Essentially non-branchingness of Riemannian/Lorentzian mfds w. lower regularity metrics

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Essentially non-branchingness of

Riemannian/Lorentzian mfds w. lower regularity metrics

$(+,+,\dots,+)$ $(-,+,+,\dots,+)$

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Riemannian mfd

hard

$$(\overset{\in C^\infty}{M}, \overset{\in C^\infty}{g_{ij}})$$

Riemannian mfd

hard

$(M^{\in C^\infty}, g_{ij}^{\in C^\infty})$

Riemannian mfd

hard

Metric sp

$$(M^{\in C^\infty}, g_{ij}^{\in C^\infty})$$

Riemannian mfd

hard

Metric sp

easy

$$(M^{\in C^\infty}, g_{ij}^{\in C^\infty})$$

Riemannian mfd

hard

$$(X, d)$$

Metric sp

easy

Curvature dimension bounds on (X, d, m)

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K, N)$

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

$\Downarrow$:

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

$\in C^\infty$

$\Downarrow$: Φ $\quad (M^n, g_{ij}, e^{-V} d\mathrm{vol}_g)$

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

$\Updownarrow$: $\exists$ $(M^n, g_{ij}, e^{-V} dvol_g)$ is $CD^e(K,N)$

$\in C^\infty$

Curvature dimension bounds on (X,d,m)

Ex: $CD^e(K,N)$

$\Downarrow$: ① $(M^n, g_{ij}, e^{-V} dvol_g)$ is $CD^e(K,N)$ $\quad V \in C^\infty$

$\Leftrightarrow$ $Ric^{(N,V)} \geq K$ and $n \leq N$

Curvature dimension bounds on (X,d,m)

Ex: $CD^e(K,N)$

$\Downarrow$: ① $(M^n, g_{ij}, e^{-V} d\mathrm{vol}_g)$ is $CD^e(K,N)$, $V \in C^\infty$

$\Leftrightarrow \mathrm{Ric}^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

$\Downarrow$: ① $(M^n, g_{ij}, e^{-V} dvol_g)$ is $CD^e(K,N)$, $V \in C^\infty$

$\Leftrightarrow Ric^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH . . . ..

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

$\Downarrow$: ① $(M^n, g_{ij}, e^{-V} d\mathrm{vol}_g)$ is $CD^e(K,N)$ ($g_{ij} \in C^\infty$)

$\Leftrightarrow Ric^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH . . . . .

Curvature dimension bounds on (X,d,m)

Ex: $CD^e(K,N)$

$\in C^\infty$

☺ : ① $(M^n, g_{ij}, e^{-V} d\mathrm{vol}_g)$ is $CD^e(K,N)$

$\Leftrightarrow \mathrm{Ric}^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH $\cdot \quad \cdot \quad \cdot \quad \cdot\cdot\cdot$

$CD^e(K,N)$

Curvature dimension bounds on (X,d,m)

Ex: $CD^e(K,N)$

① $(M^n, g_{ij}, e^{-V} d\mathrm{vol}_g)$ is $CD^e(K,N)$, $V \in C^\infty$

$\Leftrightarrow$ $Ric^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH

$CD^e(K,N)$

$CD^e(K,N)$

Curvature dimension bounds on (X, d, m)

Ex: $CD^e(K,N)$

① $\underbrace{(M^n, g_{ij}, e^{-V} dvol_g)}_{\circledast}$ is $CD^e(K,N)$, $\quad V \in C^\infty$

$\Longleftrightarrow \quad Ric^{(N,V)} \geq K$ and $n \leq N$

② Stable under GH $\quad \cdot \quad \cdot \quad \cdot \quad \cdot\cdot\; \to$ $CD^e(K,N)$ (red); $CD^e(K,N)$ (blue)

③ $\circledast$ is dense in $CD^e(K,N)$

Motivation: Lazy Gas Experiment

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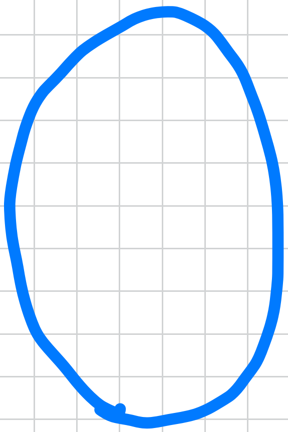
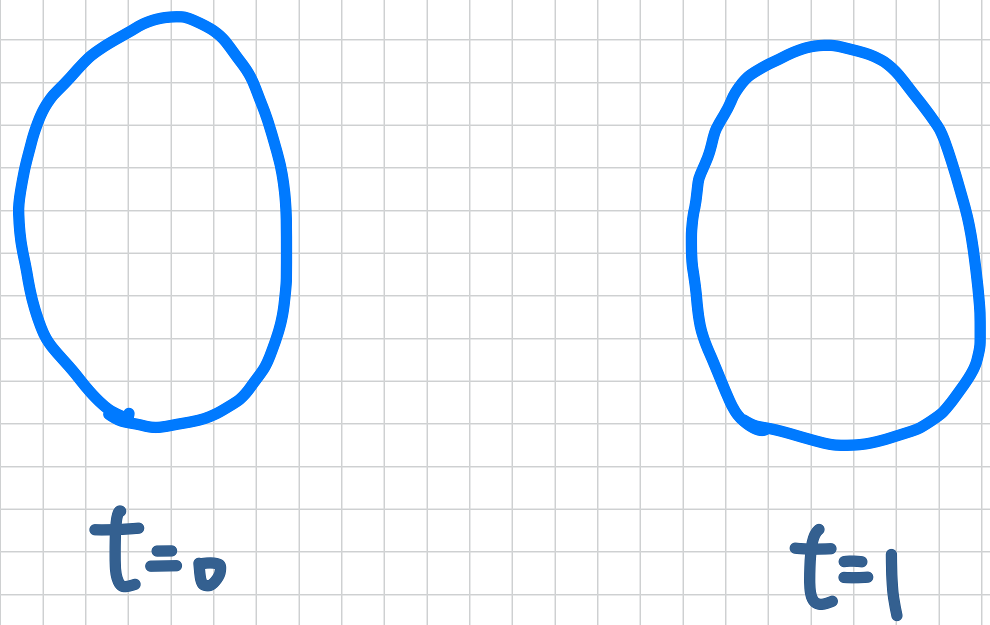
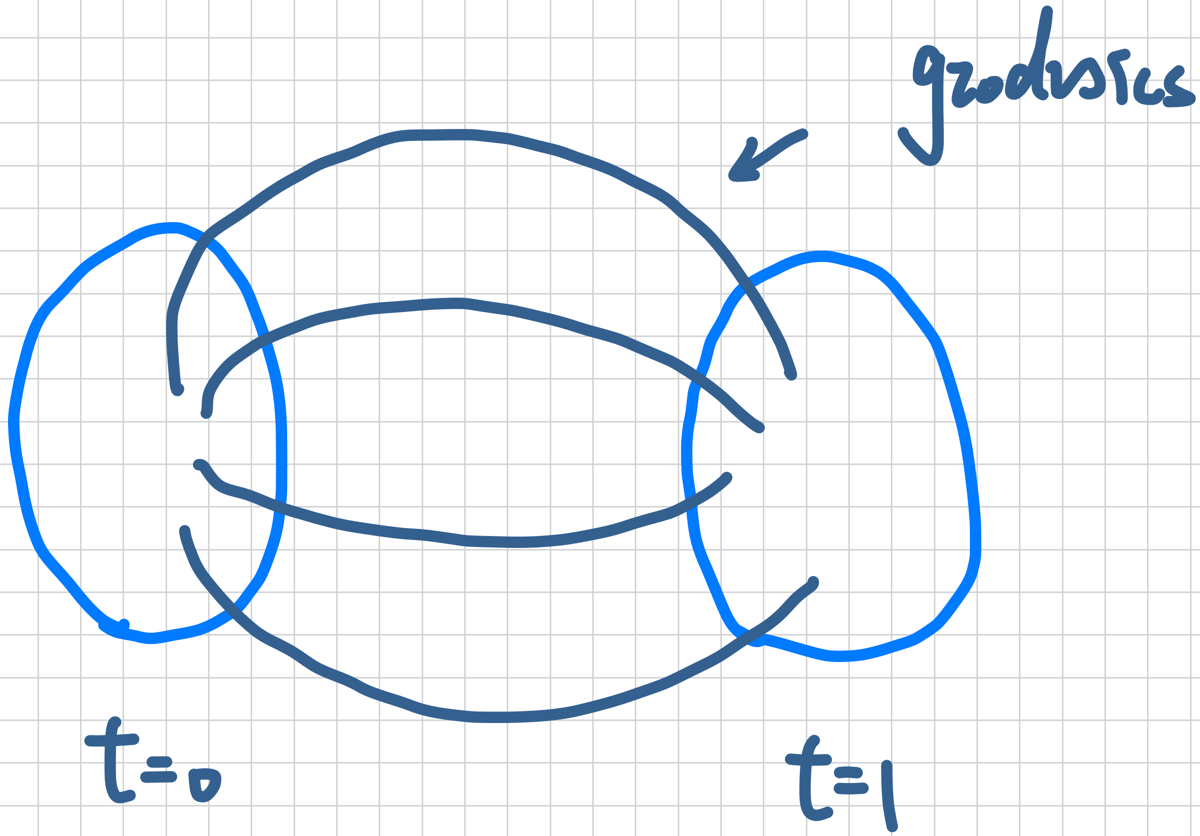


$t=0$

Motivation: Lazy Gas Experiment



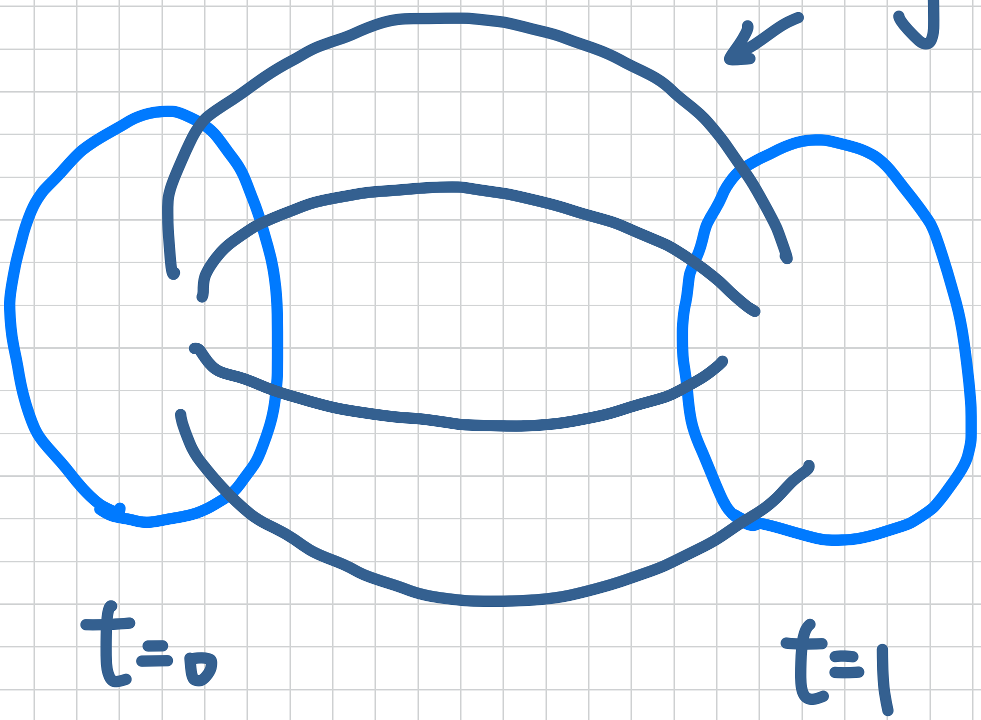
Motivation: Lazy Gas Experiment



Motivation: Lazy Gas Experiment

On smooth Riemannian mfd

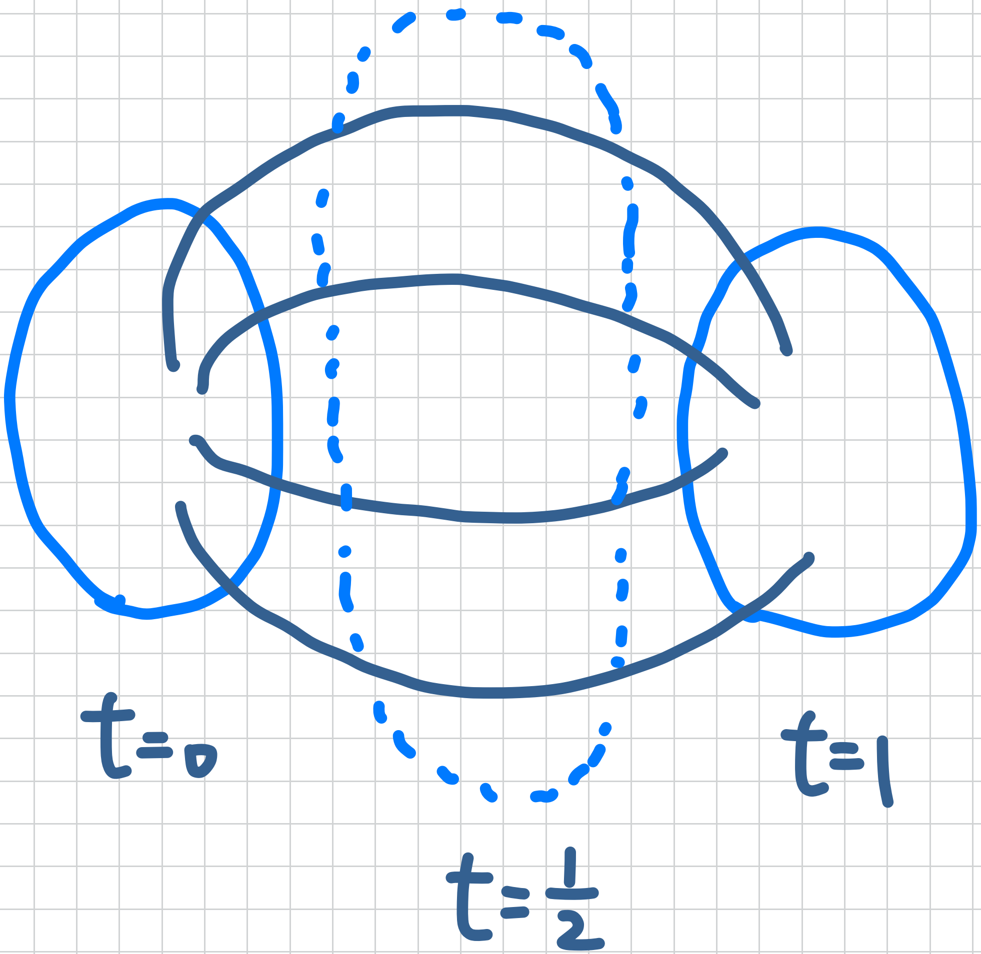
$\mathrm{Ric} \geq 0$.



Motivation: Lazy Gas Experiment

On smooth Riemannian mfd

$Ric \geq 0$.



Motivation: Lazy Gas Experiment

On smooth Riemannian mfd

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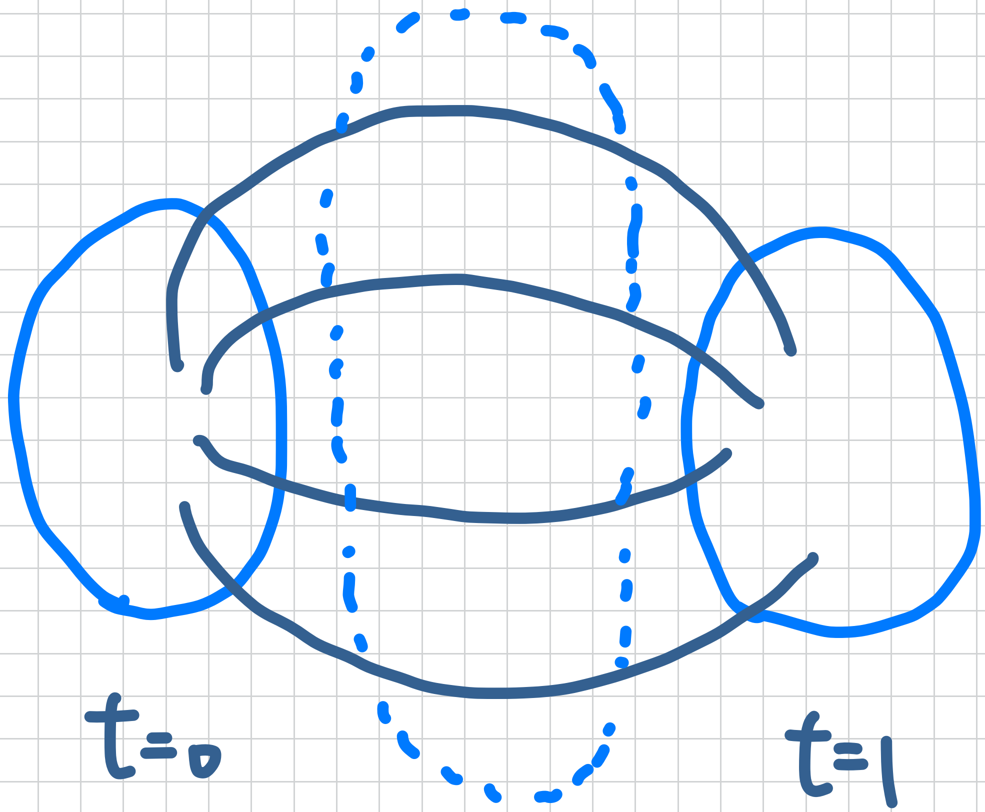
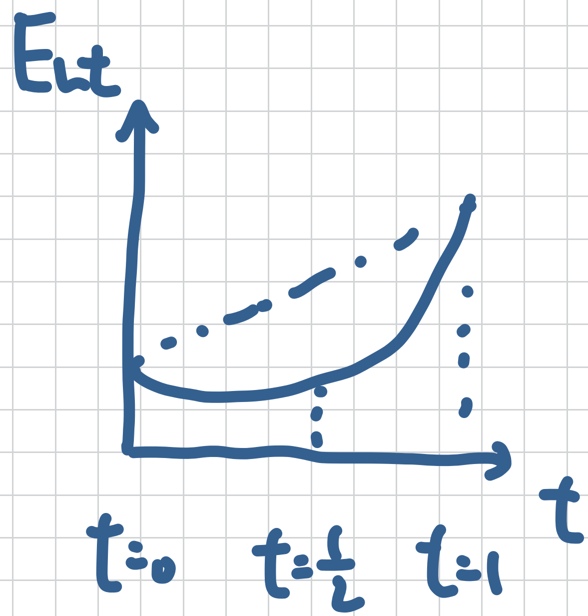
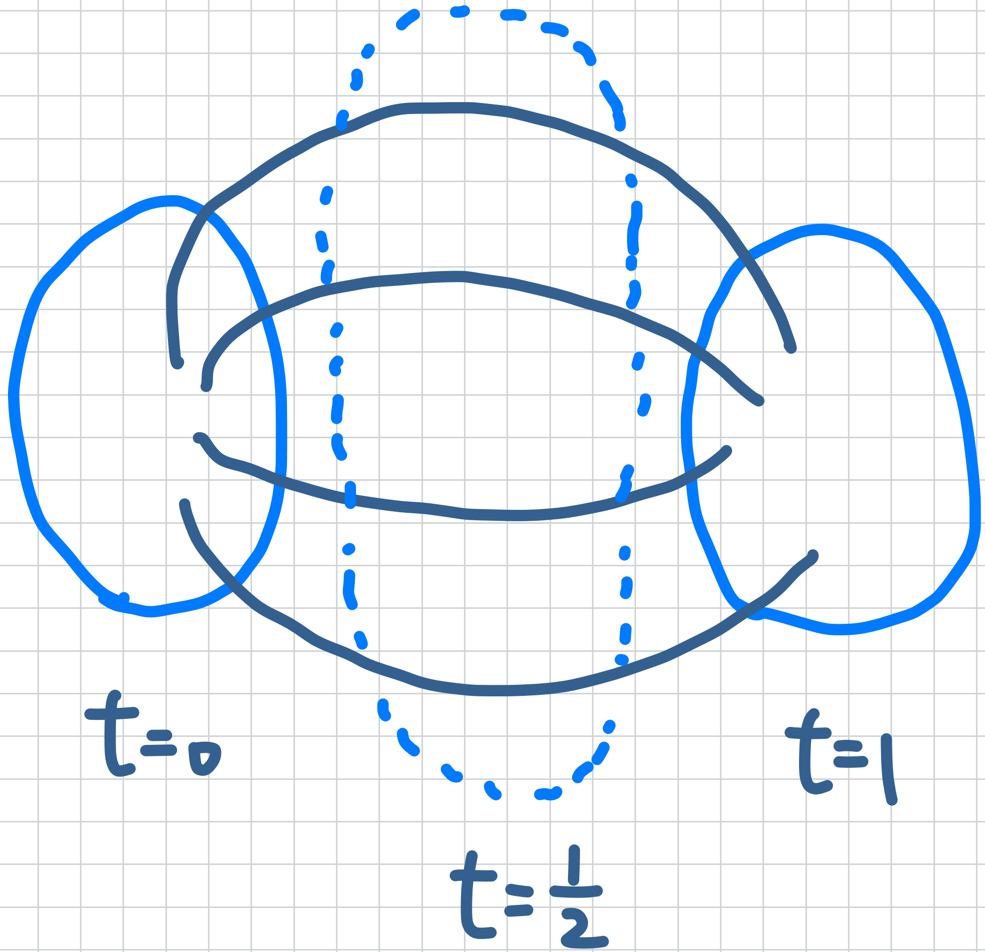


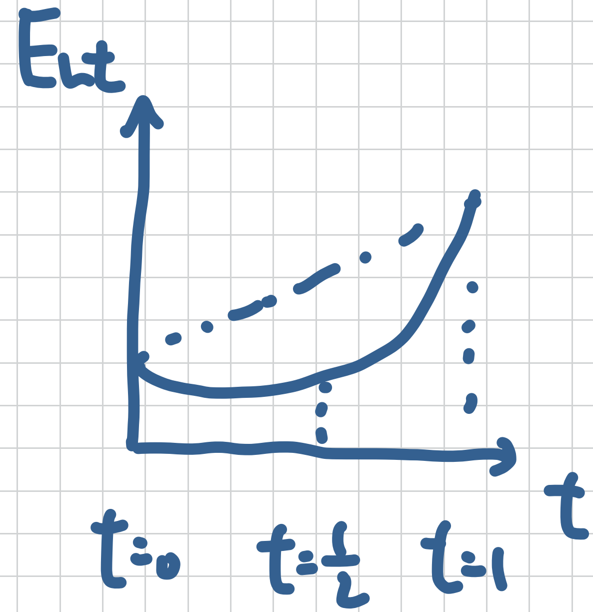


Motivation: Lazy Gas Experiment

On smooth Riemannian mfd

$Ric \geq 0$.



