Math CCS 117: Practice Final

(Not to be turned in)

Question 1

Determine whether the following statements are true or false. If they are true, prove them. If they are false, provide a counterexample and justify your counterexample.

(i) If $\limsup_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

(ii) If $\limsup_{n\to+\infty} x_n < \liminf_{n\to+\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

Question 2

Given $S \subset \mathbb{R}$, a function $f : S \to \mathbb{R}$ is *increasing* if, for all $x, y \in S$,

$$x \leq y \iff f(x) \leq f(y).$$

We define a function $f : \mathbb{R} \to \mathbb{R}$ to be *right continuous* if, for all $a \in \mathbb{R}$,

$$\lim_{x \to a^+} f(x) = f(a).$$

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is right continuous but not continuous. Justify your example with a proof.

(b) Suppose f_n is a sequence of continuous, increasing functions. Suppose that, for each $x \in \mathbb{R}$, $f_n(x)$ is a decreasing sequence and f_n converges pointwise to f. Prove that f is right continuous.

This problem shows that, while the pointwise limit of continuous functions is not generally continuous, as long as suitable monotonicity hypotheses hold, the limit will be right continuous.

Question 3

Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and convex. For $\lambda > 0$, define the Moreau envelope by

$$f_\lambda(x) = \inf_{y \in \mathbb{R}} \left\{ f(y) + \frac{1}{2\lambda}|x - y|^2 \right\}.$$

(a) Prove that $f_\lambda(x) \leq f(x)$ for all $x \in \mathbb{R}$.

(b) Prove that if $0 < \lambda < \mu$, then

$$f_\mu(x) \leq f_\lambda(x), \quad \forall x \in \mathbb{R}.$$

(c) Prove that $\lim_{\lambda \to 0^+} f_\lambda(x) = f(x)$ for every $x \in \mathbb{R}$.

Hint: Parts (a) and (b) are straightforward, and part (c) requires more argument. You may use the fact, proved in class, that for any f proper, convex, and lower semicontinuous, there exist $\alpha, \beta \in \mathbb{R}$ so that $f(y) \geq \alpha y + \beta$ for all $y \in \mathbb{R}$.

Question 4 - Extra Credit

Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and convex, and consider the Moreau envelope from Question 3.

(a) Prove that, for each $x \in \mathbb{R}$, the infimum is attained at a unique point. Denote this point by $\text{prox}_{\lambda f}(x)$.

(b) Prove that

$$\frac{x - \text{prox}_{\lambda f}(x)}{\lambda} \in \partial f(\text{prox}_{\lambda f}(x)).$$

(c) Prove that f_λ is differentiable and that

$$(f_\lambda)'(x) = \frac{x - \text{prox}_{\lambda f}(x)}{\lambda}.$$