Math CCS 117: Practice Final Solutions

Question 1

(i) This statement is false. Let $x_n = 0$ and $y_n = 0$ for all $n \in \mathbb{N}$. Then $\limsup_{n\to+\infty} x_n = 0$ and $\liminf_{n\to+\infty} y_n = 0$, so $\limsup_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n$. However, $x_n \geq y_n$ for every $n \in \mathbb{N}$. Therefore $x_n \geq y_n$ for infinitely many n, so the statement is false.

(ii) This statement is true. Let $L = \limsup_{n\to+\infty} x_n$ and $M = \liminf_{n\to+\infty} y_n$, and suppose $L < M$. Choose $c \in \mathbb{R}$ such that $L < c < M$. Recall that $L = \lim_{N\to+\infty} a_N$ for $a_N = \sup_{n\geq N} x_n$. If $L \in \mathbb{R}$, take $\epsilon = (c-L)/2 > 0$, so, by the definition of convergence, there exists $N_1 \in \mathbb{N}$ such that, for all $N \geq N_1$, $a_N < L + \epsilon = (L+c)/2 < c$ for all $N \geq N_1$. If $L = -\infty$, the definition of divergence to $-\infty$ gives N_1 such that $a_N < c$ for all $N \geq N_1$. In either case, $\sup_{n\geq N_1} x_n < c$, and hence $x_n < c$ for all $n \geq N_1$.

Similarly, $M = \lim_{N\to+\infty} b_n$ for $b_N = \inf_{n\geq N} y_n$. If $M \in \mathbb{R}$, take $\epsilon = (M-c)/2 > 0$, and, by the definition of convergence, there exists $N_2 \in \mathbb{N}$ such that, for all $N \geq N_2$, $|b_N - M| < \epsilon$. Thus $b_N > M - \epsilon = (M+c)/2 > c$ for all $N \geq N_2$. If $M = +\infty$, the definition of divergence to $+\infty$ instead gives N_2 such that $b_N > c$ for all $N \geq N_2$. In either case, $\inf_{n\geq N_2} y_n > c$, and hence $y_n > c$ for all $n \geq N_2$. Therefore, for all $n \geq \max\{N_1, N_2\}$, we have $x_n < c < y_n$. Hence $x_n \geq y_n$ can occur only for finitely many n.

Question 2

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

We claim that f is right continuous. Suppose $a < 0$. If $z_k \in (a, +\infty)$ and $z_k \to a$, then, for $\epsilon = -a > 0$, we have $|z_k - a| < \epsilon$ for all k sufficiently large. Since $z_k - a \leq |z_k - a|$, this implies $z_k - a < -a$, and hence $z_k < 0$, for all k sufficiently large. Thus $f(z_k) = 0 = f(a)$ for all k sufficiently large, and $f(z_k) \to f(a)$. If $a \geq 0$, then $z_k \in (a, +\infty)$ implies $f(z_k) = 1 = f(a)$ for every k, so again $f(z_k) \to f(a)$. Therefore, by the sequential definition of the right limit, $\lim_{x\to a^+} f(x) = f(a)$ for every $a \in \mathbb{R}$, so f is right continuous.

However, f is not continuous at 0. Indeed, $-1/n \to 0$, but $f(-1/n) = 0$ for all n, whereas $f(0) = 1$. Thus f is right continuous but not continuous.

(b) I should have clarified what it means for f_n to converge pointwise to f. This means that $\lim_{n\to+\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.
First note that f is increasing. Indeed, if $x \leq y$, then $f_n(x) \leq f_n(y)$ for every n, and passing to the limit gives $f(x) \leq f(y)$.

We now prove that f is right continuous. Fix $a \in \mathbb{R}$ and $\epsilon > 0$. Let $z_k \in (a, +\infty)$ be any sequence such that $z_k \to a$. Since $f_n(a)$ decreases to $f(a)$, there exists $N \in \mathbb{N}$ such that $f_N(a) < f(a) + \epsilon/2$. Since f_N is continuous at a, we have $f_N(z_k) \to f_N(a)$. Hence there exists $K \in \mathbb{N}$ such that, for all $k \geq K$, $f_N(z_k) < f_N(a) + \epsilon/2$. For such k, since $f_n(z_k)$ decreases to $f(z_k)$, we have

$$f(z_k) \leq f_N(z_k) < f_N(a) + \frac{\epsilon}{2} < f(a) + \epsilon.$$

On the other hand, since f is nondecreasing and $z_k > a$, we have $f(a) \leq f(z_k)$. Therefore, for all $k \geq K$, $f(a) \leq f(z_k) < f(a) + \epsilon$. Thus $f(z_k) \to f(a)$. Since the sequence $z_k \in (a, +\infty)$ with $z_k \to a$ was arbitrary, the sequential definition of the right limit gives $\lim_{x \to a^+} f(x) = f(a)$.