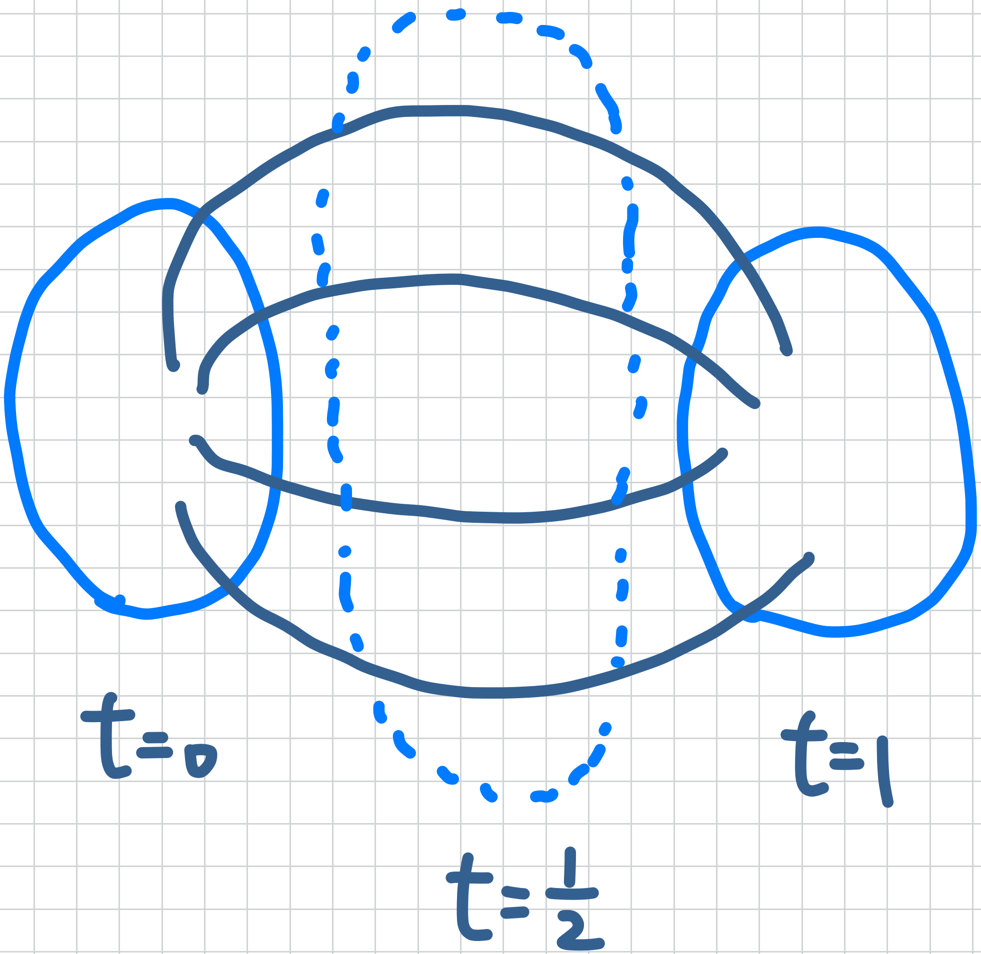


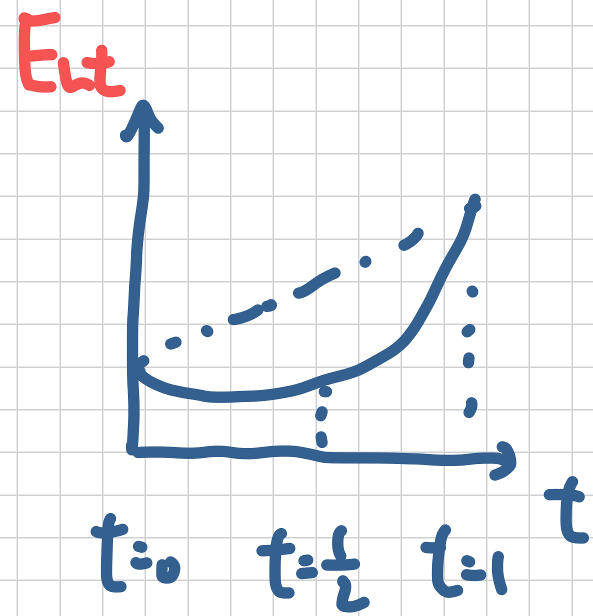
Convex

Motivation: Lazy Gas Experiment

On smooth Riemannian mfd

$Ric \geq 0$.





Convex

Curvature dimension bounds on (X,d,m)

Ex: $CD^e(K,N)$

$\in C^\infty$

① $\underbrace{(M^n, g_{ij}, e^{-V}\, d\mathrm{vol}_g)}_{\circledast}$ is $CD^e(K,N)$

$\Leftrightarrow \mathrm{Ric}^{(N,V)} \geq K$ and $n \leq N$

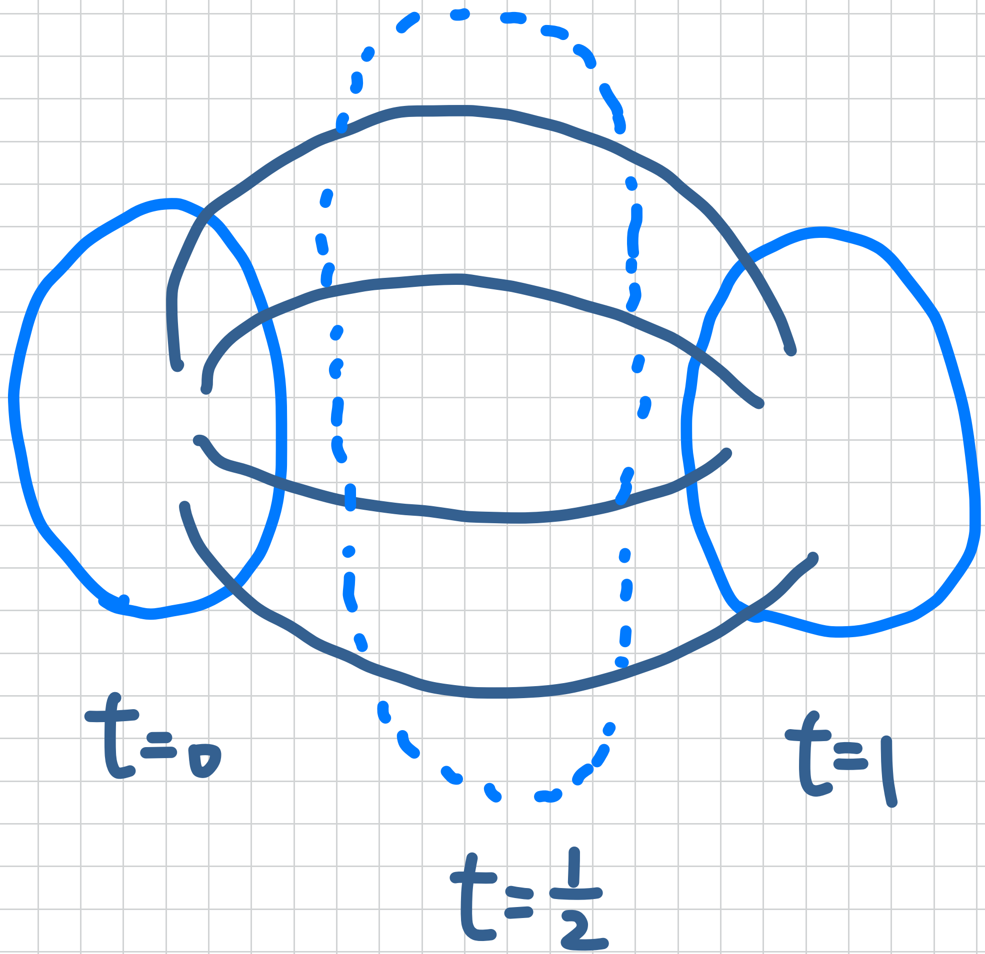
$CD^e(K,N)$

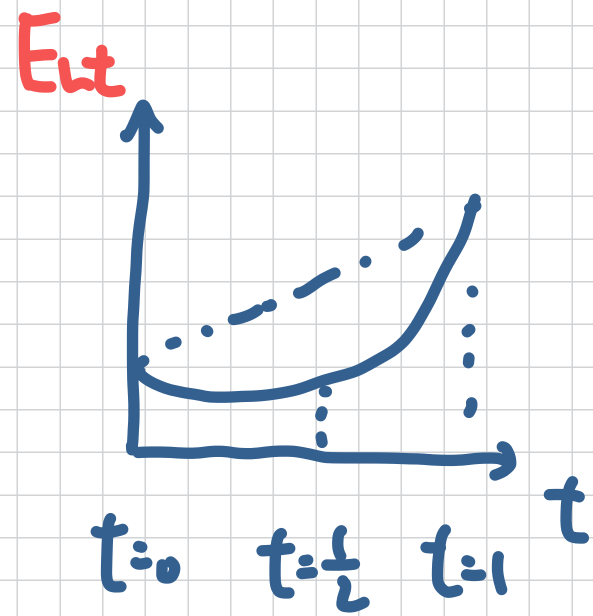
② Stable under GH

$CD^e(K,N)$

③ $\circledast$ is dense in $CD^e(K,N)$

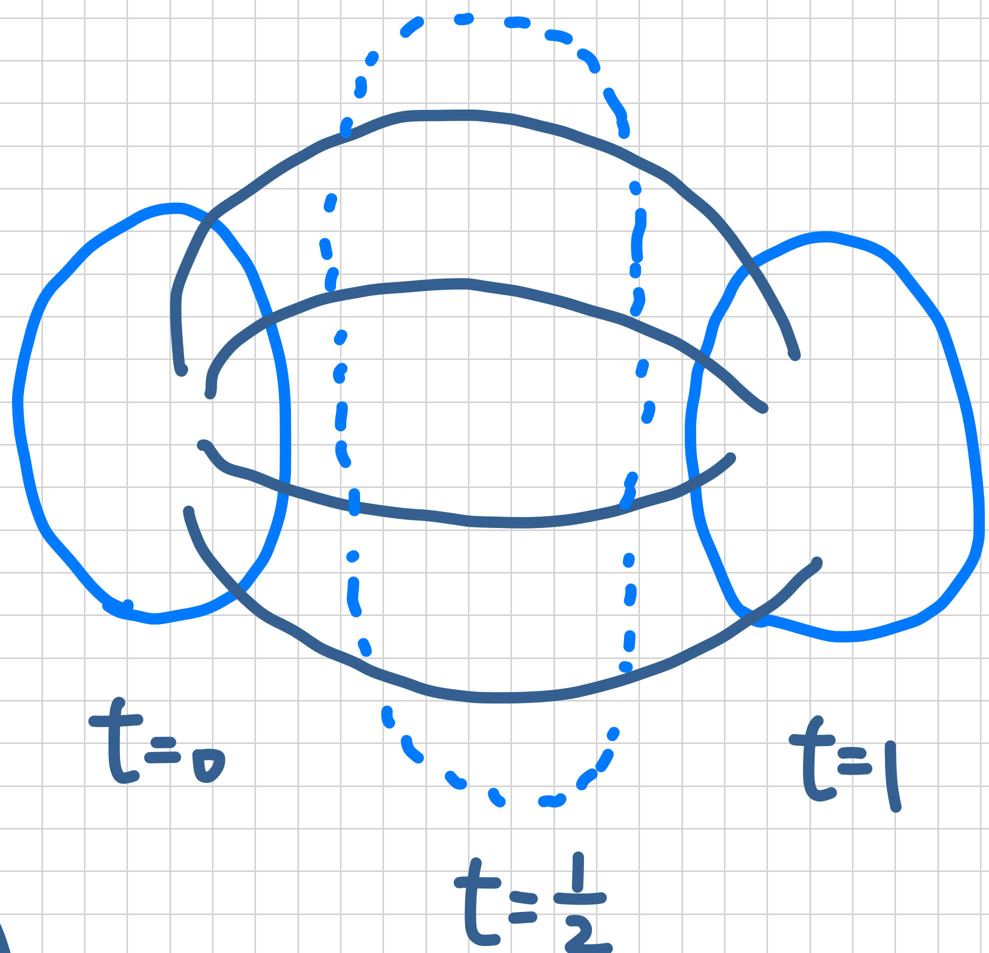
Motivation: Lazy Gas Experiment

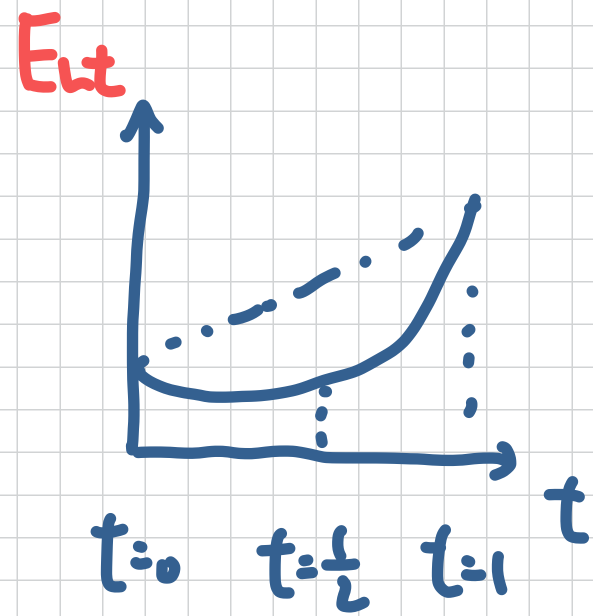




Convex

Motivation: Lazy Gas Experiment



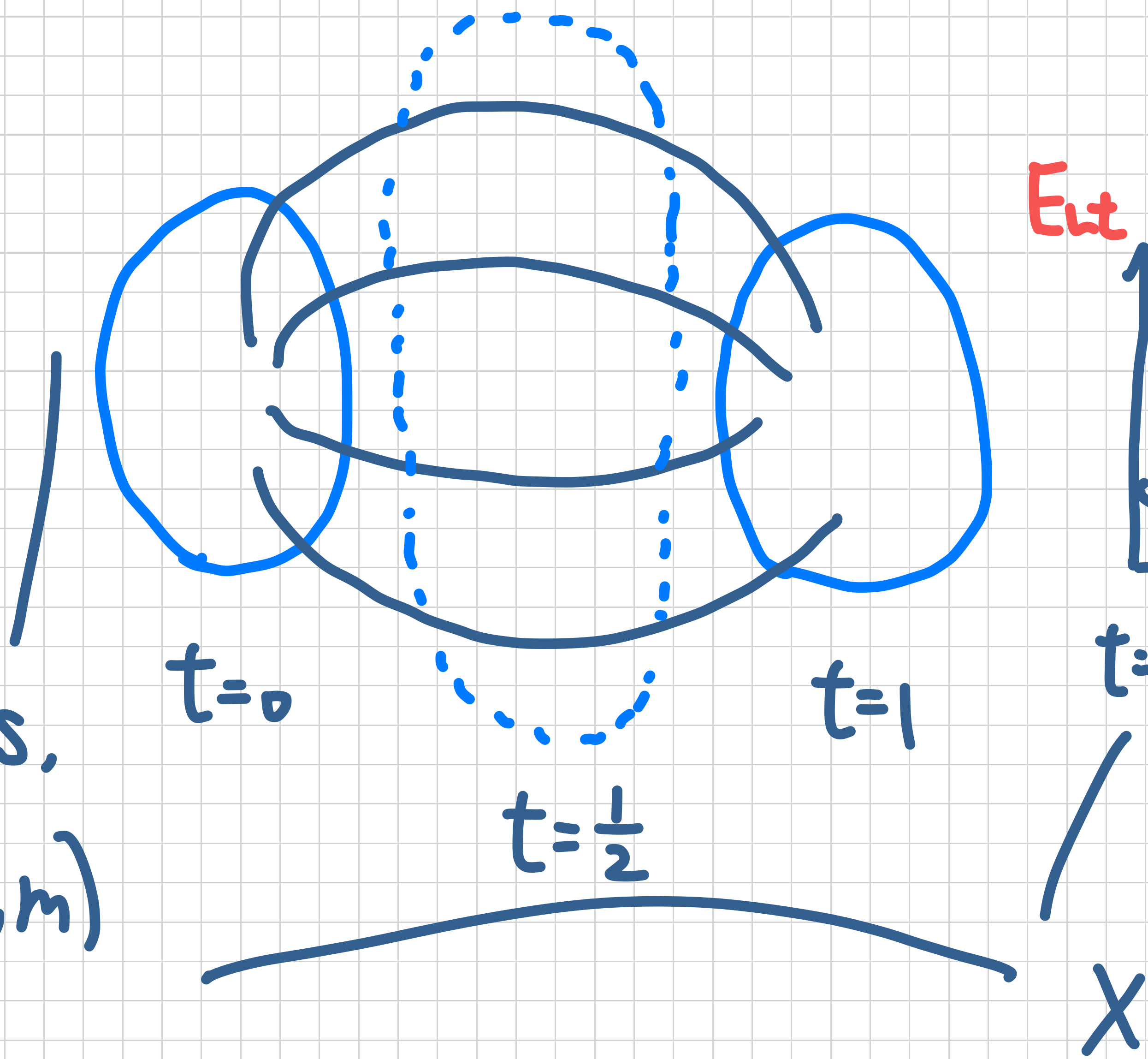


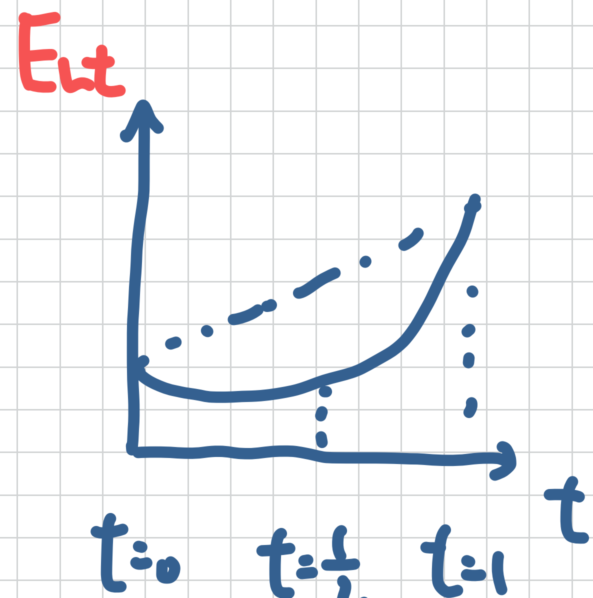
Convex

m.m.s, (X,d,m)

Motivation: Lazy Gas Experiment

m.m.s,
(X, d, m)

