Math CCS 117: Practice Midterm 2 Solutions

Question 1

(a) If $\lim_{x\to a^+} f(x) = L$, then, by definition, for any $x_n \searrow a$ we have $\lim_{n\to+\infty} f(x_n) = L$.

Conversely, suppose $\lim_{x\to a^+} f(x) \neq L$. Then, there exists some sequence $x_n : \mathbb{N} \to (a, +\infty)$ so that $x_n \to a$ but $\lim_{n\to+\infty} f(x_n) \neq L$. We must have either $\limsup_{n\to+\infty} f(x_n) \neq L$ or $\liminf_{n\to+\infty} f(x_n) \neq L$. Since lim sup and lim inf are subsequential limits, there exists a subsequence $f(x_{n_k})$ so that its limit exists and $\lim_{k\to+\infty} f(x_{n_k}) \neq L$. Since every sequence has a monotone subsequence and $x_n > a$ for all $n \in \mathbb{N}$, there exists a subsequence $x_{n_{k_l}}$ that is decreasing with $\lim_{l\to+\infty} x_{n_{k_l}} = a$. In this way, we have shown that there exists a sequence $y_n : \mathbb{N} \to (a, +\infty)$ satisfying $y_n \searrow a$ with $\lim_{n\to+\infty} f(y_n) \neq L$.

(b) The result follows from part (a), taking the function $h(t) = f(-t)$ at the point $-a$. In this case, $t_n \searrow -a$ with $t_n > -a$ is equivalent to $x_n = -t_n \nearrow a$ with $x_n < a$, and $h(t_n) = f(x_n)$.

Question 2

(a) False. Let $s_{2n} = 0$ and $s_{2n-1} = 2$ for $n \in \mathbb{N}$. Since the sequence is bounded, the only subsequential limits are values $s \in \mathbb{R}$ so that $|\{n : |s_n - s| < \epsilon\}| = +\infty$ for all $\epsilon > 0$. Thus, the subsequential limits are $\{0, 2\}$, so $\limsup s_n = 2$. On the other hand, there are infinitely many terms less than 1.99.

(b) False. For any finite b, let $s_n = b + 1/n$. Then $\limsup s_n = \lim s_n = b$, but $s_n > b$ for every n.

Question 3

(a) Let $h = f \Box g$.

Fix $x_1, x_2 \in \mathbb{R}$. If $h(x_1)$ or $h(x_2)$ is $+\infty$, the convexity inequality is immediate. Otherwise, for $\epsilon > 0$ arbitrary, $h(x_i) + \epsilon$ is strictly larger than $h(x_i)$, so by the definition of infimum there exists $y_i \in \mathbb{R}$ so that $f(y_i) + g(x_i - y_i) \leq h(x_i) + \epsilon$ for $i = 1, 2$. Fix $\lambda \in [0, 1]$. Then, defining $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ and $y_\lambda = \lambda y_1 + (1-\lambda)y_2$, convexity of f and g and the fact that the infimum is a lower bound gives

$$h(x_\lambda) \leq f(y_\lambda) + g(x_\lambda - y_\lambda) \leq \lambda h(x_1) + (1-\lambda)h(x_2) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this gives the result.

(b) Let $z_x = x - y_x$ and $h = f \Box g$.

First suppose $p \in \partial f(y_x) \cap \partial g(z_x)$. For any $w, y \in \mathbb{R}$, the definition of the subdifferential implies

$$f(y) + g(w - y) \geq f(y_x) + p(y - y_x) + g(z_x) + p((w - y) - z_x) = h(x) + p(w - x).$$

Since the infimum is greater than any lower bound, this shows

$$h(w) \geq h(x) + p(w - x),$$

so $p \in \partial h(x)$.

Conversely, suppose $p \in \partial h(x)$. For any $u \in \mathbb{R}$, the definition of h implies $h(u + z_x) \leq f(u) + g(z_x)$. By definition of the subdifferential, $h(u+z_x) \geq h(x)+p((u+z_x)-x)$. Combining the previous inequalities with the fact that the inf convolution is exact at x gives

$$f(u) + g(z_x) \geq h(x) + p((u + z_x) - x) = f(y_x) + g(x - y_x) + p((u + z_x) - x)$$

so $f(u) \geq f(y_x) + p(u - y_x)$. Since $u \in \mathbb{R}$ was arbitrary, this shows $p \in \partial f(y_x)$.

It remains to show that $p \in \partial g(z_x)$. Let $v \in \mathbb{R}$ be arbitrary, and set $w = y_x + v$. By definition of the infimum, $h(w) \leq f(y_x) + g(w - y_x) = f(y_x) + g(v)$. On the other hand, since $p \in \partial h(x)$, $h(w) \geq h(x) + p(w - x)$. Combining these previous inequalities with exactness, $h(x) = f(y_x) + g(z_x)$, we obtain

$$f(y_x) + g(v) \geq h(w) \geq f(y_x) + g(z_x) + p(v - z_x).$$

Thus, $g(v) \geq g(z_x) + p(v - z_x)$. Since $v \in \mathbb{R}$ was arbitrary, we have $p \in \partial g(z_x)$. Thus $p \in \partial f(y_x) \cap \partial g(x - y_x)$.

Question 4 - Extra Credit

Let $h := f \Box g$. First, note that f and g are bounded below: if, for instance, $f(u_n) \to -\infty$, then boundedness of $D(f)$ gives a convergent subsequence $u_{n_k} \to u$, and lower semicontinuity would imply $f(u) \leq -\infty$, which is impossible.

Suppose $x_n \to x$. We must show $h(x) \leq \liminf h(x_n)$. If $\liminf h(x_n) = +\infty$, the result is immediate, so we may assume that $\liminf h(x_n) < +\infty$. Choose a subsequence $h(x_{n_k})$ so that $\lim_{k \to +\infty} h(x_{n_k}) = \liminf_{n \to +\infty} h(x_n)$. Discarding finitely many terms, we may assume $h(x_{n_k}) < +\infty$ for all $k \in \mathbb{N}$. Since f and g are bounded below, we also have $h(x_{n_k}) > -\infty$ for all $k \in \mathbb{N}$.

By definition of the infimum, for all $k \in \mathbb{N}$, there exists $y_{n_k} \in \mathbb{R}$ so that $f(y_{n_k}) + g(x_{n_k} - y_{n_k}) \leq h(x_{n_k}) + 1/n_k$. Since $D(f)$ is bounded, there exists a further subsequence $y_{n_{k_l}}$ and $y \in \mathbb{R}$ with $y_{n_{k_l}} \to y$, so $x_{n_{k_l}} - y_{n_{k_l}} \to x - y$. Lower semicontinuity, together with the lower bounds on f and g, gives

$$h(x) \leq f(y) + g(x-y) \leq \liminf_{l \to +\infty} \big(f(y_{n_{k_l}}) + g(x_{n_{k_l}} - y_{n_{k_l}}) \big) \leq \liminf_{l \to +\infty} \big(h(x_{n_{k_l}}) + 1/n_{k_l} \big) = \liminf_{n \to +\infty} h(x_n).$$

Thus $h(x) \leq \liminf h(x_n)$, so h is lower semicontinuous.