Question 3

(a) For any $x \in \mathbb{R}$, we may choose $y = x$ in the infimum defining $f_\lambda(x)$. This gives $f_\lambda(x) \leq f(x) + (1/2\lambda)|x - x|^2 = f(x)$. Thus $f_\lambda(x) \leq f(x)$ for all $x \in \mathbb{R}$.

(b) Suppose $0 < \lambda < \mu$. Then $(1/2\mu)|x - y|^2 \leq (1/2\lambda)|x - y|^2$ for all $x, y \in \mathbb{R}$. Hence

$$f(y) + \frac{1}{2\mu}|x - y|^2 \leq f(y) + \frac{1}{2\lambda}|x - y|^2.$$

Thus, any lower bound for the set in the infimum defining $f_\mu(x)$ must also be a lower bound for the set in the infimum defining $f_\lambda(x)$. This implies $f_\mu(x) \leq f_\lambda(x)$.

(c) We prove that $\lim_{\lambda \to 0^+} f_\lambda(x) = f(x)$ for every $x \in \mathbb{R}$.

By the hint, there exist $\alpha, \beta \in \mathbb{R}$ such that $f(y) \geq \alpha y + \beta$ for all $y \in \mathbb{R}$.

Let $\lambda_j \to 0^+$. By part (a), $f_\lambda(x) \leq f(x)$ for every $\lambda > 0$, so

$$\limsup_{j \to +\infty} f_{\lambda_j}(x) \leq f(x). \tag{1}$$

It remains to show that $f(x) \leq \liminf_{j \to +\infty} f_{\lambda_j}(x)$.

First suppose $f(x) < +\infty$. For each j, if $f_{\lambda_j}(x) \in \mathbb{R}$, the definition of the infimum as the greatest lower bound ensures that $f_{\lambda_j}(x) + \frac{1}{j}$ is not a lower bound, so we may choose $y_j \in \mathbb{R}$ such that

$$f(y_j) + \frac{|x - y_j|^2}{2\lambda_j} \leq f_{\lambda_j}(x) + \frac{1}{j} \leq f(x) + \frac{1}{j}. \tag{2}$$

On the other hand, if $f_{\lambda_j}(x) = +\infty$ then the above inequality holds for any choice of $y_j \in \mathbb{R}$.

Set $r_j = |x - y_j|$. Since $1/j \leq 1$ for all $j \in \mathbb{N}$, using the affine lower bound for f, we obtain

$$\alpha y_j + \beta + \frac{r_j^2}{2\lambda_j} \leq f(x) + 1.$$

Therefore, with $C = f(x) + 1 - \alpha x - \beta$,

$$\frac{r_j^2}{2\lambda_j} \leq C + |\alpha| r_j.$$

Since $f(x) \geq \alpha x + \beta$, we have $C \geq 1$. Completing the square,

$$\frac{(r_j - \lambda_j|\alpha|)^2}{2\lambda_j} \leq C + \frac{\lambda_j \alpha^2}{2} \implies (r_j - \lambda_j|\alpha|)^2 \leq 2\lambda_j \left(C + \frac{\lambda_j \alpha^2}{2} \right)$$

Since $\lambda_j \to 0$, the right-hand side converges to 0. Thus $r_j \to 0$, so $y_j \to x$.

Since f is lower semicontinuous, $f(x) \leq \liminf_{j\to+\infty} f(y_j)$. Combining this with (2) gives

$$f(x) \leq \liminf_{j\to+\infty} f(y_j) \leq \liminf_{j\to+\infty} f(y_j) + \frac{|x-y_j|^2}{2\lambda_j} \leq \liminf_{j\to+\infty} f_{\lambda_j}(x) + \frac{1}{j} = \liminf_{j\to+\infty} f_{\lambda_j}(x).$$

Finally, since $\lambda_j \to 0^+$ was arbitrary, combining this with (1), we conclude $\lim_{\lambda\to 0^+} f_\lambda(x) = f(x)$.

Now suppose $f(x) = +\infty$. We must show that $\liminf_{j\to+\infty} f_{\lambda_j}(x) = +\infty$. Suppose not. Then there exists $C \in \mathbb{R}$ such that $f_{\lambda_j}(x) \leq C$ for infinitely many j. For each such j, by definition of the infimum as the greatest lower bound, we may choose $y_j \in \mathbb{R}$ so that

$$f(y_j) + \frac{|x-y_j|^2}{2\lambda_j} \leq C + 1.$$

The same affine lower bound argument as above implies that $y_j \to x$. Since $f(y_j) \leq C+1$, lower semicontinuity gives $f(x) \leq C+1$, contradicting $f(x) = +\infty$.

Question 4 - Extra Credit

(a) Fix $x \in \mathbb{R}$. Define $q(y) := (2\lambda)^{-1}|x-y|^2$ and $g := f + q$. Note that y is a minimizer of g if and only if y is a minimizer of $2\lambda f(x) + |x-y|^2$. Since $h(x) := 2\lambda f(x)$ is proper, lower semicontinuous, and convex, the result from class on the existence and uniqueness of the proximal point sequence ensures there exists a unique minimum of $2\lambda f(x) + |x-y|^2$. This gives the result.

