Math CCS 117: Practice Midterm 1

(Not to be turned in)

Question 1

Consider a nonempty set $A \subseteq \mathbb{R}$.

(a) Suppose A is bounded above. Prove that there exists a sequence a_n, satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and $\lim_{n\to\infty} a_n = \sup A$.

(b) Now suppose A is not bounded above. Prove that there exists a sequence a_n satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and $\lim_{n\to+\infty} a_n = \sup A$

In summary, you have proved the following important result: for any nonempty set $A \subseteq \mathbb{R}$, we may always find a sequence of elements a_n in A so that $\lim_{n\to+\infty} a_n = \sup A$.

Question 2

Define a sequence s_n as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$.

(a) Use induction to prove that $s_n \geq 1/2$ for all n.

(b) Use the definition of the sequence and part (a) to show that the sequence is decreasing.

(c) Prove that $\lim s_n = s$ for some $s \in \mathbb{R}$.

(d) Use part (d), the definition of s_n, and the limit theorems to find the value of s.

You may use, without proof, the fact that $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} s_{n+1}$.

Question 3

(a) State the definition of what it means for a sequence s_n to converge to a limit $s \in \mathbb{R}$.

(b) Consider a sequence s_n satisfying $s_n \neq 0$ for all n and for which $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L. If $L < 1$, show that $\lim_{n\to+\infty} s_n = 0$.

Question 4 - Extra Credit

Given a sequence s_n of real numbers, define its arithmetic mean by

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

(a) If s_n converges, prove that σ_n converges.

(b) Give an example to show that the converse of part (a) is not true.

(c) Let $a_n = s_{n+1} - s_n$. Assume that $\lim_{k\to+\infty} k a_k = 0$ and σ_n converges. Prove that s_n converges.

Hint: First, show that $s_{n+1} - \sigma_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$.

Moral of the problem: while the convergence of σ_n is not, in general, sufficient to imply the convergence of s_n, if we also know that the increments of s_n converge to zero sufficiently quickly, it is sufficient.

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