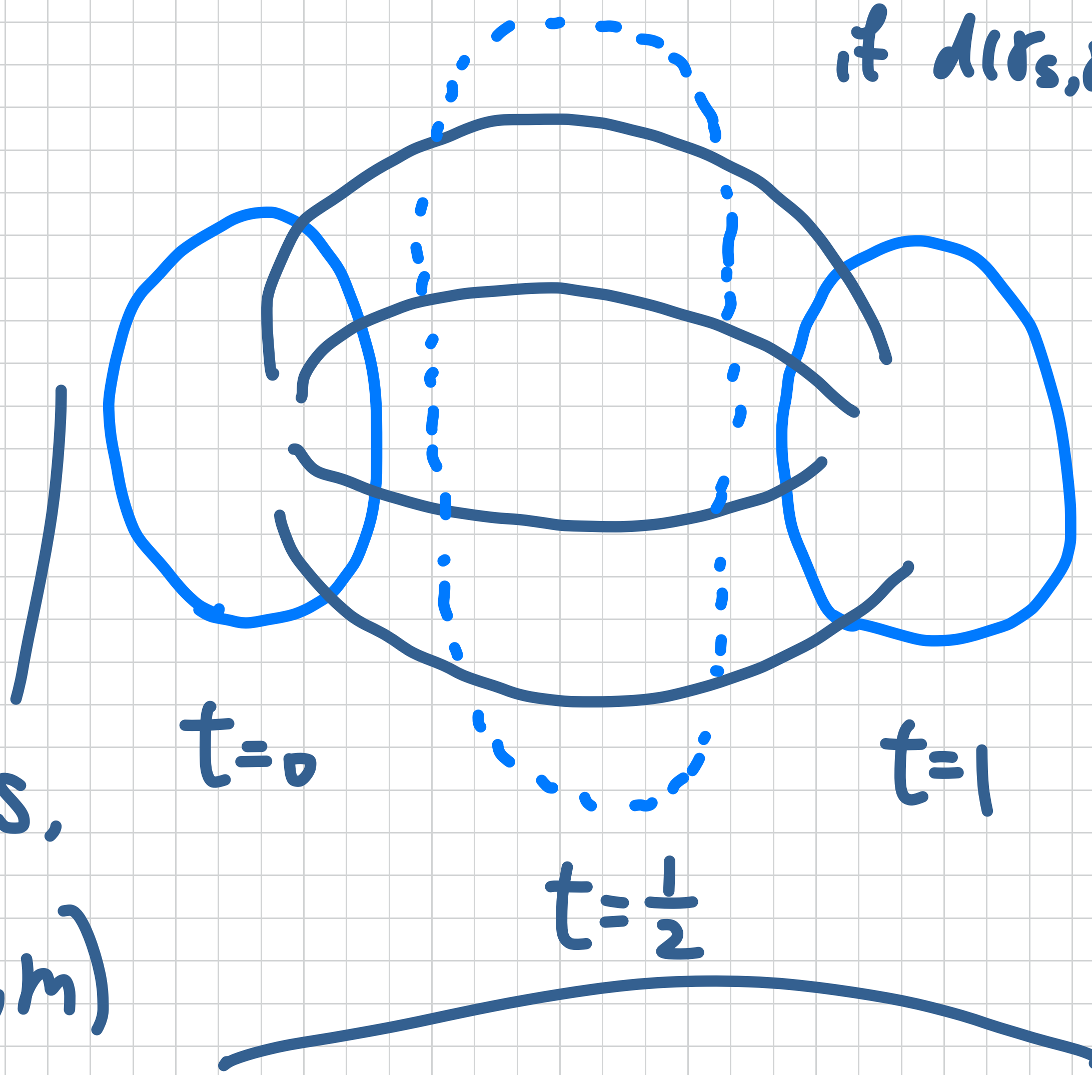


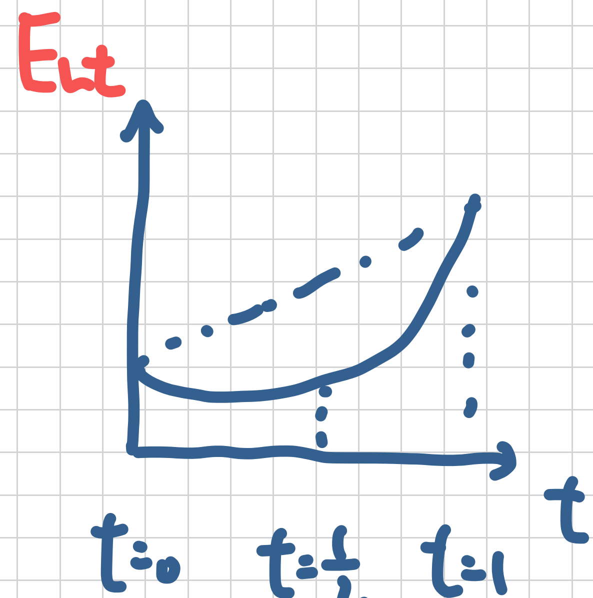


Convex

Motivation: Lazy Gas Experiment

$\gamma:[0,1]\to X$ is a geodesic X
if $d(\gamma_s,\gamma_t)=|s-t|\,d(\gamma_0,\gamma_1)\ \forall s,t$





m.m.s,
(X,d,m)

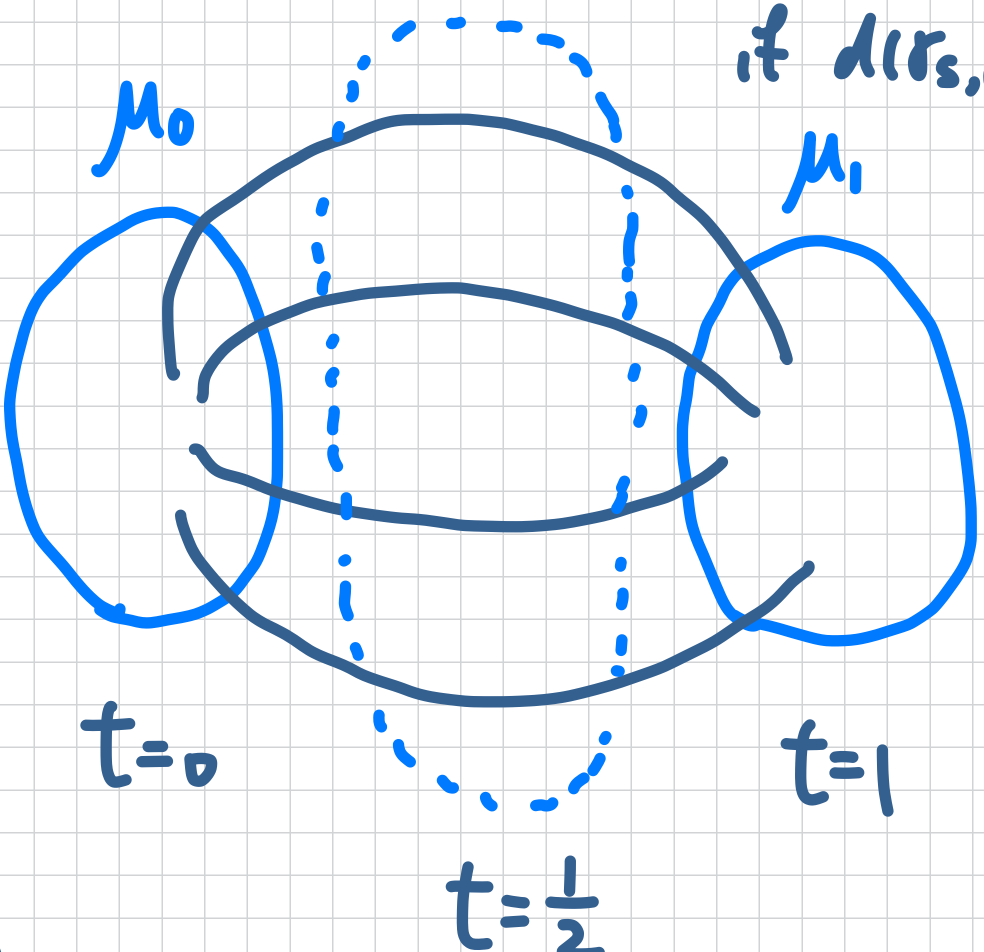
Convex

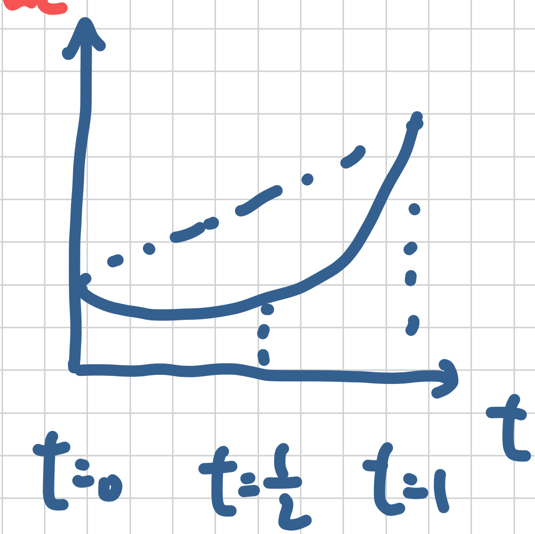
X

Motivation: Lazy Gas Experiment

$\gamma:[0,1]\to X$ is a geodesic X
if $d(\gamma_s,\gamma_t)=|s-t|\,d(\gamma_0,\gamma_1)\ \forall s,t$

$P_2(X)$





m.m.s,
(X,d,m)

Convex

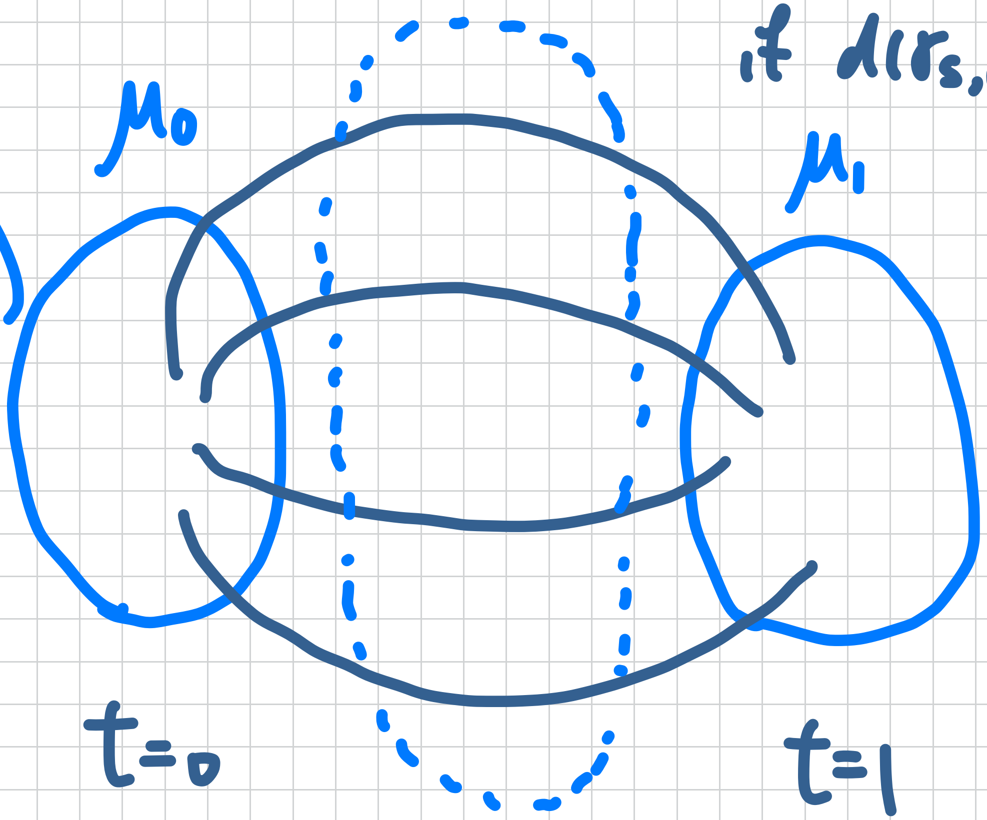
X

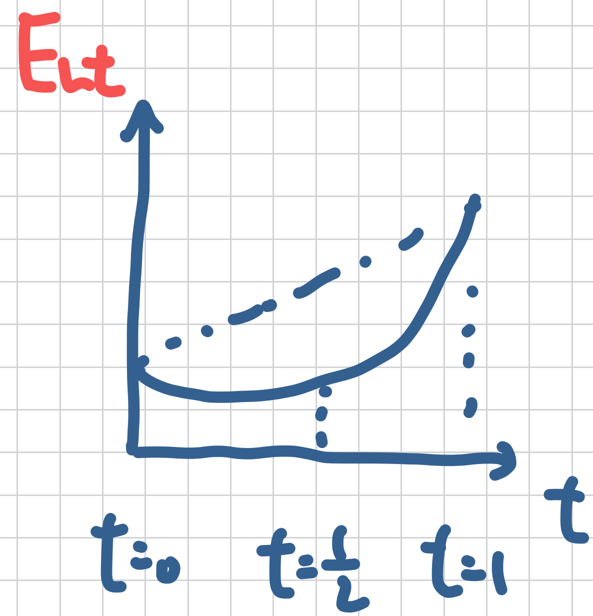
Motivation: Lazy Gas Experiment

$\gamma : [0,1] \to X$ is a geodesic X
if $d(\gamma_s, \gamma_t) = |s-t|\, d(\gamma_0, \gamma_1)$ $\forall s,t$

$(\mathcal{P}_2(X), d_W)$

$c = \dfrac{d_g^2}{2}$

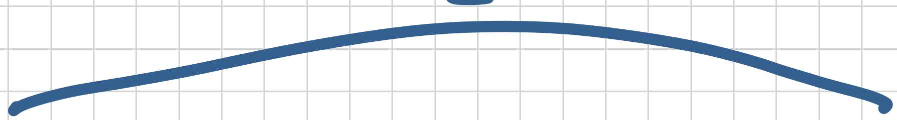


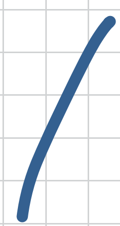


m.m.s,

(X, d, m)

$t = \frac{1}{2}$





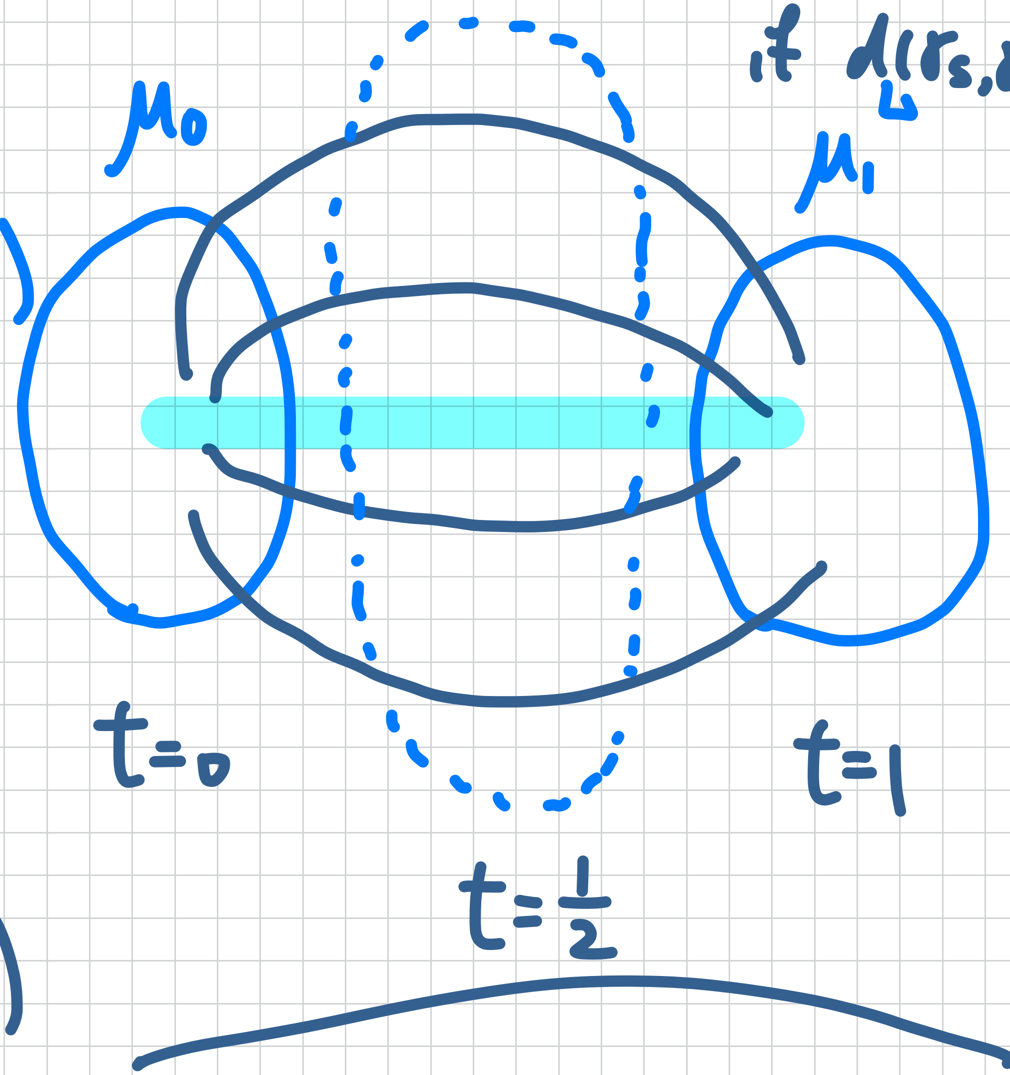
Convex

X

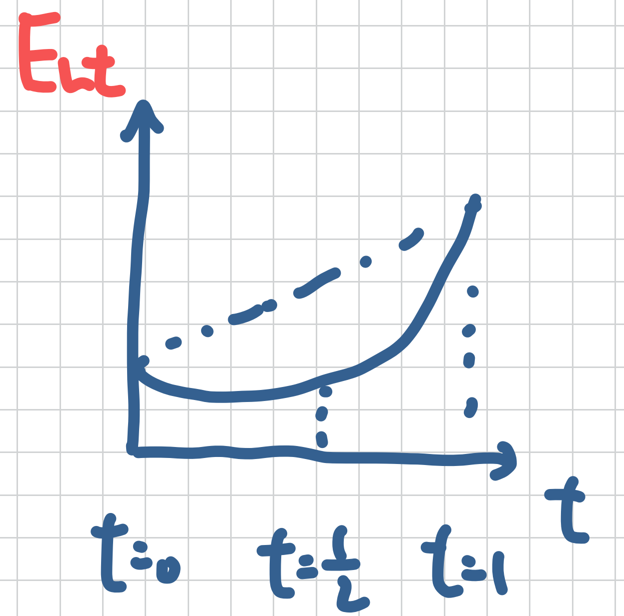
Motivation: Lazy Gas Experiment

$\gamma:[0,1]\to P_2(X)$ is a geodesic $P_2(X)$
if $d_W(\gamma_s,\gamma_t)=|s-t|\, d_W(\gamma_0,\gamma_1)$ $\forall s,t$

$(P_2(X), d_W)$



m.m.s.
(X,d,m)

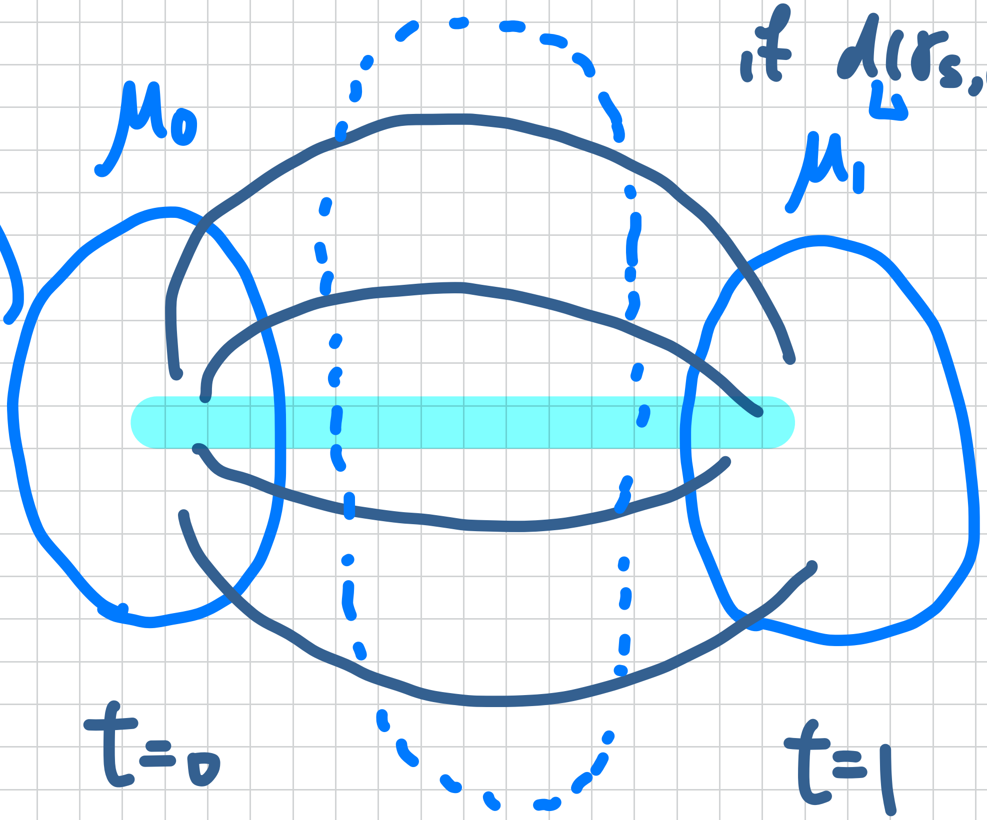


Convex

Motivation: Lazy Gas Experiment

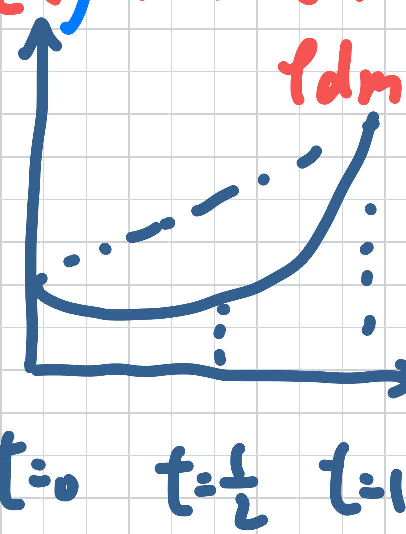
$\gamma:[0,1]\to P_2(X)$ is a geodesic $P_2(X)$ if $d_W(\gamma_s,\gamma_t)=|s-t|\,d_W(\gamma_0,\gamma_1)\ \forall s,t$

$(P_2(X), d_W)$



$\mathrm{Ent}(\mu|m)=\int \rho\log\rho\,dm$

$\rho\,dm=\mu$



m.m.s.

