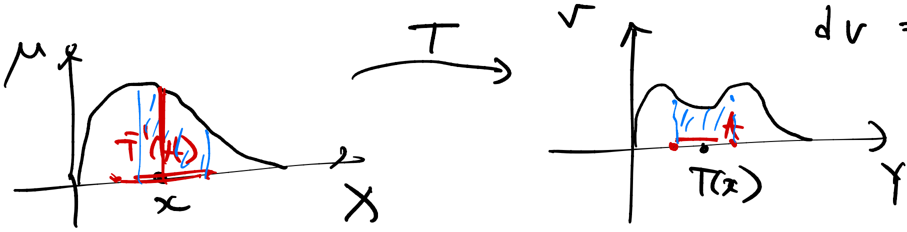
Intro to Optimal Transport

Let $\mu \in P(X)$, $\nu \in P(Y)$. $d\mu = f(x)\,dx$ $\int f(x)\,dx = 1$

$d\nu = g(x)\,dx$ $\int g(y)\,dy = 1$



$T : X \to Y$

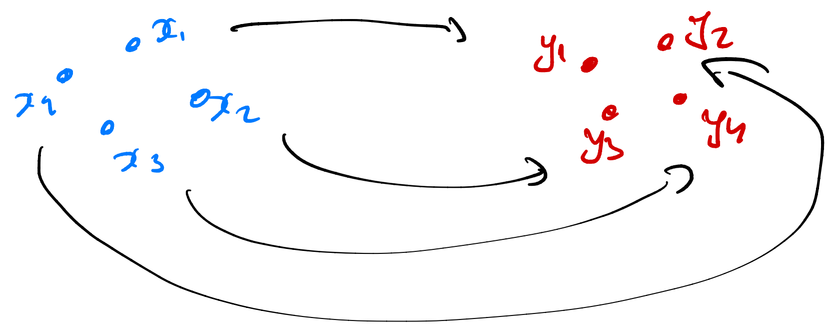
$A \subseteq Y$ $\nu(A) = \mu(T^{-1}(A)) = \int_Y 1\!\!1_A(y)\, \underbrace{g(y)\,dy}_{d\nu}$

$\boxed{\nu = T_\# \mu}$ push-forward T - transport map

$\boxed{T_\# \mu(A) = \mu(T^{-1}(A))}$

$$g(T(x)) = \underbrace{\det(DT(x))}_{JT(x)} f(x)$$

Ex $\mu = \sum_{i=1}^{n} \frac{1}{n} \delta_{x_i}$ $\nu = \sum_{j=1}^{n} \frac{1}{n} \delta_{y_j}$



Let $c : X \times Y \to [0, +\infty)$ be a continuous map.

(Usually $X = Y = \mathbb{R}^d$, $c(x,y) = |x-y|^p$)

$$\underset{T_\#\mu = \nu}{\min / \inf} \int_X \underbrace{c(x, T(x))}_{\text{cost of moving from } x \text{ to } T(x)} \, \underbrace{d\mu(x)}_{\text{mass that we are moving}} \qquad d\mu(x) = f(x)\,dx$$

(Monge Problem)

Issues:

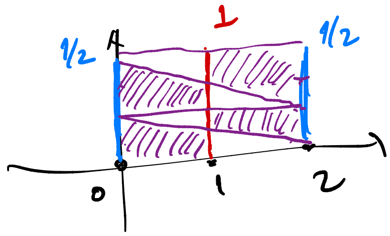
(i) Transport map may not exist.

$$\delta_{x_1} \to \tfrac{1}{2}\delta_{y_1}, \quad \delta_{x_1} \to \tfrac{1}{2}\delta_{y_2}$$

$x \mapsto T(x)$

$\boxed{\text{Mass can't split.}}$

(ii)

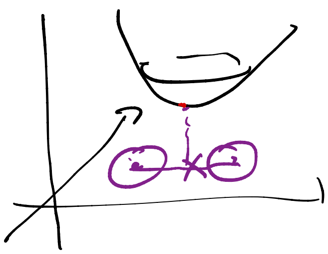


$c(x,y) = |x-y|$

$1 < \text{cost}(T_n) \le 1 + \frac{1}{n}$

inf exist, but minimizer doesn't

(iii) $T_{\#}\mu = \nu$ constraint set may not be convex



$$\begin{array}{ccc} X & \times & Y \\ {\scriptstyle P_X}\downarrow & & \downarrow{\scriptstyle P_Y} \\ X & & Y \end{array}$$

$$\min_{\gamma \in \Pi(\mu,\nu)} \int_{X\times Y} c(x,y)\, d\gamma(x,y)$$

(Kantorovich Problem)

$P_{X\#}\gamma = \mu$

$\Pi(\mu,\nu) = \{ \gamma \in P(X\times Y) \mid$ first marginal of γ is μ, second marginal of γ is $\nu \}$

$P_{Y\#}\gamma = \nu$

$$\int_{X\times Y} \varphi(x)\, d\gamma(x,y) = \int_X \varphi(x)\, d\mu$$

$$\int_{X\times Y} \psi(y)\, d\gamma(x,y) = \int_Y \psi(y)\, d\nu$$

(i) $d\gamma(A\times B)$ gives us how much mass we can move from set A to set B

(We can split the mass.)

(ii) If X, Y are metric spaces, $\mu \in P_2(X)$, $\nu \in P_2(Y)$,

$\left(P_2(X) = \left\{ \mu \in P(X) \;\middle|\; \int_X |x|^2 \, d\mu < \infty \right\} \right)$

and if $c(x,y) = \frac{1}{2}(d(x,y))^2$, then (KP) minimizer is attained.

(iii) This is a convex problem.

$(1-\lambda)\gamma_0 + \lambda \cdot \gamma_1 \in \Pi(\mu, \nu)$

This allows us to talk about a dual problem.