Math CCS 117: Midterm 2 Solutions

Question 1

(a) Suppose $\lim_{n\to+\infty} s_n = s$, where $s \in \{\pm\infty\}$. Let s_{n_k} be an arbitrary subsequence of s_n. Since every subsequence of a sequence with limit s also has limit s, we have

$$\lim_{k\to+\infty} s_{n_k} = s.$$

Thus, taking the further subsequence to be the entire subsequence s_{n_k}, property (*) holds.

(b) We prove the contrapositive. First suppose $s = +\infty$ and $\lim_{n\to+\infty} s_n \neq +\infty$. Then there exists $M \in \mathbb{R}$ so that, for all $N \in \mathbb{R}$, there is some $n > N$ so that $s_n \leq M$. Thus, there exists a subsequence s_{n_k} so that

$$s_{n_k} \leq M \quad \text{for all } k \in \mathbb{N}.$$

No further subsequence of s_{n_k} can converge to $+\infty$, contradicting property (*). Hence property (*) implies $\lim_{n\to+\infty} s_n = +\infty$.

Now suppose $s = -\infty$ and $\lim_{n\to+\infty} s_n \neq -\infty$. Then $\lim_{n\to+\infty} -s_n \neq +\infty$, so there exists a subsequence $-s_{n_k}$ so that for all further subsequences $-s_{n_{k_l}}$ its limit is not $+\infty$. Multiplying by -1, we obtain a contradiction to property (*).

Question 2

(a) For each $N \in \mathbb{N}$, define

$$a_N = \inf_{n\geq N} s_n, \qquad b_N = \inf_{n\geq N} t_n, \qquad c_N = \inf_{n\geq N} (s_n + t_n).$$

For every $n \geq N$, we have $s_n \geq a_N$ and $t_n \geq b_N$. Therefore

$$s_n + t_n \geq a_N + b_N \quad \text{for every } n \geq N.$$

Taking the infimum over all $n \geq N$, gives $c_N \geq a_N + b_N$. The fact that s_n and t_n are bounded ensures that all three sequences converge as $N \to +\infty$. Thus, since the limit of the sum is the sum of the limits,

$$\liminf_{n\to+\infty} (s_n + t_n) = \lim_{N\to+\infty} c_N = \lim_{N\to+\infty} a_N + b_N \geq \liminf_{n\to+\infty} s_n + \liminf_{n\to+\infty} t_n.$$

(b) Let

$$s_{2n} = 1, \quad s_{2n-1} = -1, \qquad t_{2n} = -1, \quad t_{2n-1} = 1.$$

Then $s_n + t_n = 0$ for every $n \in \mathbb{N}$, so $\liminf_{n\to+\infty}(s_n + t_n) = 0$.

On the other hand, $\liminf_{n\to+\infty} s_n = -1$ and $\liminf_{n\to+\infty} t_n = -1$. Therefore

$$\liminf_{n\to+\infty}(s_n + t_n) = 0 > -2 = \liminf_{n\to+\infty} s_n + \liminf_{n\to+\infty} t_n.$$

Question 3

First, suppose f is convex. For any $x, y \in \mathbb{R}^d$, positive homogeneity gives

$$f(x+y) = 2f\left(\frac{x+y}{2}\right).$$

By convexity,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Therefore

$$f(x+y) \leq f(x) + f(y).$$

Conversely, suppose

$$f(x+y) \leq f(x) + f(y) \quad \forall x, y \in \mathbb{R}^d.$$

Let $x, y \in \mathbb{R}^d$ and $\alpha \in [0,1]$. If $\alpha = 0$ or $\alpha = 1$, the convexity inequality is immediate. If $0 < \alpha < 1$, then by subadditivity and positive homogeneity,

$$f(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y).$$

Thus f is convex.

Question 4 - Extra Credit

Note that there was a small type-o in the Extra Credit. I intended to require that $[p,q] \subseteq D(f)$. However, since I wrote $(p,q) \subseteq D(f)$, one could have also chosen $\tilde{p}, \tilde{q}$ so that $[\tilde{p}, \tilde{q}] \subseteq (p,q) \subseteq D(f)$, and the argument goes through as follows. Full credit was given either way.

Choose $a, b \in (p,q)$ so that $a < b$. Fix $x, y \in (a,b)$, and suppose without loss of generality that $x < y$. By the slope monotonicity for convex functions,

$$\frac{f(a) - f(p)}{a - p} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(q) - f(b)}{q - b}.$$

Indeed, the first inequality follows by applying slope monotonicity to $p < a < x$ and to $p < x < y$, while the second follows by applying slope monotonicity to $x < y < q$ and to $y < b < q$.

Define

$$L_{a,b} = \max\left\{\left|\frac{f(a) - f(p)}{a - p}\right|, \left|\frac{f(q) - f(b)}{q - b}\right|\right\}.$$

Then

$$|f(y) - f(x)| \leq L_{a,b}|y - x|.$$

Since $x, y \in (a,b)$ were arbitrary, f is locally Lipschitz on $\operatorname{int} D(f)$.