(X,d,m)

X

Convex

Motivation: Lazy Gas Experiment

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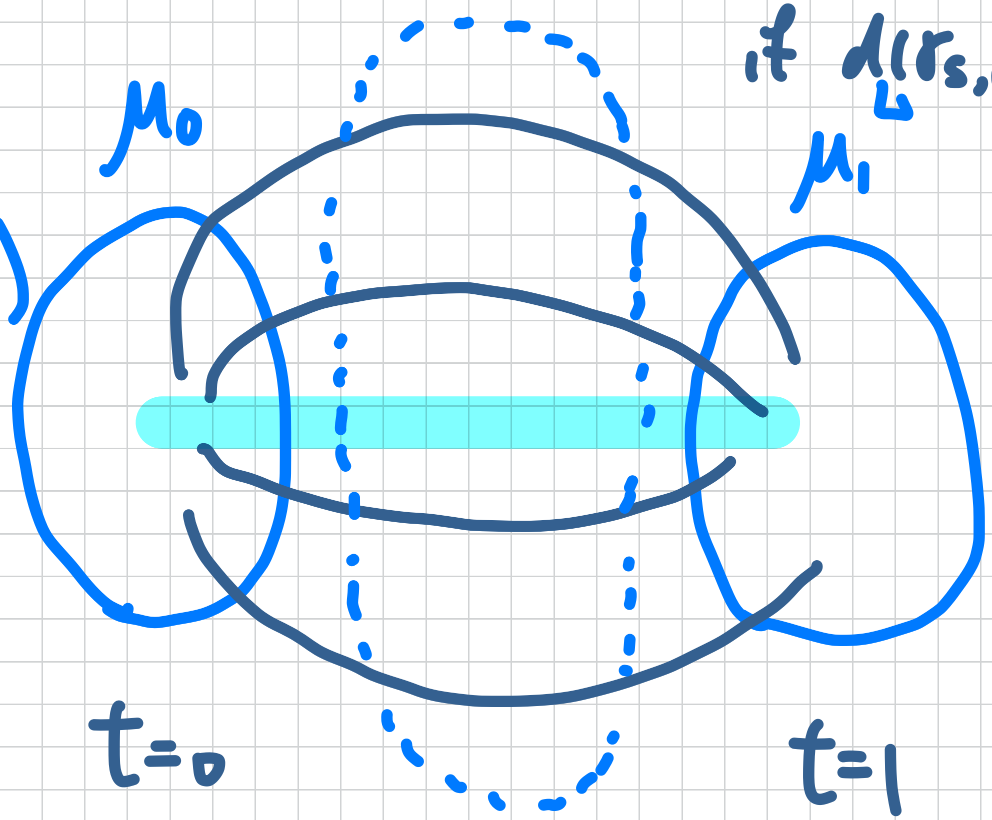
$(P_2(X), d_W)$



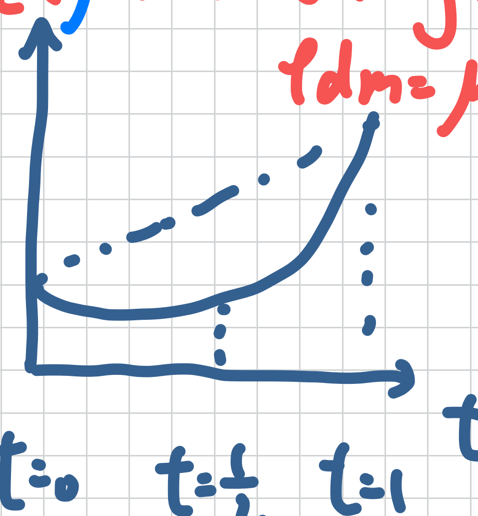
$t=\frac{1}{2}$

m.m.s,

(X,d,m)

$\mathrm{Ent}(\mu|m)=\int \rho\log\rho\,dm$

$\rho\,dm=\mu$



u is K-N convex if

X $\quad u''\ge K+\frac{1}{N}(u')^2$

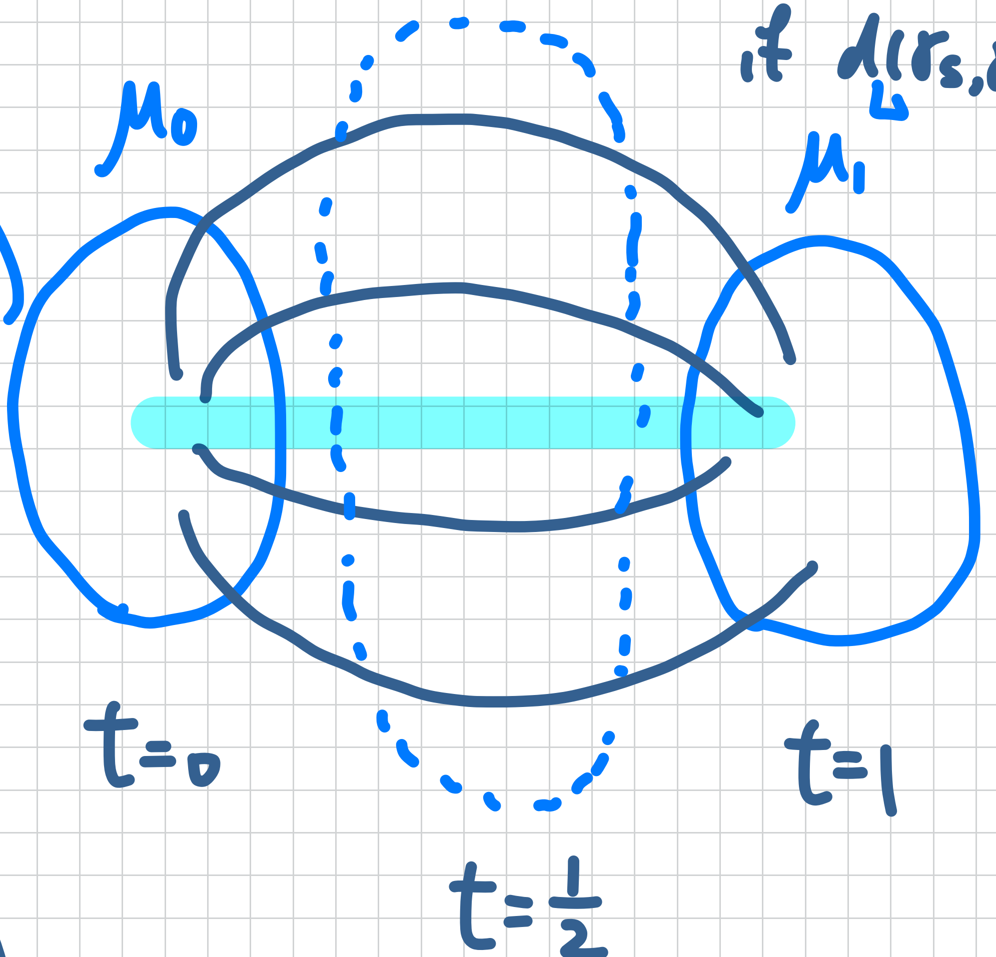
Similarly, we can define

$CD(K,N)$, $CD^*(K,N)$, $CD_1(K,N)$, ...

$\gamma : [0,1] \to P_2(X)$ is a geodesic $P_2(X)$

if $d_W(\gamma_s, \gamma_t) = |s-t| \, d_W(\gamma_0, \gamma_1)$ $\forall s,t$

$(P_2(X), d_W)$

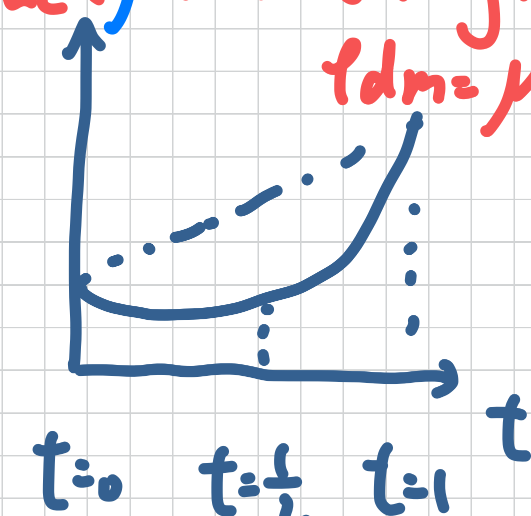


m.m.s,

(X, d, m)

$Ent(\mu | m) = \int \rho \log \rho \, dm$

$\rho \, dm = \mu$



u is K-N convex if

$u'' \geq K + \frac{1}{N}(u')^2$

Similarly, we can define

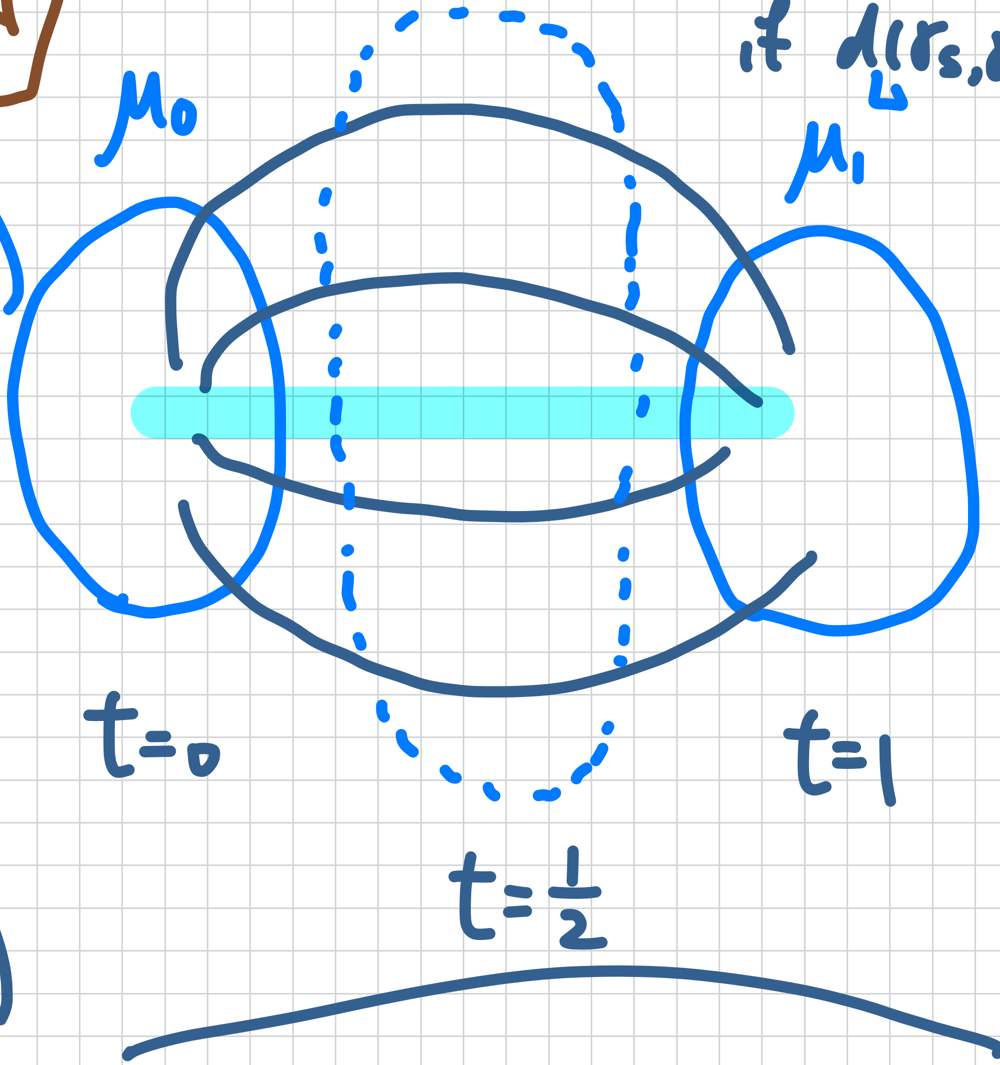
$$\boxed{CD(K,N) = CD^*(K,N) = CD_1(K,N) = \dots = CD^e(K,N)}$$

ENB

$\gamma : [0,1] \to P_2(X)$ is a geodesic $P_2(X)$

if $d_W(\gamma_s, \gamma_t) = |s-t|\, d_W(\gamma_0, \gamma_1)$ $\forall s,t$

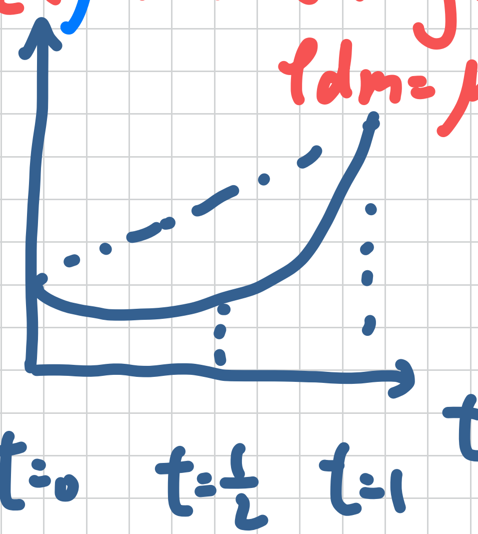
$(P_2(X), d_W)$



m.m.s. (X, d, m)

$Ent(\mu | m) = \int \rho \log \rho \, dm$

$\rho \, dm = \mu$



u is K-N convex if

$u'' \geq K + \frac{1}{N}(u')^2$

ENB

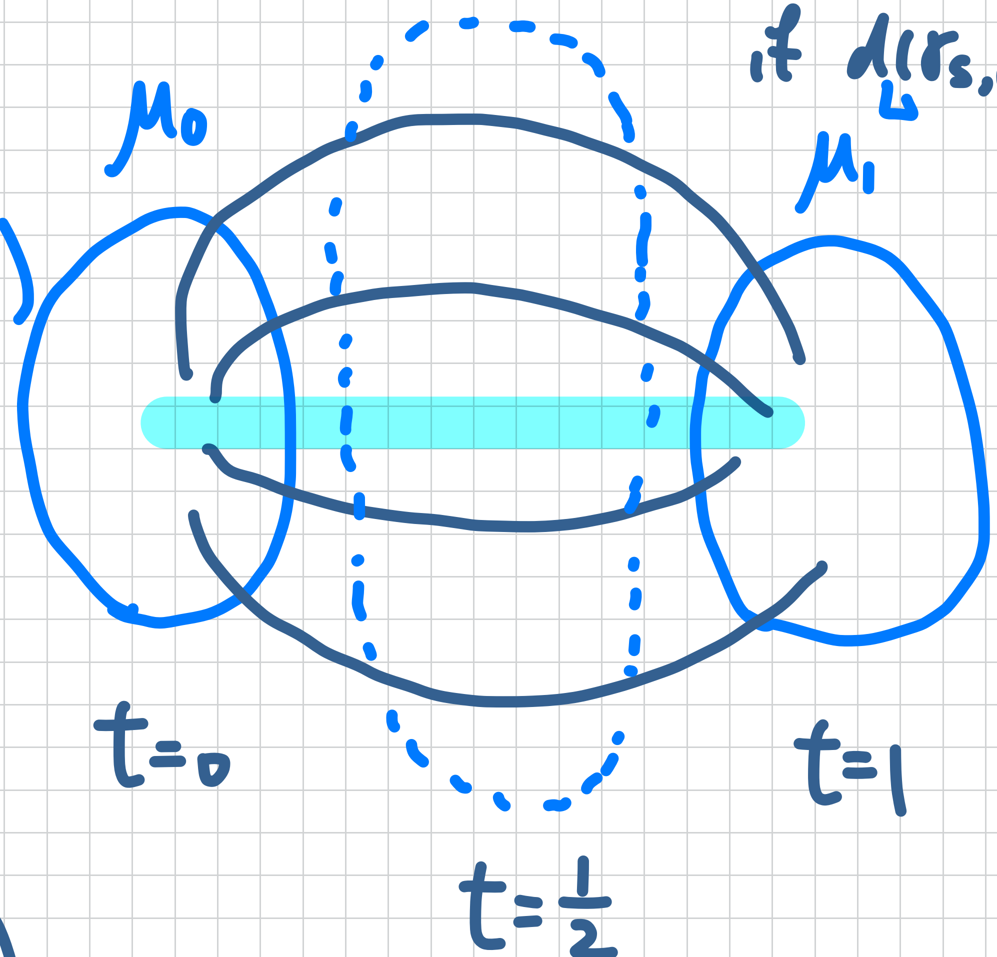
On smooth Riemannian mfd

$\mathrm{Ric} \geq 0$ ·

$(P_2(X), d_W)$

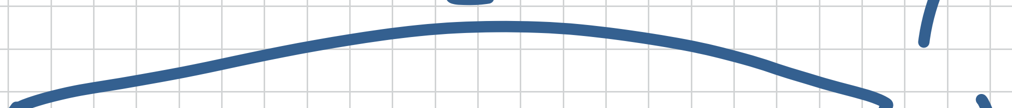
$\gamma : [0,1] \to P_2(X)$ is a geodesic $P_2(X)$

if $d_W(\gamma_s, \gamma_t) = |s-t| \, d_W(\gamma_0, \gamma_1)$ $\forall s,t$

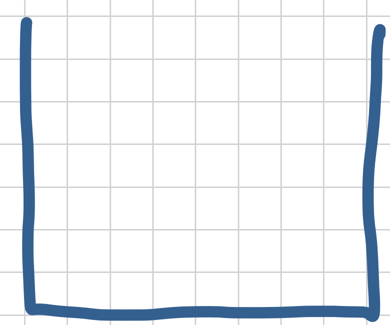


m.m.s.

(X, d, m)



X



$\mathrm{Geo}(X)$

ENB

On smooth Riemannian mfd

$Ric \geq 0.$

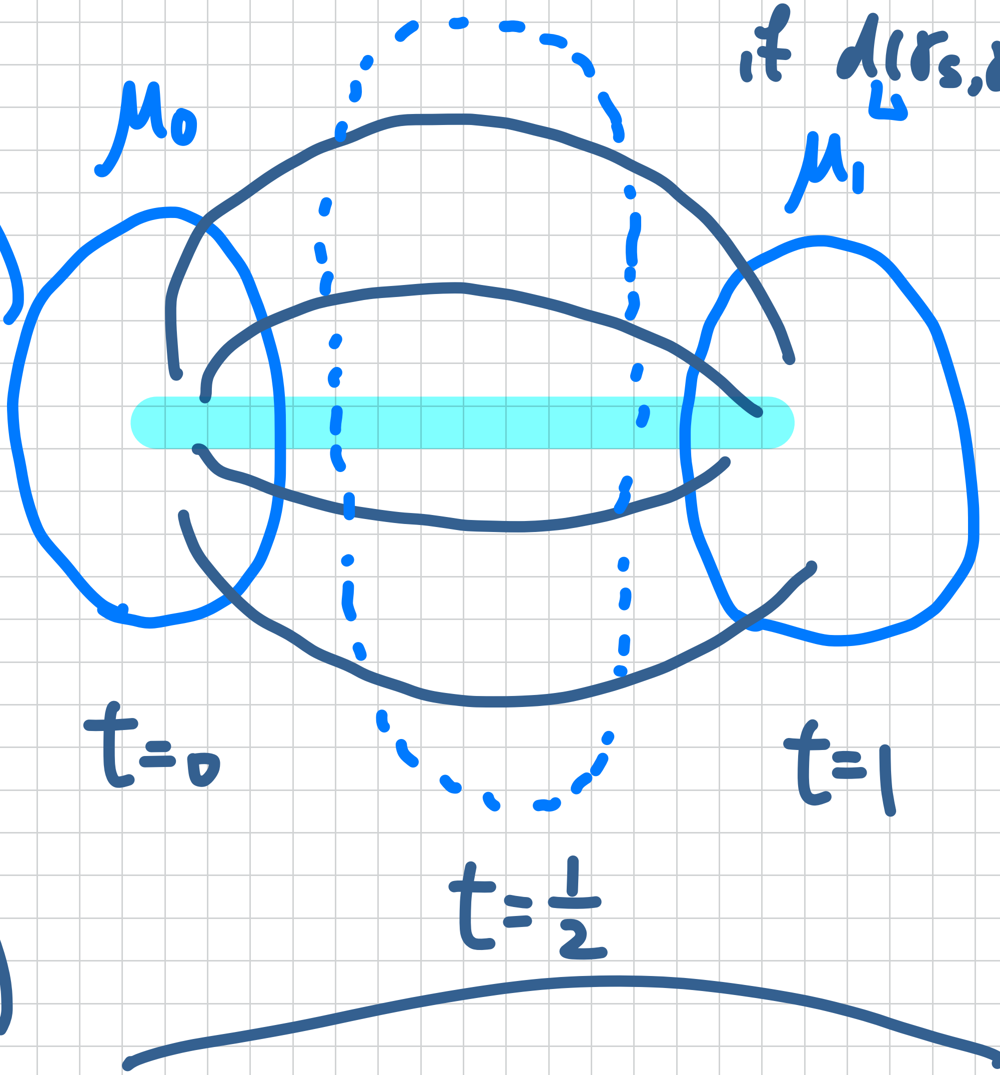
$(\mathcal{P}_2(X), d_W)$

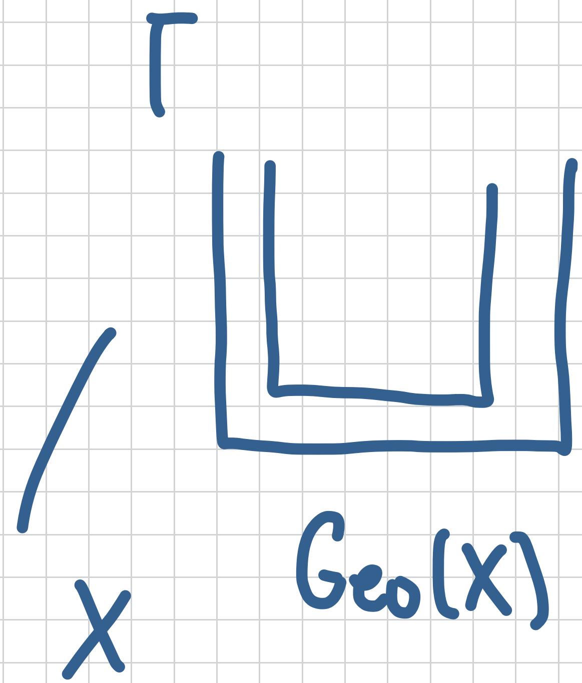
m.m.s.

