Open Problem Session

Perturbations of OT maps

Incompressible reachability

Incompressible uncertainty quantification

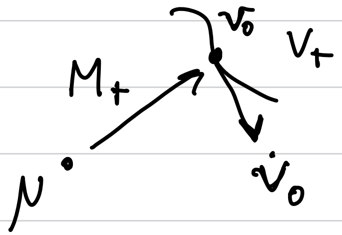
Guarantees for NN $\approx$ JKO

Benamou, Brenier, and Einstein

Rotationally Invariant OT

Sliced JKO

#1) Perturbations of OT maps



Given μ, ν_0

$\dot\nu_0$ (equiv v s.t.

$$\dot\nu_0 = -\nabla\cdot(\nu_0 v)$$

or f s.t. $v = \nabla f$)

① If $\nu_t \mapsto M_t$ $\boxed{\text{is regular enough}}$

hope to define

$$\dot M_t = D_\mu(\nu_t)[\dot\nu_t]$$

linear operator, depending on μ, ν_t, acting on $\dot\nu_t$

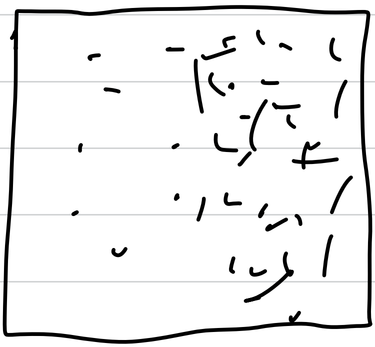
differential of log map

② Can we characterize $D_\mu(\nu_t)$

(in terms of PDE or as soln to opt problem)

Navier Stokes?
Diffusion helps?

#2) Incompressible Reachability applications in fluid control



density ρ of particles immersed in fluid

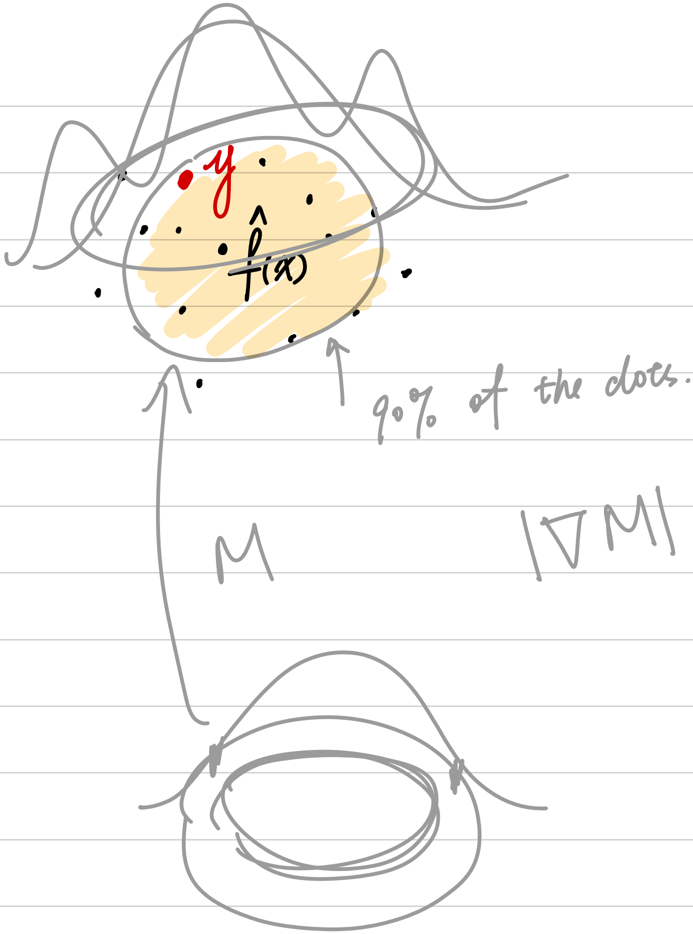
initial density ρ_I

target density ρ_T

$$(\star)\begin{cases}\dot{\rho} = -\nabla\cdot(\rho v) \\ \nabla\cdot v = 0\end{cases}$$

② If so, characterize it.

① Given ρ_I, ρ_T, does $\exists$ (ρ, v) s.t. $(\star)$ & $\rho_0 = \rho_I$, $\rho_1 = \rho_T$?



$|\nabla M| = \text{Const.}$

Why is it easier to learn transport maps than estimate densities?

Non-optimal transport?

$$\text{JKO operator} \overset{?}{=} NN$$

$$\rho^{t+\Delta t} = \underset{\rho}{\arg\min}\ W(\rho, \rho^t) + 2\Delta t\, E(\rho)$$

$$= JKO(\rho^t)$$

ρ^*

$JKO(\rho^*) = \rho^*$

$NN(\rho^*) = \rho^*$

$\forall \rho \in O(\rho^*)$

$NN(\rho^*) = JKO(\rho)$

$P(\rho) = JKO(\rho)$

Linear Linear

B – B

$$\text{crit}\left\{ \int_0^1 \int_{\mathbb{R}^d} |v|^2 \, d\rho_t \, dt \;:\; \begin{array}{l} (v, \rho) \text{ solve C.E.} \\ + \text{B.C.} \end{array} \right\}$$

Classical Physics,

$$\text{crit}\left\{ \int \frac{1}{2} m |u'|^2 - V(u) \, dt \;:\; \text{BC} \right\}$$

$u(t) \longrightarrow (t(\tau), x(\tau))$

arc length parametrization WRT

$u : \mathbb{R} \to \mathbb{R}^3$

$$\text{crit}\left\{ \int_0^T (V_{cl} + mc^2)\, d\tau \right\}$$

1. How far reaching is

BB formula $\longleftrightarrow$ principle of least action

$$\frac{dt}{d\tau}$$

$$d\tau = \gamma \, dt$$

$$\gamma = \sqrt{1 - \left(\frac{u'}{c}\right)^2}$$

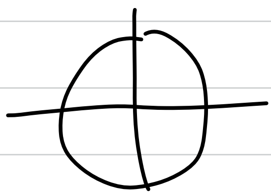
$$\Rightarrow \text{crit} \left\{ \int_0^T \underbrace{\left(V_{cR} + mc^2\right) \gamma(u')}_{f(u)} \, dt \right.$$

(2) Can "Einstein's trick" be used in the B-B formula? How?

$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^3)$ $\mu_1, \dots \mu_n$

$$\min_{R \in SO(3)} \underbrace{W_2^2}_{\text{or some other OT distance}} (R_\#(\mu), \nu)$$

$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^2)$



$SO(2) \cong S^1$

(Shi, Verbeke, Singer '25)

$$SW_2^2(\mu, \nu) = \int_{S^{n-1}} W_2^2\left(R_\#^\theta(\mu), R_\#^\theta(\nu)\right) d\theta$$

$$\mu_{t+1} \in \arg\min_{\nu} SW_2^2(\mu_T, \nu) + F(\nu)$$

1) does this have the same cnit.

2) Can we modify F?