# Lecture #3.

- **Proposition:** [Chizat, Rousillon, Léger, Vialard, Peyré (2020)]

  For all $d \geqslant 5$, if $W_2(\mu,\nu) \geq \delta > 0$, then:

$$0 \leq \mathbb{E} W_2^2(\mu,\nu_n) - W_2^2(\mu,\nu) \lesssim n^{-2/d}.$$

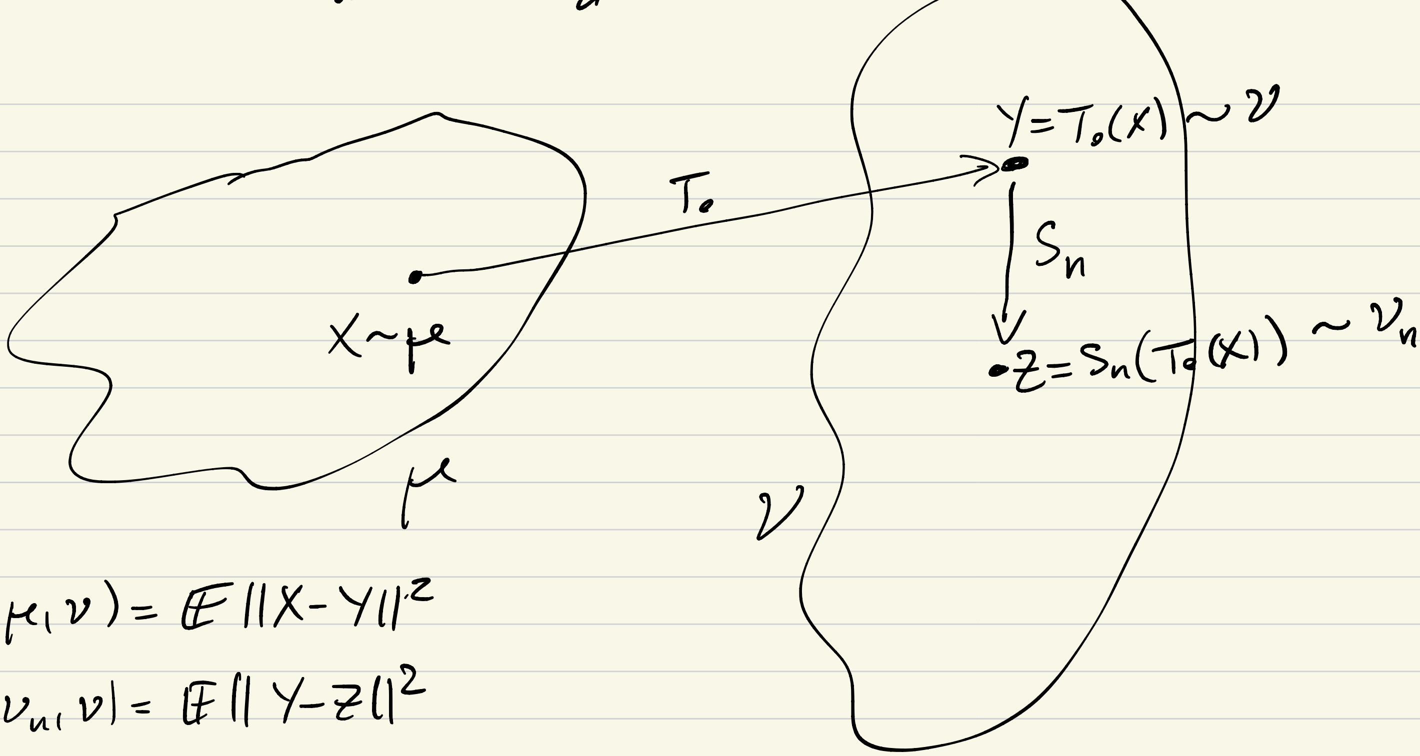
**Proof Sketch**

$$\begin{tikzcd}
\mu \arrow[rr, "T_n"] \arrow[dr, "T_0"'] & & \nu_n \\
& \nu \arrow[ur, "S_n"'] &
\end{tikzcd}$$

$$\begin{aligned}
W_2^2(\mu,\nu_n) &= \int \|T_n(x) - x\|^2 \, d\mu(x) \\
&\leq \int \|S_n(T_0(x)) - x\|^2 \, d\mu(x) \\
&= \int \|S_n(y) - T_0^{-1}(y)\|^2 \, d\nu(y) \\
&= LOT_\nu(\nu_n,\mu)
\end{aligned}$$