(X, d, m)

$\gamma : [0,1] \to \mathcal{P}_2(X)$ is a geodesic $\mathcal{P}_2(X)$

if $d_W(\gamma_s, \gamma_t) = |s-t| \, d_W(\gamma_0, \gamma_1)$ $\forall s,t$





ENB

On smooth Riemannian mfd

$Ric \geq 0$ ·

$(\mathcal{P}_2(X), d_W)$

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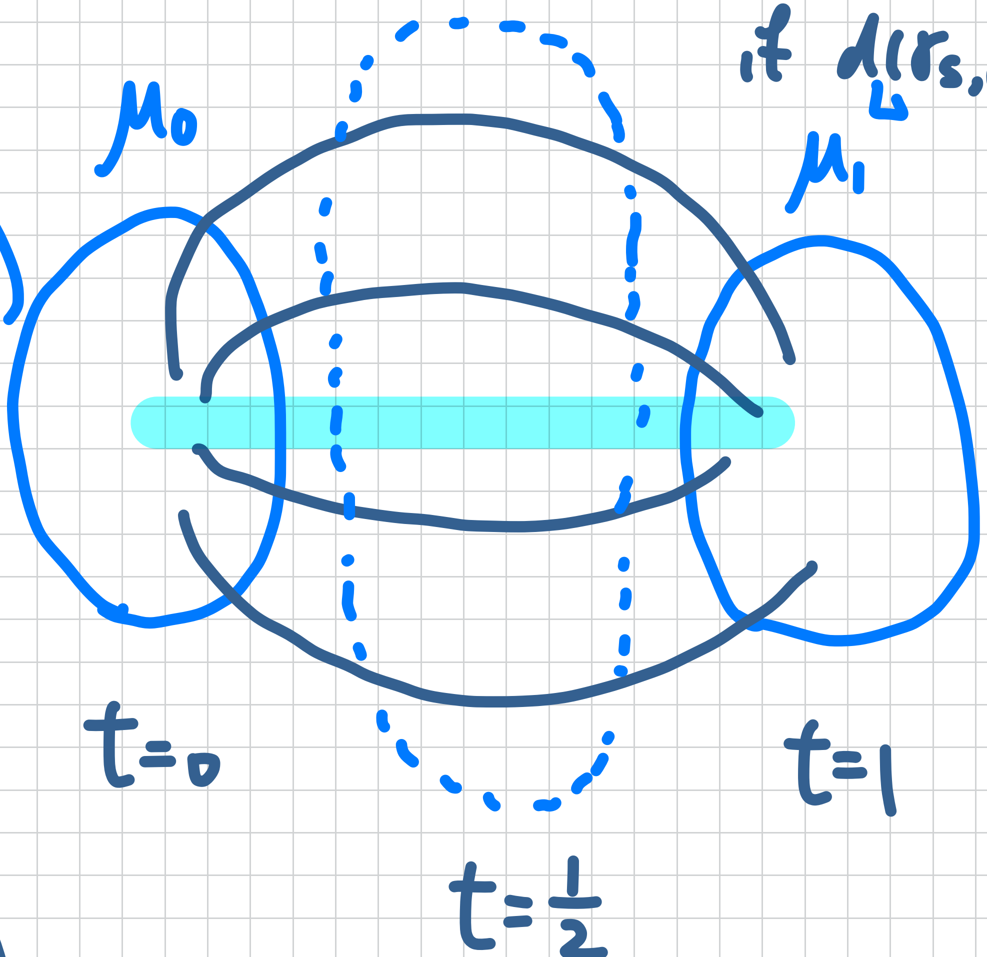
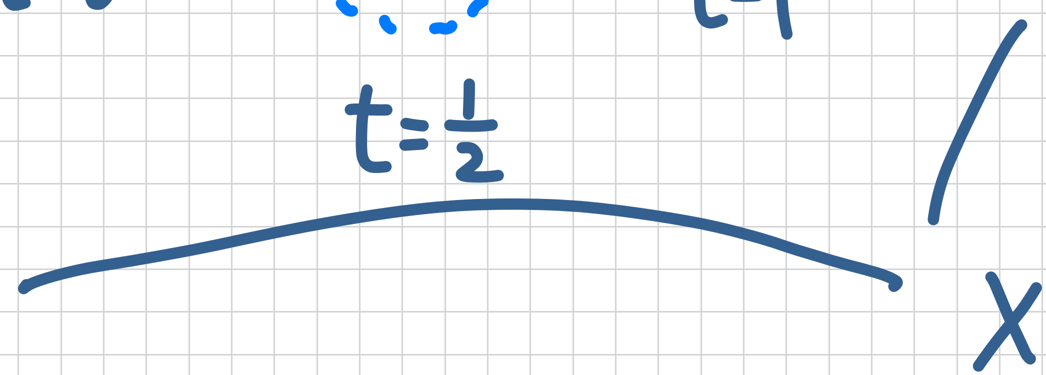
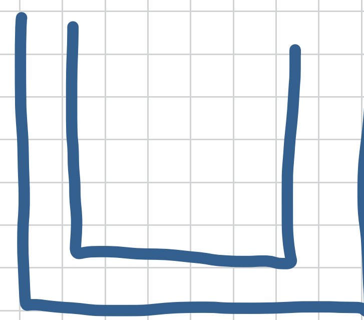


m.m.s, (X, d, m)



Γ is non branching



Geo(X)

ENB

On smooth Riemannian mfd

Ric ≥ 0 ·

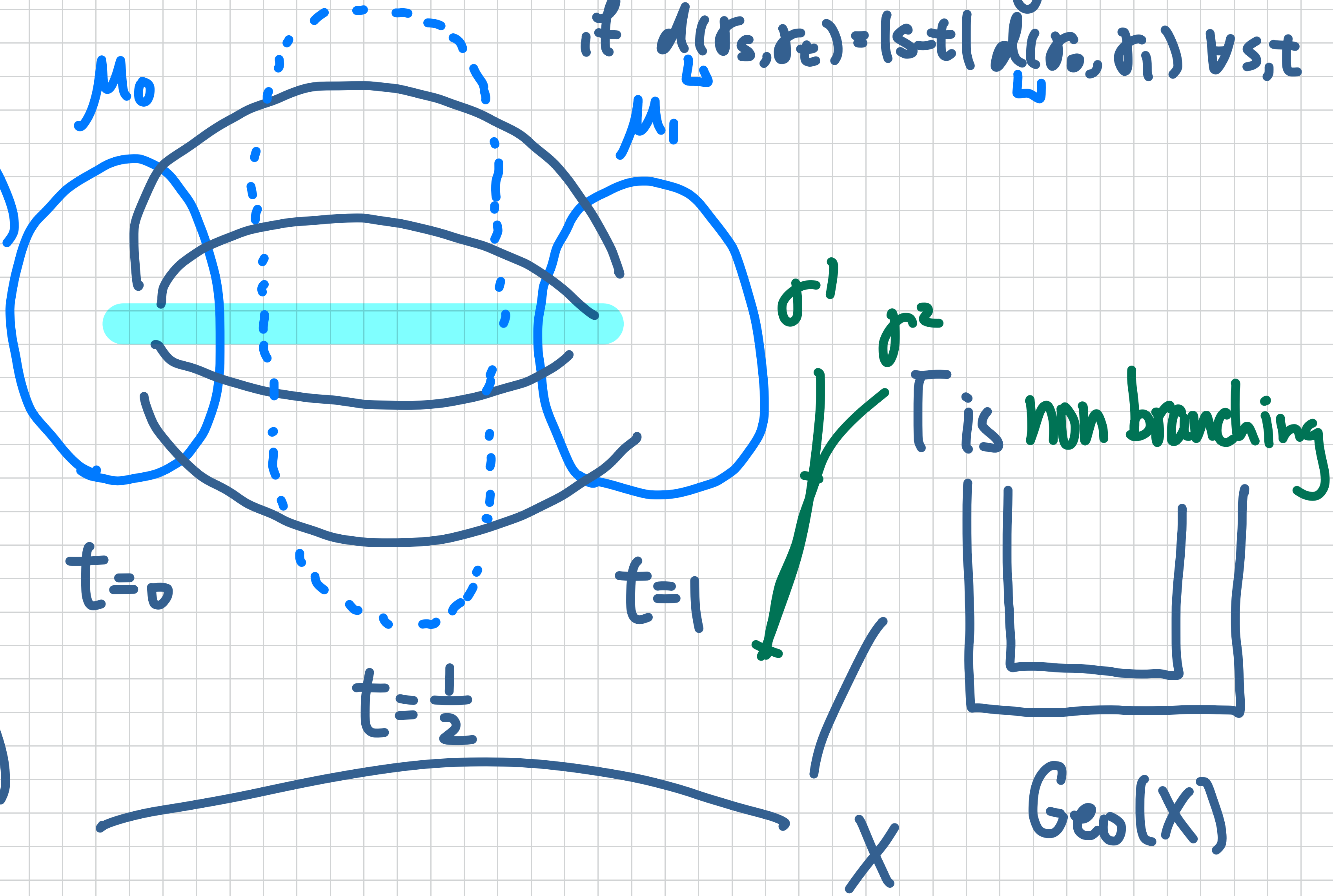
$(\mathcal{P}_2(X), d_W)$

m.m.s,

(X, d, m)

$\gamma : [0,1] \to \mathcal{P}_2(X)$ is a geodesic $\mathcal{P}_2(X)$

if $d_W(\gamma_s, \gamma_t) = |s-t| \, d_W(\gamma_0, \gamma_1) \ \forall s,t$



ENB

On smooth Riemannian mfd

Ric ≥ 0 ·

$(\mathcal{P}_2(X), d_W)$

m.m.s,

(X, d, m)

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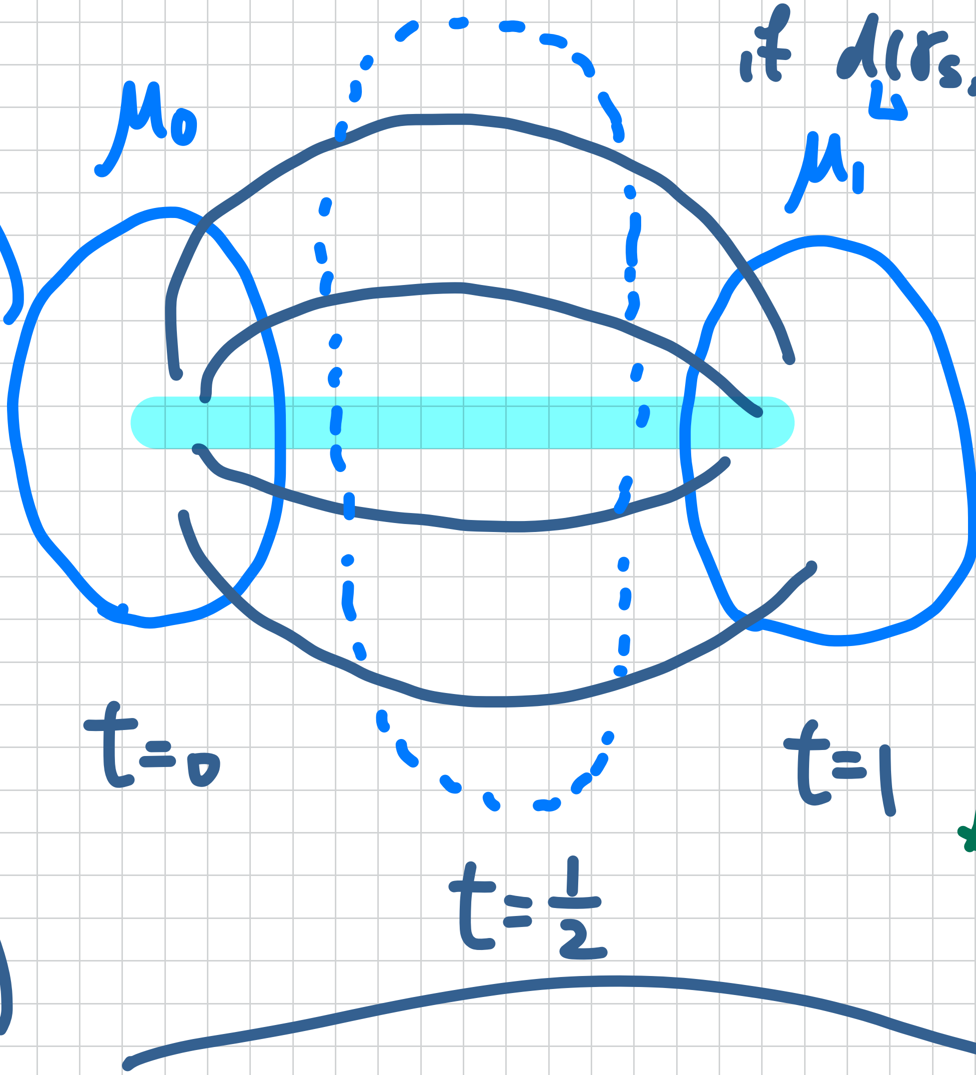
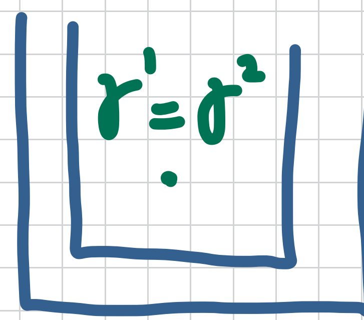


Γ is non branching



Geo(X)

ENB

On smooth Riemannian mfd $\mathrm{Ric} \geq 0$.

$(\mathcal{P}_2(X), d_W)$

m.m.s, (X, d, m)

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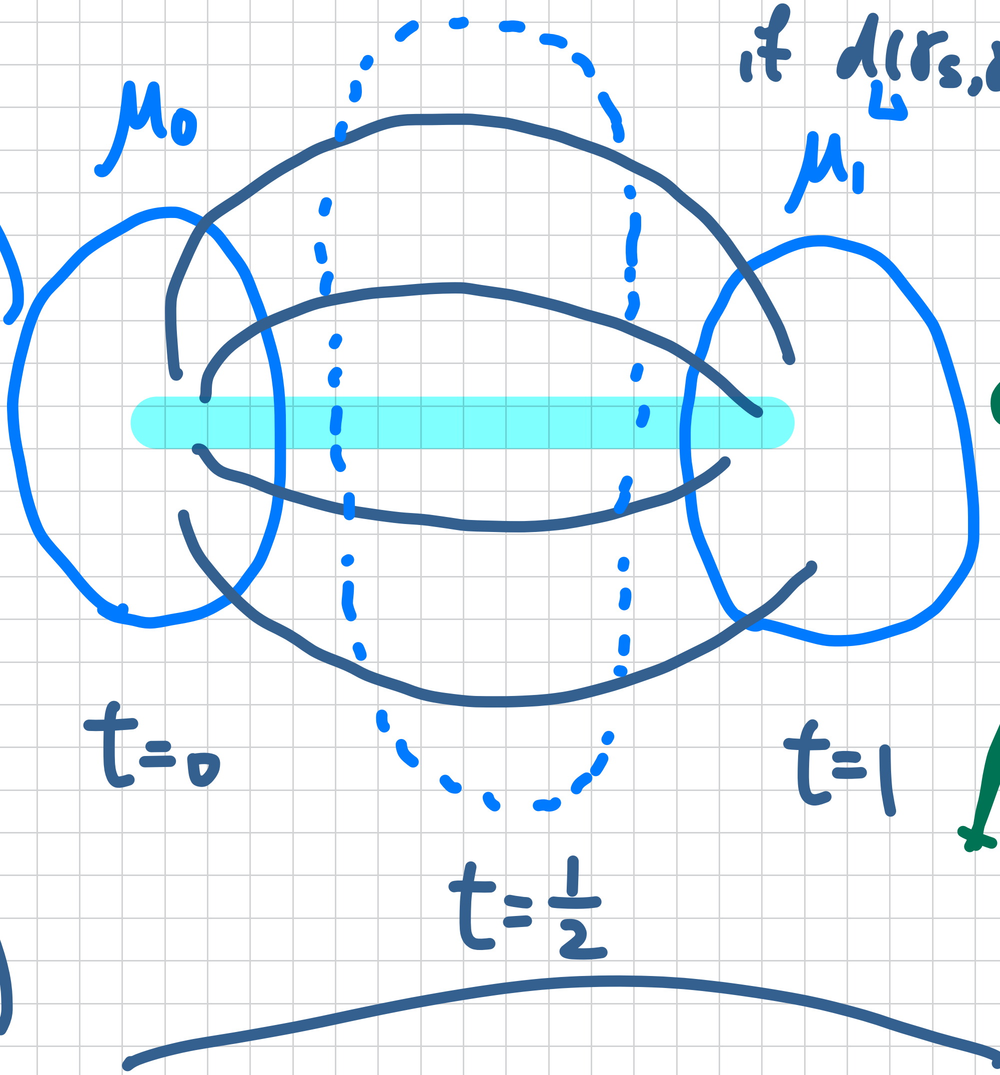

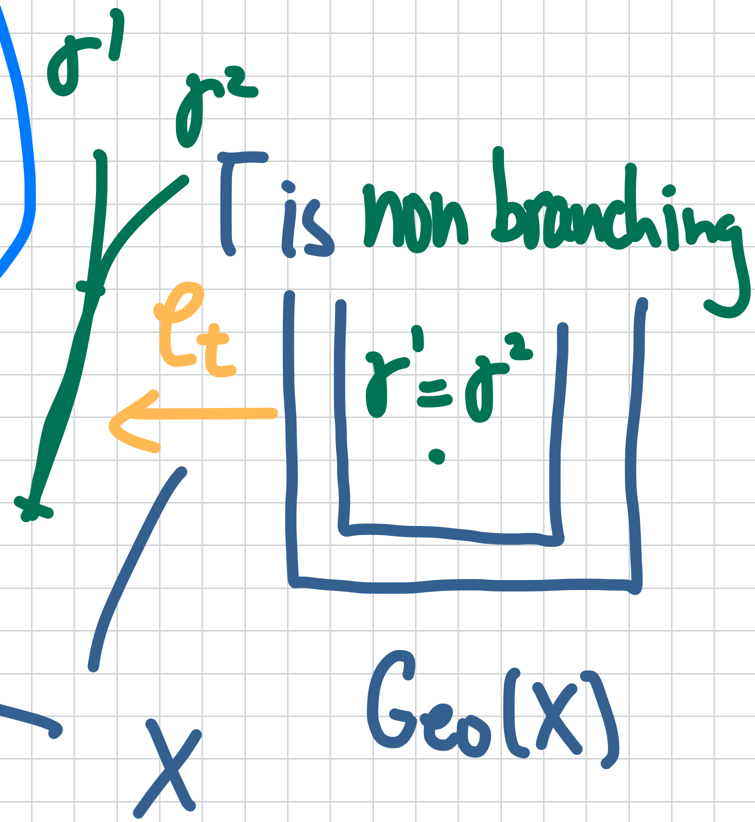

ENB

On smooth Riemannian mfd

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$(P_2(X), d_W)$

m.m.s.