(b) Let $y = \text{prox}_{\lambda f}(x)$. We prove that $(x-y)/\lambda \in \partial f(y)$. Let $v \in \mathbb{R}$ and $0 < t < 1$. By minimality of y,

$$f(y) + \frac{1}{2\lambda}|x-y|^2 \leq f(y + t(v-y)) + \frac{1}{2\lambda}|x-y-t(v-y)|^2.$$

By convexity,

$$f(y+t(v-y)) \leq (1-t)f(y) + tf(v).$$

Combining these two inequalities, subtracting, dividing by t, and letting $t \downarrow 0$, we obtain

$$f(v) \geq f(y) + \frac{x-y}{\lambda}(v-y).$$

Therefore $(x-y)/\lambda \in \partial f(y)$.

(c) For $x \in \mathbb{R}$, write $y_x = \text{prox}_{\lambda f}(x)$ and $p_x = (x-y_x)/\lambda$. By part (b), $p_x \in \partial f(y_x)$.

First, we prove that $x \mapsto y_x$ is continuous. Let $x, z \in \mathbb{R}$. By monotonicity of the subdifferential,

$$(p_x - p_z)(y_x - y_z) \geq 0.$$

Substituting the definition of p_x and p_z, we get

$$((x-z) - (y_x - y_z))\,(y_x - y_z) \geq 0.$$

Thus

$$|y_x - y_z|^2 \leq |x-z|\,|y_x - y_z|.$$

Therefore

$$|y_x - y_z| \le |x - z|.$$

In particular, $x \mapsto y_x$ is continuous. It follows that $x \mapsto p_x = (x - y_x)/\lambda$ is also continuous.

Now fix $x \in \mathbb{R}$ and $h \neq 0$. Since y_x is an admissible point in the definition of $f_\lambda(x+h)$,

$$f_\lambda(x+h) \le f(y_x) + \frac{1}{2\lambda}|x + h - y_x|^2.$$

Also, by the definition of y_x, we have $f_\lambda(x) = f(y_x) + (1/(2\lambda))|x - y_x|^2$. Subtracting this from the previous inequality gives

$$f_\lambda(x+h) - f_\lambda(x) \le \frac{1}{2\lambda}\left(|x + h - y_x|^2 - |x - y_x|^2\right).$$

Using the identity $x - y_x = \lambda p_x$, we also have $x + h - y_x = \lambda p_x + h$. Therefore

$$\frac{1}{2\lambda}\left(|x + h - y_x|^2 - |x - y_x|^2\right) = \frac{1}{2\lambda}\left(|\lambda p_x + h|^2 - |\lambda p_x|^2\right) = p_x h + \frac{h^2}{2\lambda}.$$

Thus

$$f_\lambda(x+h) - f_\lambda(x) \le p_x h + \frac{h^2}{2\lambda}.$$

On the other hand, since y_{x+h} is an admissible point in the definition of $f_\lambda(x)$,

$$f_\lambda(x) \le f(y_{x+h}) + \frac{1}{2\lambda}|x - y_{x+h}|^2.$$

Also, by the definition of y_{x+h}, we have $f_\lambda(x+h) = f(y_{x+h}) + (1/(2\lambda))|x + h - y_{x+h}|^2$. Subtracting the previous inequality from this identity gives

$$f_\lambda(x+h) - f_\lambda(x) \ge \frac{1}{2\lambda}\left(|x + h - y_{x+h}|^2 - |x - y_{x+h}|^2\right).$$

Using $x + h - y_{x+h} = \lambda p_{x+h}$, we also have $x - y_{x+h} = \lambda p_{x+h} - h$. Therefore

$$\frac{1}{2\lambda}\left(|x + h - y_{x+h}|^2 - |x - y_{x+h}|^2\right) = \frac{1}{2\lambda}\left(|\lambda p_{x+h}|^2 - |\lambda p_{x+h} - h|^2\right) = p_{x+h} h - \frac{h^2}{2\lambda}.$$

Thus

$$f_\lambda(x+h) - f_\lambda(x) \ge p_{x+h} h - \frac{h^2}{2\lambda}.$$

Hence

$$p_{x+h} h - \frac{h^2}{2\lambda} \le f_\lambda(x+h) - f_\lambda(x) \le p_x h + \frac{h^2}{2\lambda}.$$

Dividing by h, considering $h > 0$ and $h < 0$ separately, and using the continuity of p_x, we conclude that

$$\lim_{h \to 0} \frac{f_\lambda(x+h) - f_\lambda(x)}{h} = p_x.$$

Therefore f_λ is differentiable and $(f_\lambda)'(x) = p_x = (x - \mathrm{prox}_{\lambda f}(x))/\lambda$.