$\Rightarrow$ Goal: $\quad LOT_\nu(\nu_n,\mu) - W_2^2(\mu,\nu) \lesssim \, ?$

$S_n : \nu \longrightarrow \nu_n$




$$W_2^2(\mu, \nu) = \mathbb{E}\|X - Y\|^2$$

$$W_2^2(\nu_n, \nu) = \mathbb{E}\|Y - Z\|^2$$

$$LOT_\nu(\mu, \nu_n) = \mathbb{E}\|X - Z\|^2$$

$$\geq W_2^2(\mu, \nu_n)$$

IF $\mathbb{E}\left[\langle Y - X, Z - Y \rangle\right] \approx 0$ $\quad$ (*)

$$\Rightarrow \mathbb{E}\|X - Z\|^2 \approx \mathbb{E}\|X - Y\|^2 + \mathbb{E}\|Y - Z\|^2$$

$$\mathbb{E}\left[LOT_\nu(\mu, \nu_n)\right] \approx W_2^2(\mu, \nu) + \mathbb{E}\left[W_2^2(\nu, \nu_n)\right]$$

$$\Rightarrow \mathbb{E} W_2^2(\mu, \nu_n)$$

$$\leq \mathbb{E}\, LOT_\nu(\mu, \nu_n)$$

$$\leq W_2^2(\mu, \nu) + n^{-2/d}.$$

(*)

<u>Example</u>: Suppose $T_0(x) = x + \lambda$ for fixed $\lambda \in \mathbb{R}^d$

$$\mathbb{E}\left[\langle Y - X, Z - Y \rangle\right]$$

$$= \mathbb{E}\left[\langle \lambda, Z - Y \rangle\right]$$

$$= \mathbb{E}\left[\int \langle \lambda, x \rangle \, d(\nu_n - \nu)(y)\right] = 0.$$

<u>Fact</u>: This argument continues to work if $T_0$ is bi-Lipschitz.

↳ If $\rho, P, Q$ are measures over $[0,1]^d$ such that $\rho, Q$ a.c Lebesgue, and the OT map from $\rho$ to $Q$ is bi-Lipschitz, then:

$$W_2(P,Q) \le LOT_\rho(P,Q) \lesssim W_2(P,Q).$$

↳ When these smoothness assumptions fail to hold, the above proof is not known to work.

↳ Chizat et al. developed the following argument instead:

<u>Lemma</u>: If $(f_n, g_n)$ & $(f_0, g_0)$ are optimal Kantorovich potentials from $\mu \to \nu_n$ & $\mu \to \nu$ then:

$$\int g_0 \, d(\nu_n - \nu) \leq W_2^2(\mu, \nu_n) - W_2^2(\mu, \nu)$$

$$\leq \int g_n \, d(\nu_n - \nu).$$

$$\shortparallel$$

$$\frac{1}{n} \sum_{i=1}^{n} \left\{ g_n(Y_i) - \mathbb{E}\left[ g_n(Y) \,\middle|\, Y_1, \dots, Y_n \right] \right\}$$

Let $\mathcal{G}$ be the set of maps $\|\cdot\|^2 - 2\varphi$, with $\varphi$ cvx and $L$-Lipschitz. Note that $g_n \in \mathcal{G}$ for large enough $L$.

$$\mathcal{G}, \qquad g \in \mathcal{G} \longmapsto \int g \, d(\nu_n - \nu)$$ "Empirical Process".

$$\mathbb{E}\left[ \int g_n \, d(\nu_n - \nu) \right] \leq \mathbb{E}\left[ \sup_{g \in \mathcal{G}} \int g \, d(\nu_n - \nu) \right]$$

$$\lesssim n^{-2/d}$$ -> Dudley's Chaining Bound

Bronshtein '79. $\square$

<u>Remark</u>: [Hundrieser, Staudt, Munk '24 AIHP]

"Lower Complexity Adaptation Principle".

# III. Central Limit Theorems.

[del Barrio, Loubes '19 AOP]

<u>Theorem</u>: Let $\mu, \nu \in \mathcal{P}(\Omega)$ which are a.c, and such that $\nu$ has a density $q$ s.t.: $q(y) \geq c > 0$ $\forall y \in \Omega$.

Let $(f_0, g_0)$ be a pair of K. potentials from $\mu$ to $\nu$. Then:

$$\sqrt{n}\left(W_2^2(\mu, \nu_n) - \mathbb{E}W_2^2(\mu, \nu_n)\right) \xrightarrow{d} \mathcal{N}\left(0, \mathrm{Var}_\nu[g_0(Y)]\right)$$

<u>Remark</u>: When $d \leq 3$:

$$\sqrt{n}\left(W_2^2(\mu, \nu_n) - W_2^2(\mu, \nu)\right) \xrightarrow{d} \mathcal{N}\left(0, \mathrm{Var}_\nu[g_0(Y)]\right)$$