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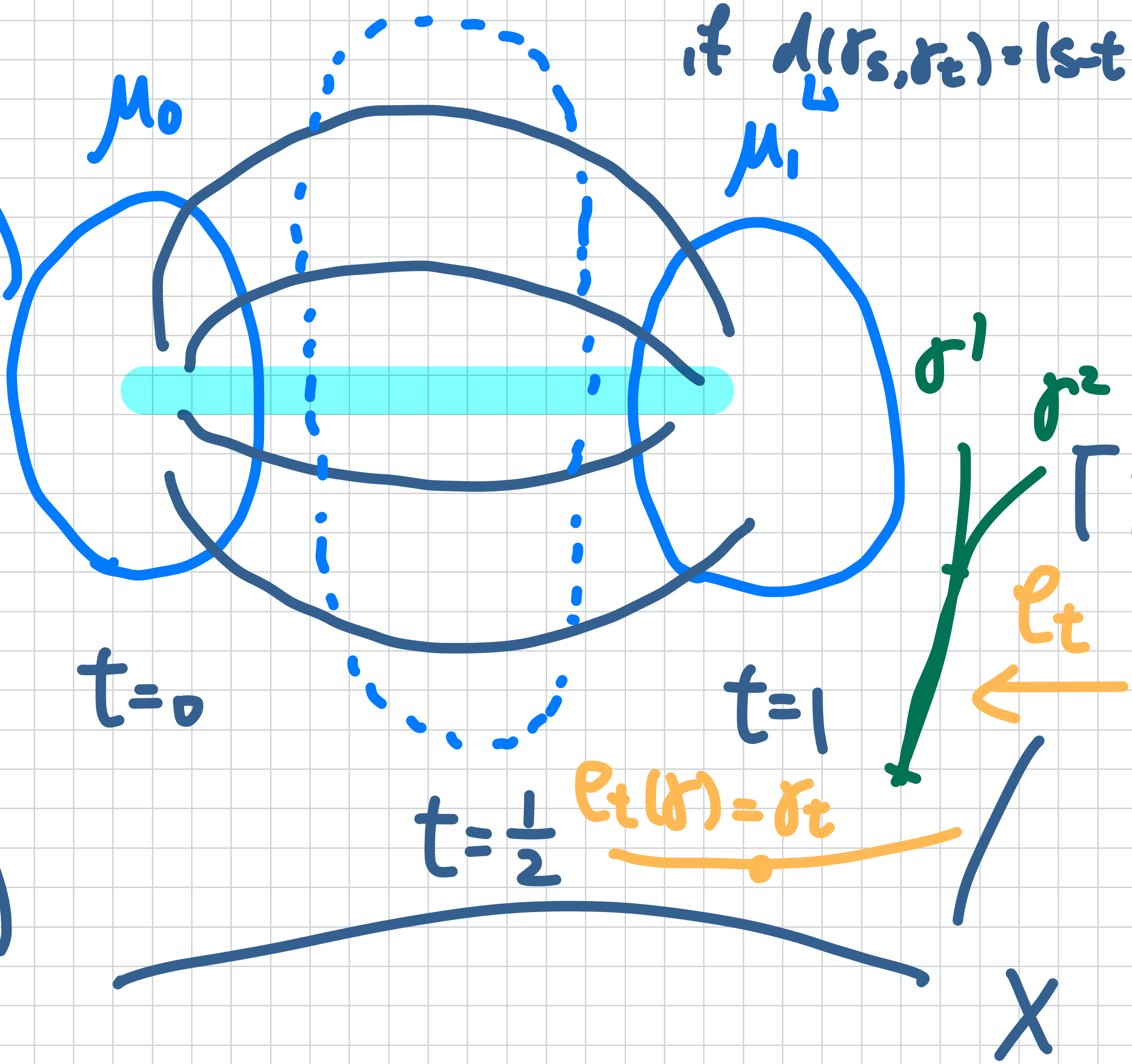

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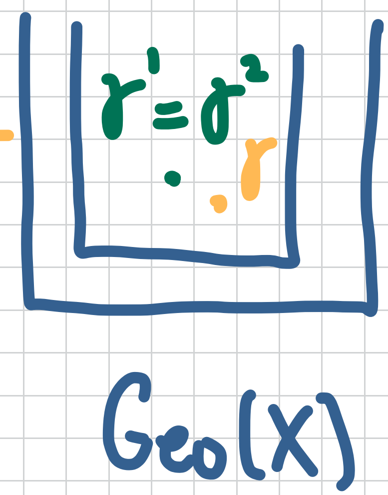

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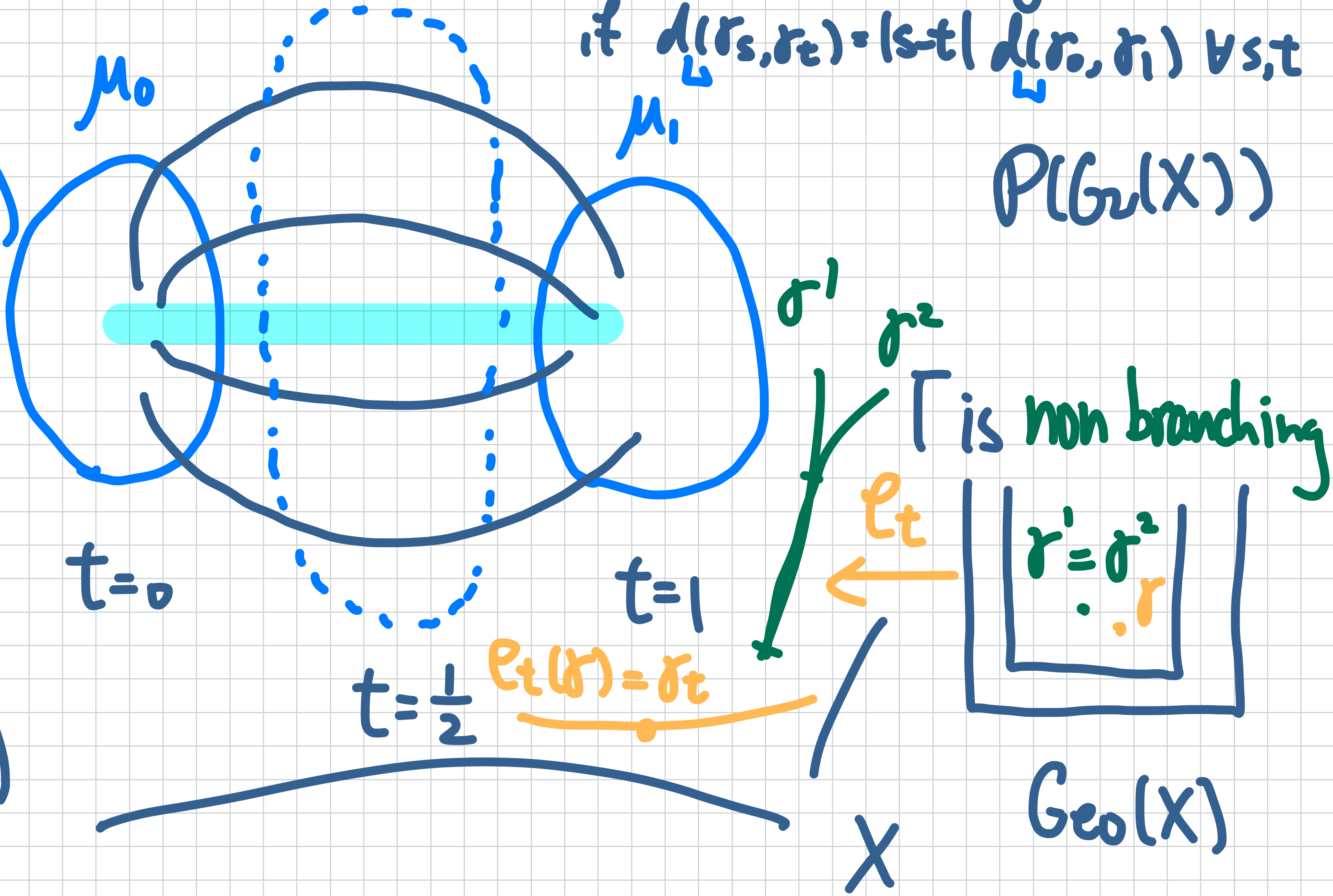

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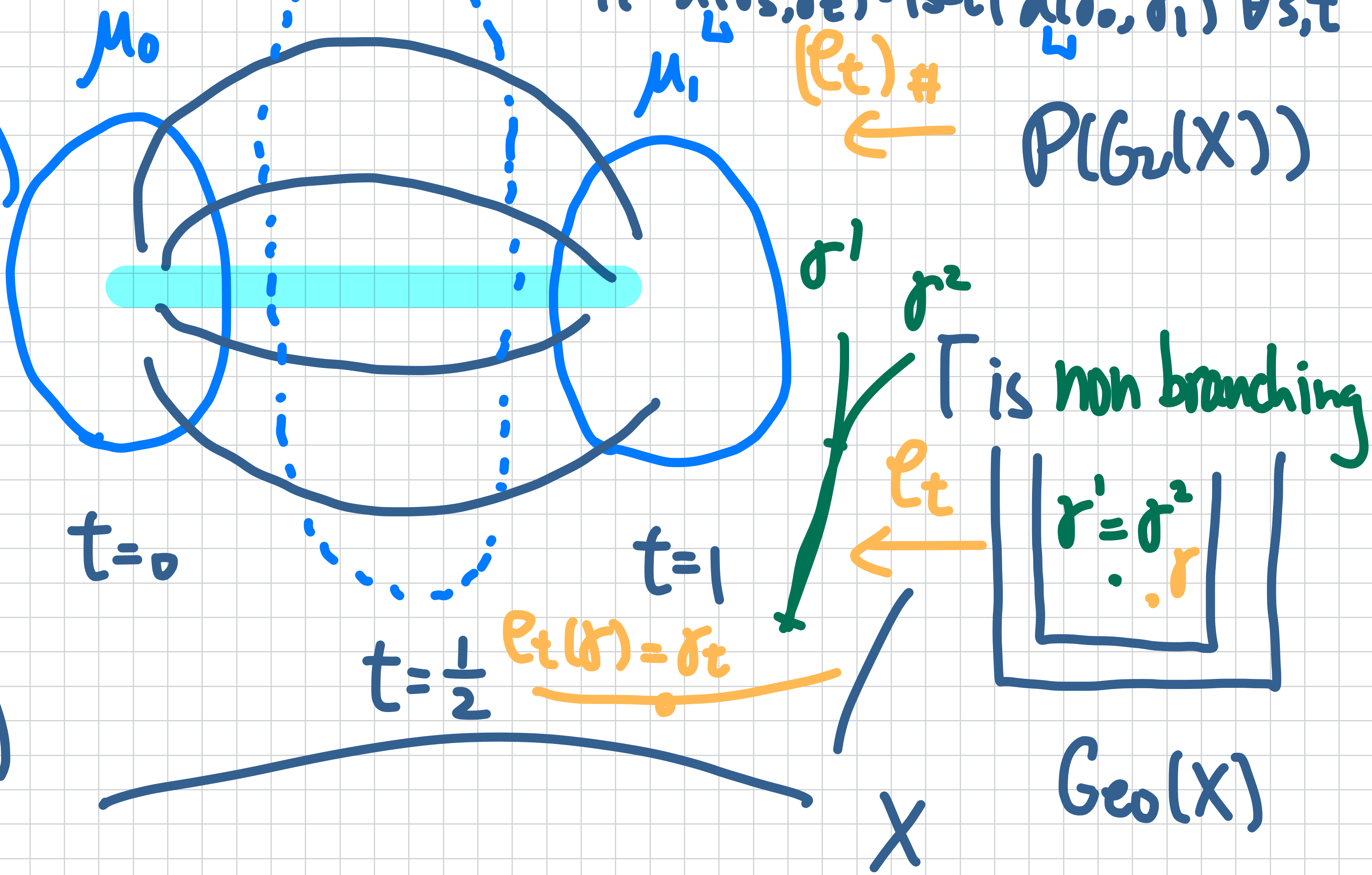
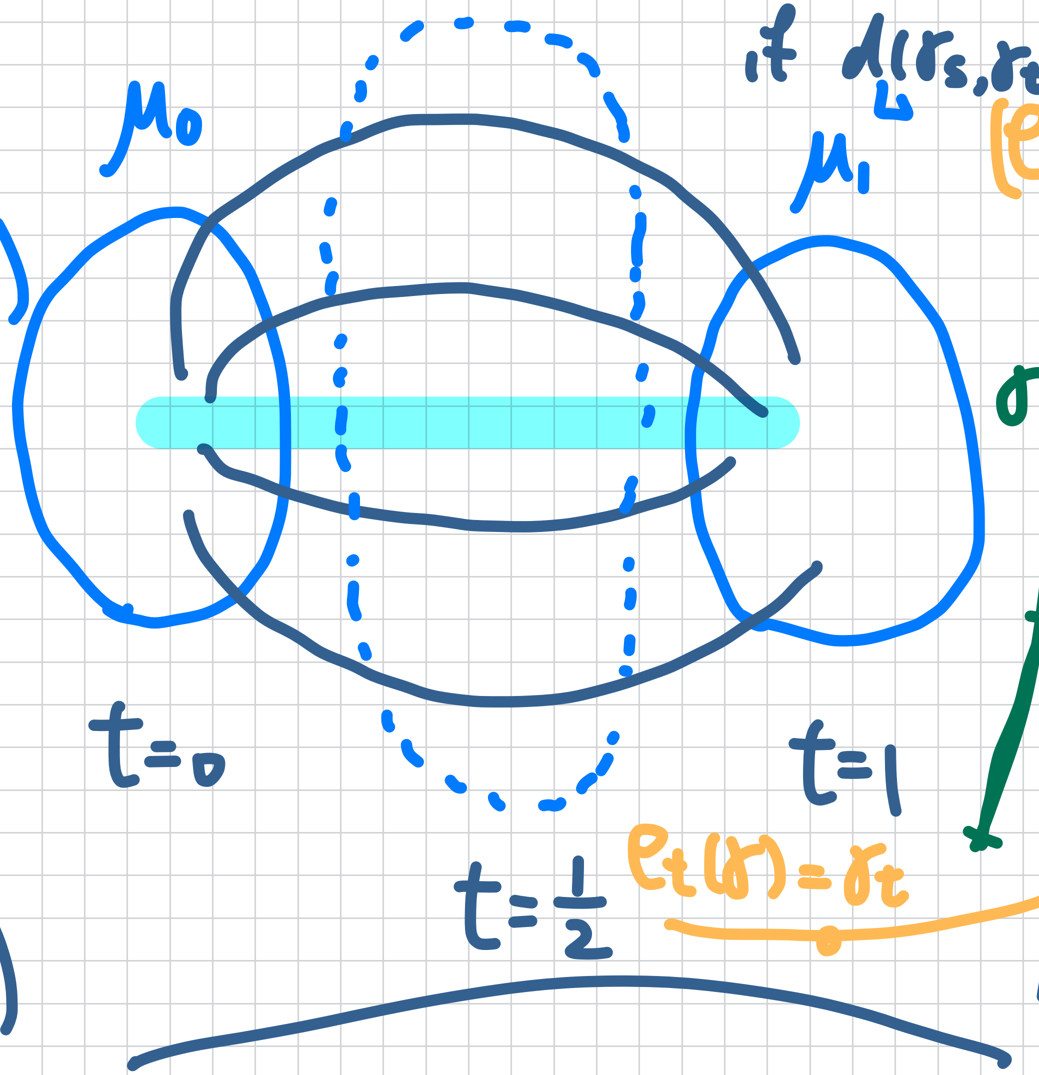


ENB

On smooth Riemannian mfd

Ric ≥ 0.

$(\mathcal{P}_2(X), d_W)$



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$\mathcal{P}(\mathrm{Geo}(X)) \xrightarrow{(e_t)_\#}$

$\gamma^1, \gamma^2 \in \mathrm{OptGeo}(\mu_0, \mu_1)$

Γ is non branching

$\xleftarrow{e_t}$

$\gamma^1 = \gamma^2$

γ

$\mathrm{Geo}(X)$

ENB

On smooth Riemannian mfd

Ric ≥ 0.

$(P_2(X), d_W)$

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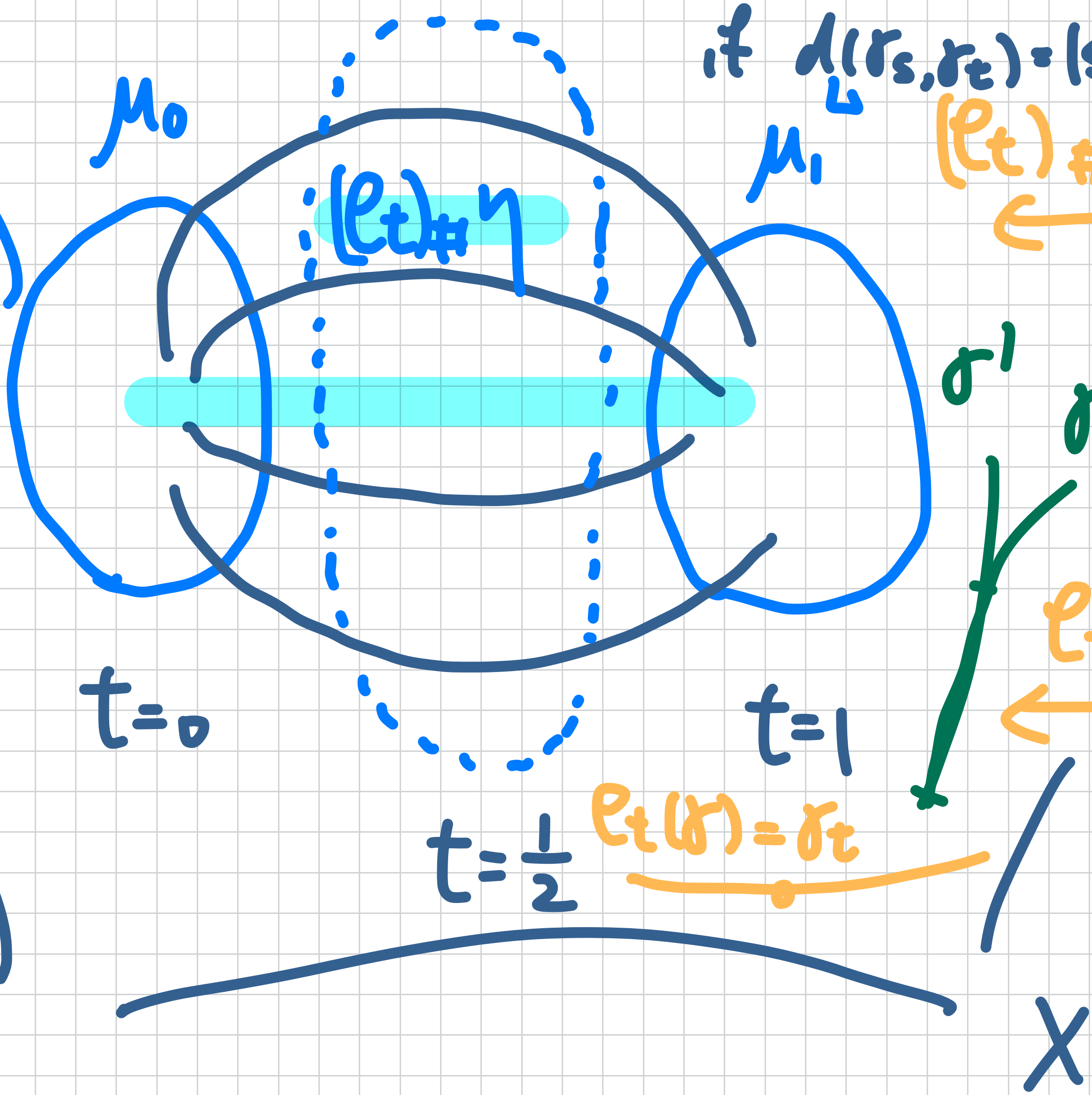

$(e_t)_\#$ $\leftarrow$ $P(\mathrm{Geo}(X))$

γ^1 γ^2 $\eta \in \mathrm{OptGeo}(\mu_0, \mu_1)$

Γ is non branching

e_t $\leftarrow$

$\gamma^1 = \gamma^2$ γ

Geo(X)

(X, d, m) is ENB

On smooth Riemannian mfd

$\mathrm{Ric} \geq 0$.

