $M_0 \geq 0$ s.t. $s_n \le M_0$ $\forall n \in \mathbb{N}$. Then

$\sup\{s_n : n > N\} \le \sup\{s_n : n \in \mathbb{N}\} \le M_0$

which is a contraction, so by main subsequences thm, $+\infty$ is a subsequential limit.

$\boxed{\text{Case 3}}$ $\limsup s_n = t$, for $t \in \mathbb{R}$ ("$\lim_{N\to\infty} a_N$")

Fix $\varepsilon > 0$. It suffices to show $|\{n : |s_n - t| < \varepsilon\}| = +\infty$

$$|\{n : |s_n - t| < \varepsilon\}| = |\{n : t-\varepsilon < s_n < t+\varepsilon\}|$$

There exists N_0 s.t. $N > N_0$ ensures

$$t - \varepsilon < a_N < t + \varepsilon$$

$$a_N = \sup\{s_n : n > N\}$$

For $n > N$,

$$s_n \leq a_N < t + \varepsilon$$

Since a_N is the least upper bound, $\forall \tilde{\varepsilon} > 0$, $\exists\, n > N$ s.t.

$$t - \varepsilon - \tilde{\varepsilon} < a_N - \tilde{\varepsilon} < s_n.$$

It remains to show there are infinitely many s_n s.t. $s_n > t - \varepsilon$. Assume, for the sake of contradiction that $s_n > t - \varepsilon$ for at most finitely many values of n. Let $M = \max\{s_n : s_n > t - \varepsilon\}$

Assume, for the sake of contradiction

$$|\{n : t - \varepsilon < s_n < t + \varepsilon\}| < +\infty,$$

WLOG, $N_1 > N_0$

that is, $\exists\ N_1$ s.t. $n > N_1$ implies $s_n \le t - \varepsilon$.

Then $a_{N_1} \le t - \varepsilon$.

This is a contradiction.

$$\boxed{\text{We have shown } \limsup s_n \text{ is always a subsequential limit}}$$

Using $-\liminf s_n = \limsup -s_n$, the same holds for liminf.

Finally, we will show $\liminf s_n$, $\limsup s_n$ are smallest/largest subseq. limits

Let $t^{\in \mathbb{R}\cup\{\pm\infty\}}$ be a subseq. limit, with s_{n_k} satisfying

$$\lim_{k\to\infty} s_{n_k} = t. \quad \{s_k : k>K\} \supseteq \{s_{n_k} : k>K\}$$

Then

$$t = \limsup_{k\to\infty} s_{n_k} \qquad \{n_k \geq k$$

$$= \lim_{K\to\infty} \sup\{s_{n_k} : k>K\}$$

$$\leq \lim_{K\to\infty} \sup\{s_k : k>K\}$$

$$= \limsup_{n\to\infty} s_n$$

Analogous argument shows $t \geq \liminf s_n$ $\square$