$(\mathcal{P}_2(X), d_W)$

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(X, d, m)

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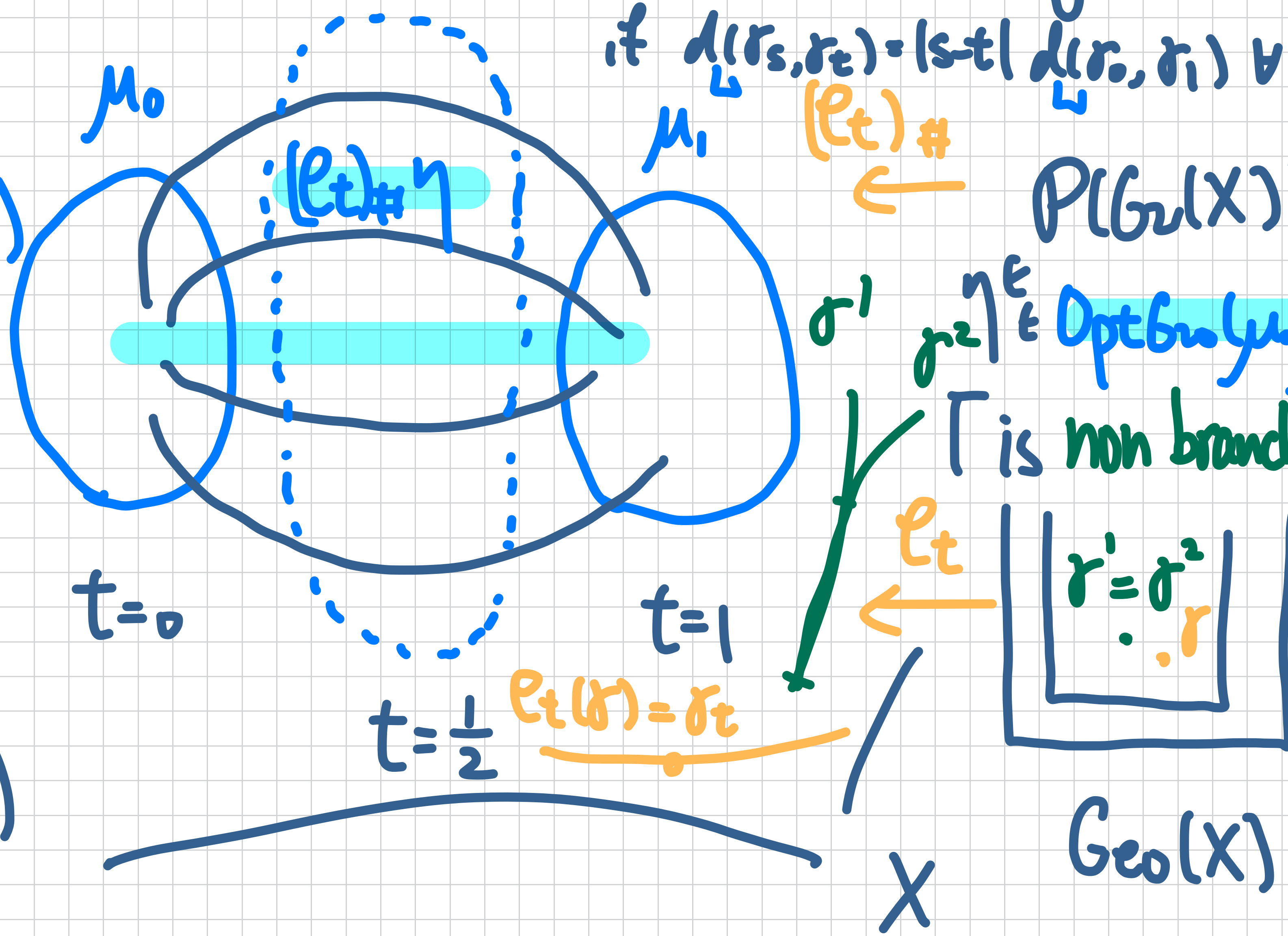

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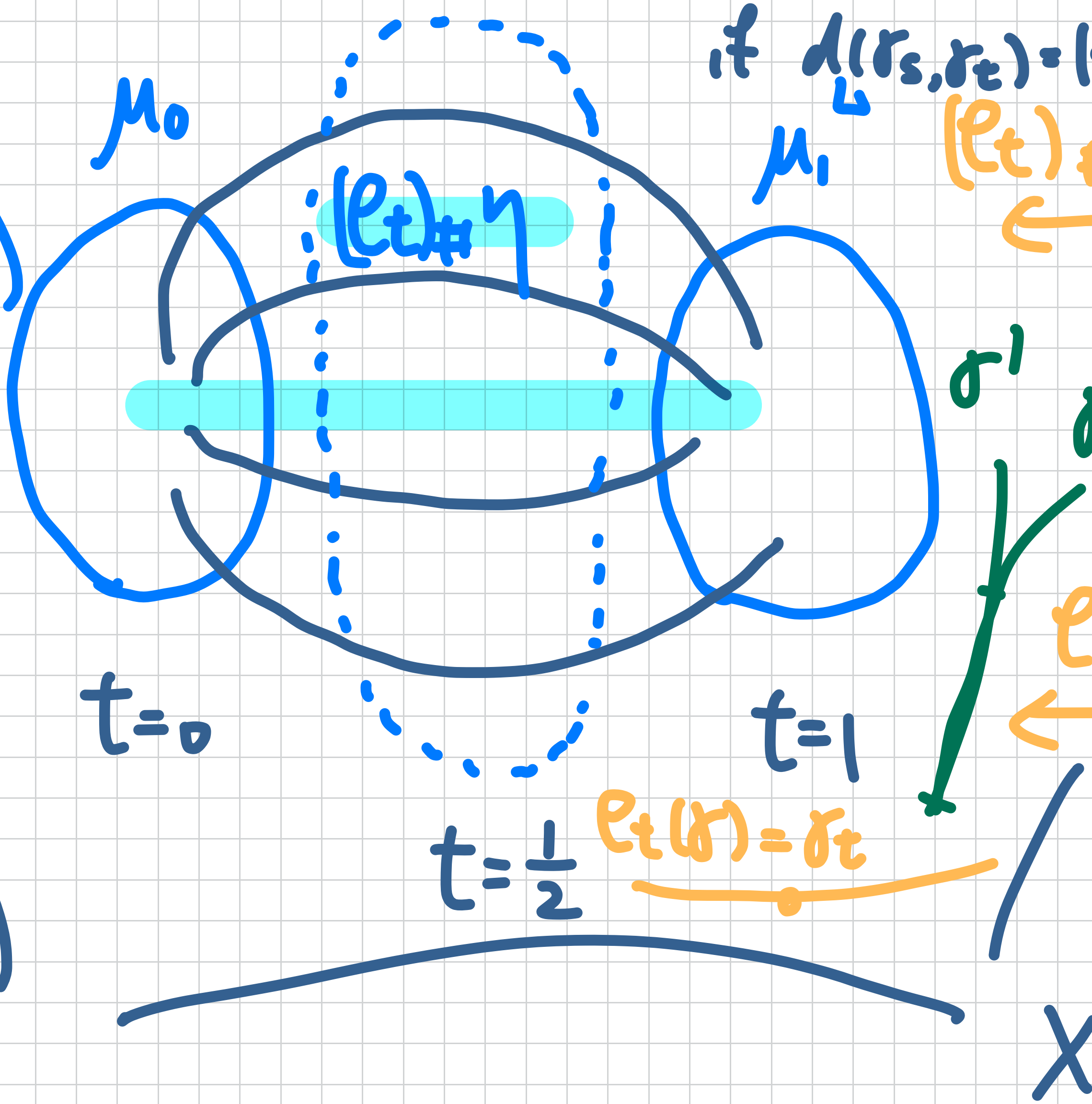
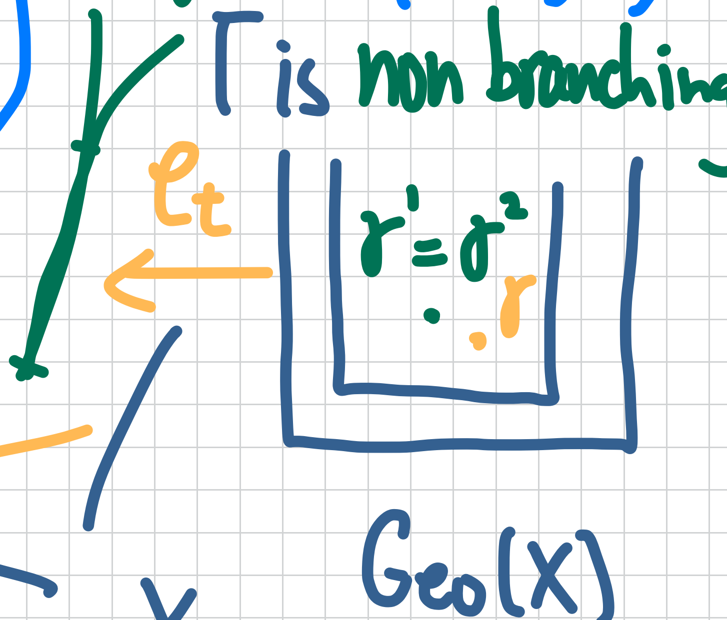
$(e_t)_\# \leftarrow \mathcal{P}(G_{eo}(X))$

$\forall$

$\gamma^1, \gamma^2 \; \eta \in \mathrm{OptGeo}(\mu_0, \mu_1)$

Γ is non branching





(X, d, m) is ENB

On smooth Riemannian mfd

$Ric \geq 0$.

$(P_2(X), d_W)$

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$(e_t)_\#$ $\longleftarrow$ $P(Geo(X))$

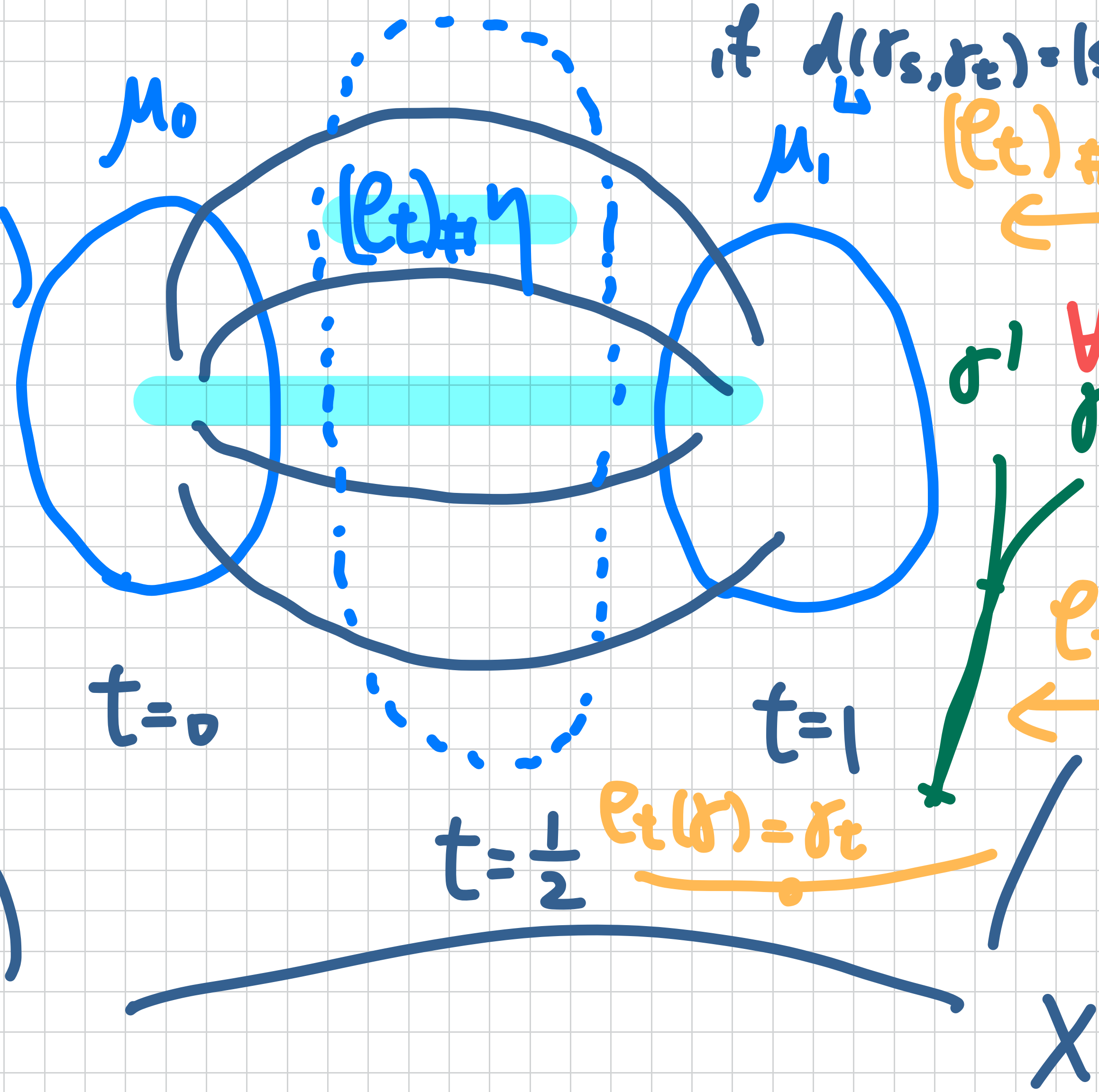
γ^1 $\forall \gamma^2$ $\forall \eta \in OptGeo(\mu_0, \mu_1)$

Γ is non branching

e_t $\longleftarrow$ $Geo(X)$

$\gamma^1 = \gamma^2$

γ



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On smooth Riemannian mfd

$\mathrm{Ric} \geq 0$.

$(\mathcal{P}_2(X), d_W)$

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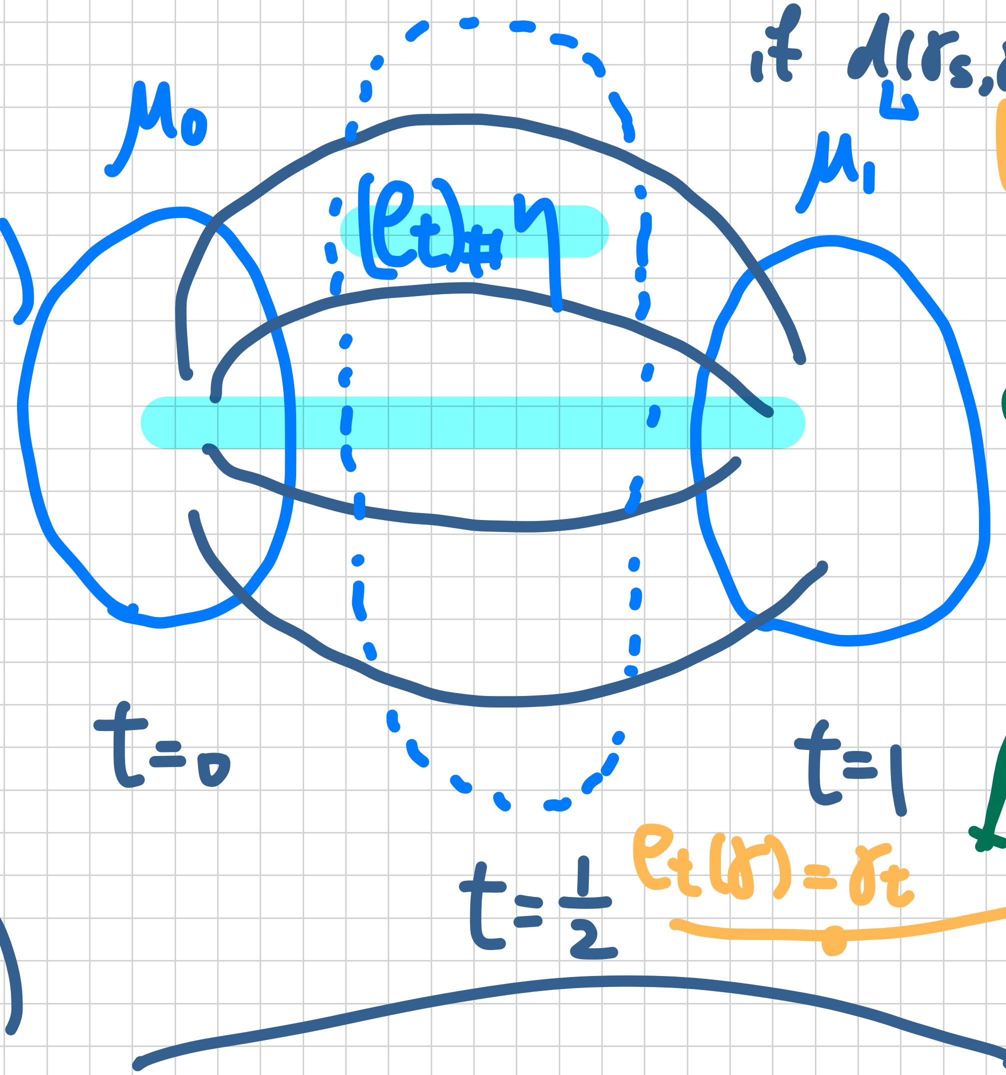

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$(e_t)_\#$ $\leftarrow$ $\mathcal{P}(\mathrm{Geo}(X))$

γ^1 $\forall$ $\cancel{\gamma^2}$ $\eta \in \mathrm{OptGeo}(\mu_0, \mu_1)$ $\forall$

$\exists \Gamma$ is non branching

e_t $\leftarrow$ $\gamma^1 = \gamma^2$, γ

X

$\mathrm{Geo}(X)$

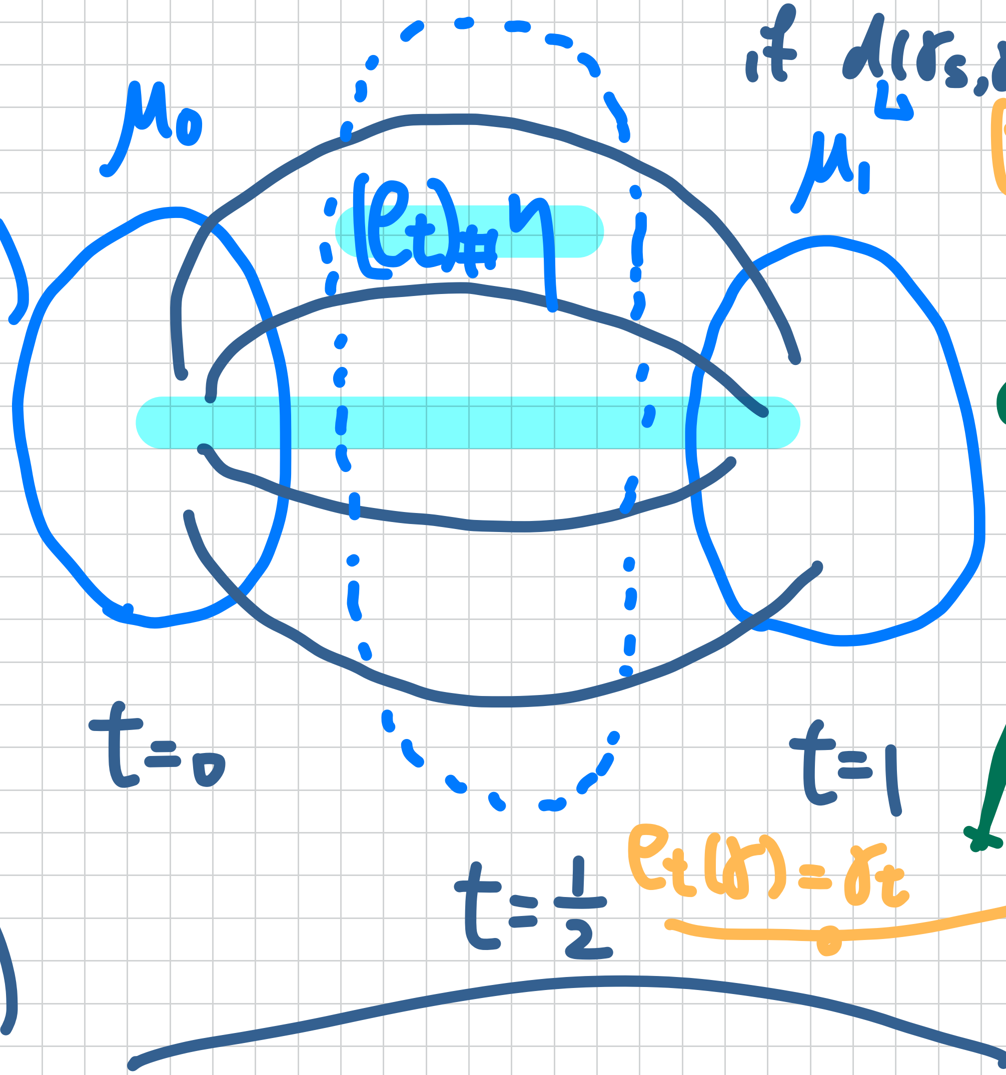
(X, d, m) is ENB

On smooth Riemannian mfd

$\mathrm{Ric} \geq 0$.

$(\mathcal{P}_2(X), d_W)$

m.m.s, (X, d, m)



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$(e_t)_\#$

$\mathcal{P}(\mathrm{Geo}(X))$

$\forall \eta \in \mathrm{OptGeo}(\mu_0, \mu_1)$ γ^1 γ^2

$\exists \Gamma$ $\eta(\Gamma) = 1$

Γ is non branching

e_t

$\gamma^1 = \gamma^2$

γ

$e_t(\gamma) = \gamma_t$

X

$\mathrm{Geo}(X)$

$$(M, \underset{\in C^{1,1}}{g_{ij}}, m) \text{ is ENB}$$

On smooth Riemannian mfd

Ric ≥ 0 .

$(P_2(X), d_W)$

m.m.s,

(X, d, m)

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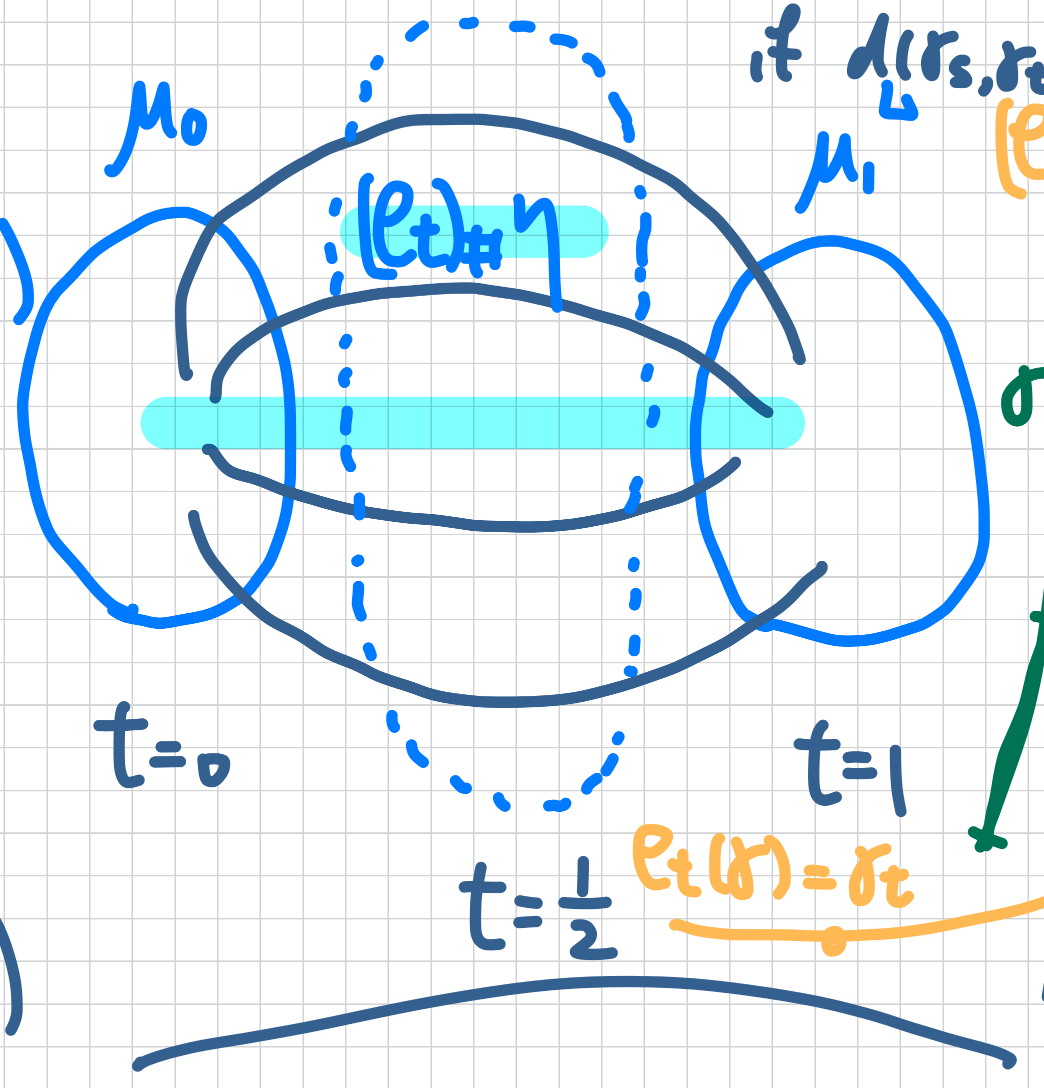
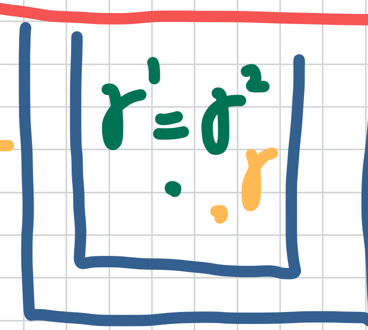


$(e_t)_\#$ $\leftarrow$ $P(Geo(X))$

$\forall \eta \in OptGeo(\mu_0, \mu_1)$ $\exists \Gamma$, $\eta(\Gamma) = 1$, Γ is non branching

γ^1 γ^2

e_t $\leftarrow$



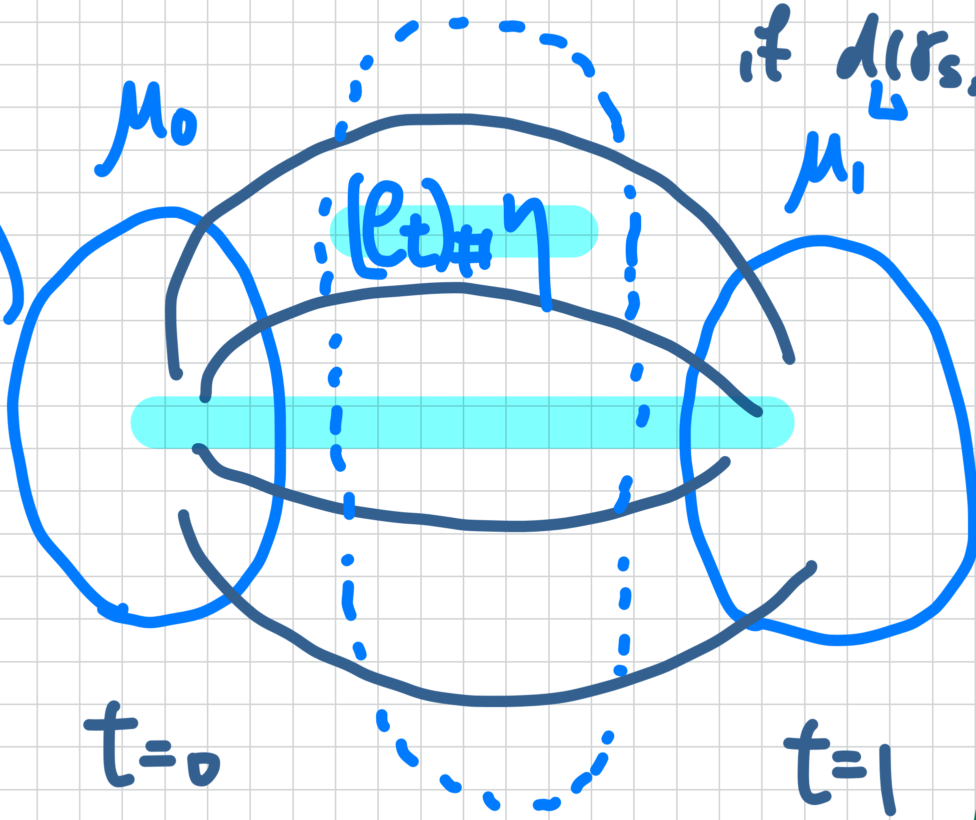
Geo(X)

(M, g_{ij}, m) $\in \cancel{C^{1,1}}$ $C^{0,1}$ is ENB

On smooth Riemannian mfd

$Ric \geq 0.$

$(\mathcal{P}_2(X), d_W)$



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$(e_t)_\#$

$\longleftarrow$ $P(Geo(X))$

γ^1 η γ^2 $\forall \eta \in$ $\forall$ $OptGeo(\mu_0, \mu_1)$

$\exists \Gamma$, $\eta(\Gamma) = 1$, Γ is non branching

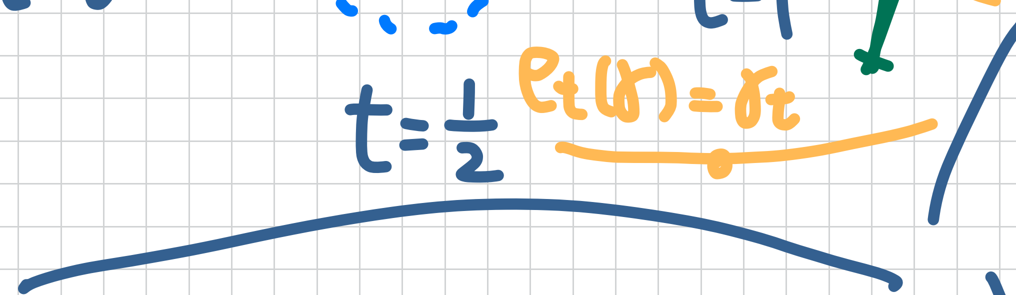
e_t $\longleftarrow$

$\gamma^1 = \gamma^2$

γ

Geo(X)

$e_t(\gamma) = \gamma_t$



X

Dimension Reduction of Distributionally Robust Optimization Problems

Optimal Transport Summer School 2025

Brandon Tam[1]
Silvana Pesenti[1]

[1]Department of Statistical Sciences
University of Toronto

July 2025

Problem Introduction

- We study an application of optimal transport to distributionally robust optimization (DRO) problems.

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- Our problem takes the following form:

$$\sup_{\boldsymbol{X} \in \mathcal{U}} \rho(g(\boldsymbol{X})).$$

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Problem Introduction

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- Our problem takes the following form:

$$\sup_{\boldsymbol{X} \in \mathcal{U}} \rho(g(\boldsymbol{X})).$$

- Consider the worst-case over the set of plausible distributions for the random vector $\boldsymbol{X}$.
- This optimization addresses the issue that the exact distribution of $\boldsymbol{X}$ is often unknown.
- For this talk, I will focus on uncertainty sets $\mathcal{U}$ defined by the Wasserstein distance.

Notation

- We use the following notation:

$$W^{n}(F,G) := \inf_{F_{\boldsymbol{X}}=F, F_{\boldsymbol{Y}}=G} (\mathbb{E}[\|\boldsymbol{X} - \boldsymbol{Y}\|_a^p])^{\frac{1}{p}}.$$

- As the dimension plays an important role in our problem, we will denote it with a superscript n for $n > 1$.
- The dependence on a and p are not important to our problem, so we omit them in the notation whenever there is no confusion.

Wasserstein Uncertainty Sets for Random Variables (rvs)

For $\varepsilon \geq 0$, we define the following uncertainty sets:

1. The univariate Wasserstein uncertainty set (for rvs) around the rv $X \in \mathcal{L}^p$ is given by

$$\mathcal{U}_\varepsilon(X) := \{Z \in \mathcal{L}^p \mid W(F_Z, F_X) \leq \varepsilon\}.$$

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2. The multivariate Wasserstein uncertainty set (for random vectors) around the random vector $\boldsymbol{X} \in \mathcal{L}^p_n$ is given by

$$\mathcal{U}^n_\varepsilon(\boldsymbol{X}) := \{\boldsymbol{Z} \in \mathcal{L}^p_n \mid W^n(F_{\boldsymbol{Z}}, F_{\boldsymbol{X}}) \leq \varepsilon\}.$$

Two Approaches for Introducing Uncertainty in $g(\boldsymbol{X})$

1. Uncertainty in the Risk Factors:

$$g\big(\mathcal{U}_\varepsilon^n(\boldsymbol{X})\big) := \{g(\boldsymbol{Z}) \mid \boldsymbol{Z} \in \mathcal{U}_\varepsilon^n(\boldsymbol{X})\}$$

2. Uncertainty in the Aggregate:

$$\mathcal{U}_\varepsilon(g(\boldsymbol{X}))$$

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Two Approaches for Introducing Uncertainty in $g(\boldsymbol{X})$

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2. Uncertainty in the Aggregate:

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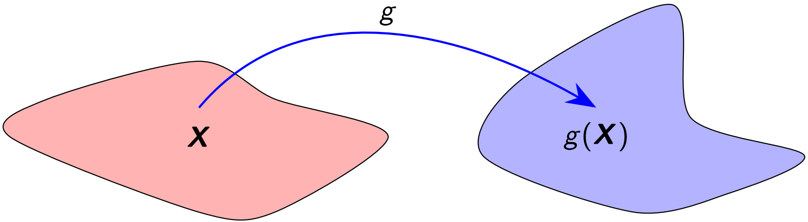
- In general, these two approaches for introducing uncertainty are not the same, i.e., $g\big(\boldsymbol{\mathcal{U}}_\varepsilon^n(\boldsymbol{X})\big) \neq \mathcal{U}_\varepsilon(g(\boldsymbol{X}))$.

Visualization of $g\left(\mathcal{U}^n_\varepsilon(\boldsymbol{X})\right)$



$$\mathcal{U}^n_\varepsilon(\boldsymbol{X}) = \{\boldsymbol{Z} \mid W^n(F_{\boldsymbol{Z}}, F_{\boldsymbol{X}}) \le \varepsilon\} \qquad g\left(\mathcal{U}^n_\varepsilon(\boldsymbol{X})\right)$$

Figure: Visualization of $g\left(\mathcal{U}^n_\varepsilon(\boldsymbol{X})\right)$

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First Major Result

- Idea: We want g to be sufficiently nice so that the image of the multivariate uncertainty set under g is still a Wasserstein ball.

Theorem 1

Let $\mathfrak{g} : \mathbb{R}^n \to \mathbb{R}$ be L-Lipschitz. Assume $\boldsymbol{X} \in \mathcal{L}_n^p$ and $\mathfrak{g}(\boldsymbol{X}) \in \mathcal{L}^p$. Then, for any $\varepsilon \geq 0$,

$$\mathfrak{g}\big(\boldsymbol{\mathcal{U}}_\varepsilon^n(\boldsymbol{X})\big) \subseteq \mathcal{U}_{L\varepsilon}(\mathfrak{g}(\boldsymbol{X})).$$

Proposition 1

Let ρ be a law invariant risk functional. Under the conditions of Theorem 1,

$$\sup_{\boldsymbol{Y} \in \boldsymbol{\mathcal{U}}_\varepsilon^n(\boldsymbol{X})} \rho\big(\mathfrak{g}(\boldsymbol{Y})\big) \leq \sup_{Y \in \mathcal{U}_{L\varepsilon}(\mathfrak{g}(\boldsymbol{X}))} \rho(Y)\,.$$

- Under slightly stronger conditions on $\mathfrak{g}$, the inequality becomes an equality.

Signed Choquet Integrals

Signed Choquet Integral

A signed Choquet integral is given by

$$I_h(X) = \int_{-\infty}^{0} [h(\mathbb{P}(X \geq x)) - h(1)]dx + \int_{0}^{\infty} h(\mathbb{P}(X \geq x))dx,$$

where $h : [0,1] \to \mathbb{R}$ such that h has bounded variation and $h(0) = 0$.

Signed Choquet Integral

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where $h : [0,1] \to \mathbb{R}$ such that h has bounded variation and $h(0) = 0$.

- If h is absolutely continuous, then there exists $\gamma : [0,1] \to \mathbb{R}$ such that

$$I_h(X) = \int_0^1 \gamma(u) F_X^{-1}(u) du.$$

- The function γ is called a distortion weight function.

Explicit Bound for Worst-Case Signed Choquet Integrals

Proposition 7

Let I_h be a signed Choquet integral with square integrable, non-decreasing distortion weight function γ satisfying $\|\gamma\|_2 > 0$.

Under the conditions of Theorem 1,

$$\sup_{\boldsymbol{Y} \in \mathcal{U}^n_\varepsilon(\boldsymbol{X})} I_h\big(\mathfrak{g}(\boldsymbol{Y})\big) \leq I_h(\mathfrak{g}(\boldsymbol{X})) + L\varepsilon \|\gamma\|_2 .$$

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Numerical Example - Aggregation Function

- The aggregation function is given by

$$g(\boldsymbol{x}) = -(x_1 + 2\max\{x_2 - 5, 0\} + 3\max\{35 - x_3, 0\} + 4x_4).$$

Numerical Example - Aggregation Function

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$$g(\boldsymbol{x}) = -(x_1 + 2\max\{x_2 - 5, 0\} + 3\max\{35 - x_3, 0\} + 4x_4).$$

- This aggregation function corresponds to the negative payoff of a portfolio consisting of:
  - 1 unit of a risky asset X_1
  - 2 units of a call option on X_2 with strike price 5
  - 3 units of a put option on X_3 with strike price 35
  - 4 units of a risky asset X_4.
- Clearly, g is Lipschitz with Lipschitz constant $L = 4$.

Numerical Example - Worst-Case Expected Shortfall (ES)

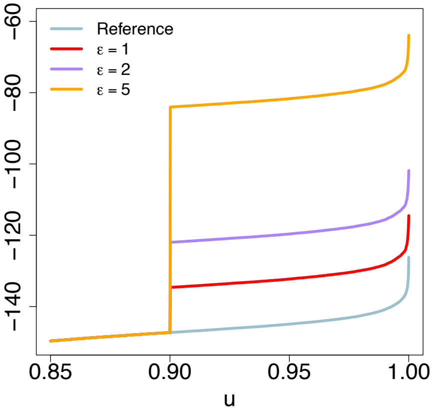

Figure: Worst-case quantile functions for the ES.

Questions?

Thank you for your attention! Any questions?

Figure: QR code for my preprint.

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Solution to Optimization Problem

- If γ is non-decreasing and $\|\gamma\|_2 > 0$, then the supremum in the bound of Proposition 1 is attained by the rv with quantile function

$$F^{-1}_{\mathfrak{g}(\boldsymbol{X})}(u) + \frac{L\varepsilon}{\|\gamma\|_2}\gamma(u).$$

- The assumptions on γ ensures that the proposed quantile function satisfies the properties of a quantile function.
- The non-decreasing assumption can be relaxed, but the form of the solution is more complicated.

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UNIVERSITY OF TORONTO

Optimal transport-based denoising and applications to collider physics

Benjamin Faktor, UCLA

In collaboration with:
Katy Craig, UCSB
Benjamin Nachman, SLAC/CMS

UCSB Summer School on Optimal Transport and Applications
22 July 2025

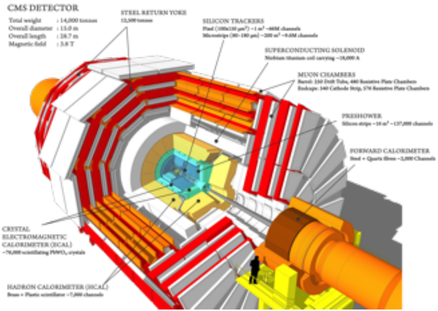
Outline

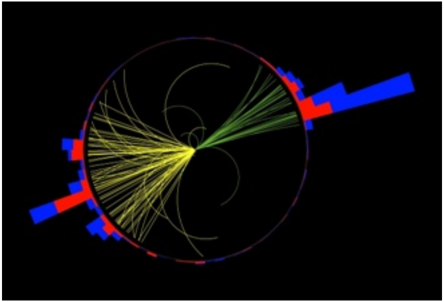
1. Large Hadron Collider and Motivation

2. Formulation

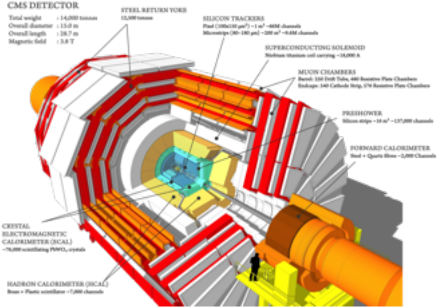
3. Numerical Solution

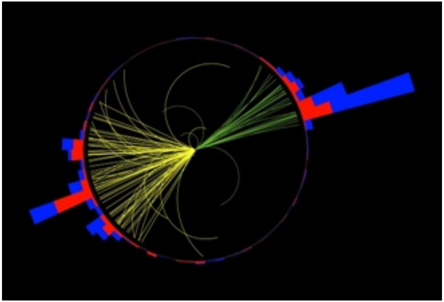
Large Hadron Collider





Large Hadron Collider



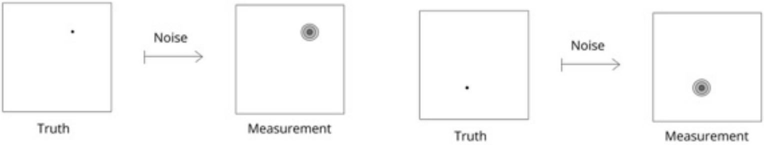


Moral: Collider measurement is a probability measure on $\mathbb{R}^2$

Motivation

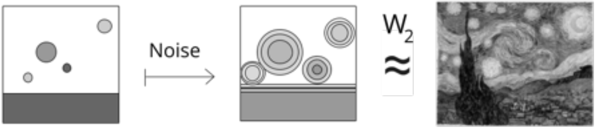
Given are a fixed measurement

and a noise model:



etc.

Question: Which image's noisification is closest?



Formulation

Given: fixed measurement $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and noise model $\rho : \mathbb{R}^d \to \mathscr{P}(\mathbb{R}^d)$

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Thm [Craig, F. '23] If ρ is W_2-continuous and $x \mapsto M_2(\rho_x)$ has compact sublevel sets then a solution exists. Moreover, if $\sigma \mapsto \nu_\sigma$ is injective and $\nu \ll \mathscr{L}^d$ then the solution is unique.

Numerical Solution

The denoising problem has poor stability;

e.g. if $\nu = \nu_{\sigma^*}$ there may be $\bar{\sigma}$ with $W_2(\nu_{\bar{\sigma}}, \nu_{\sigma^*})$ small but $W_2(\bar{\sigma}, \sigma^*)$ large.

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Instead, we add entropic regularization:

$$\sigma^* := \underset{(\sigma,\Gamma):\, \Gamma \in \Pi(\nu_\sigma, \nu)}{\arg\min} \left\{ \iint |y - y'|^2 d\Gamma + \varepsilon \int \sigma \log \sigma + \varepsilon \iint \Gamma \log \Gamma \right\}$$

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E-L equations allow us to find equivalent Sinkhorn-type iterations

Thank you!

Why OT?

– In the LHC data, the supports of ν and the ρ_{x_i} are not the same

– To compare ν_σ and ν using e.g. KL, ν must be "binned" to fit the support of the ρ_{x_i}, losing information

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