A Blob Method for the Aggregation Equation

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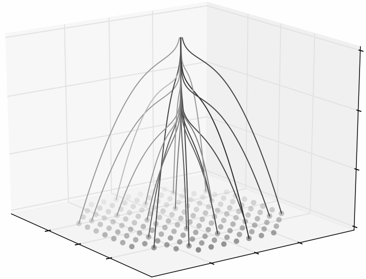
joint with Andrea Bertozzi



Univeristy of California, Santa Barbara

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Plan

- aggregation equation
- numerical methods
- blob method
- blob method converges
- sketch of proof
- numerics

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Aggregation Equation

$$\begin{cases} \frac{d\rho}{dt} + \nabla \cdot (v\rho) = 0 & \rho(0,t) = \rho_0(t) \geq 0 \\ v = -\nabla K * \rho\ . \end{cases}$$

Applied interest:

- $K(x) = |x|^a/a - |x|^b/b,\ -d < b < a,$ social aggregation in biology
- $K(x) = -\log|x|/2\pi,$ evolution of vortex densities in superconductors

Kernels with low regularity

Mathematical interest:

- non-local
- blowup
- rich structure of steady states
- gradient flow in the Wasserstein metric: $\frac{d\rho}{dt} = -\nabla_W E(\rho)$

$$\nabla_W E(\rho) = -\nabla \cdot \left(\rho \nabla \frac{\delta E}{\delta \rho}\right)\ , \quad E(\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)K(x-y)\rho(y)dxdy\ .$$

Particle Approximation and Wasserstein Gradient Flow

Suppose K is radial, continuously differentiable, and convex and we seek a weak solution of the form

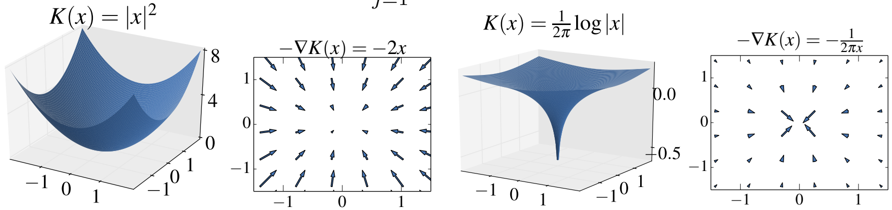
$$\rho^{\text{particle}}(x,t) = \sum_{j=1}^{N} \delta(x - X_j(t)) m_j \ .$$

Then the velocity field would be given by

$$v(x,t) = -\int \nabla K(x-y)\rho(y,t)dy = -\sum_{j=1}^{N} \nabla K(x - X_j(t)) m_j \ ,$$

and ρ^{particle} is a weak solution in case

$$\frac{d}{dt} X_i(t) = -\sum_{j=1}^{N} \nabla K(X_i(t) - X_j(t)) m_j \ .$$



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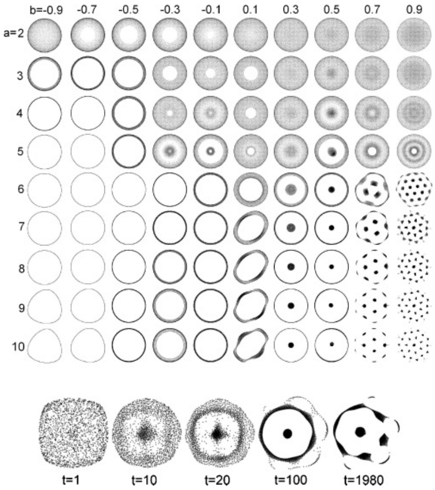

[Kolokolnikov, Sun, Uminsky, Bertozzi, 2011]

Numerical Methods: Recent Results

particle methods

- complement theoretical results: repulsive attractive steady states — [Bertozzi, Sun, Kolokolnikov, Uminsky, Von Brecht 2011], [Balagué, Carrillo, Laurent, Raoul 2012]
- used to prove theoretical results: finite time blowup, confinement — [Carrillo, DiFrancesco, Figalli, Laurent, Slepčev 2010, 2011]
- convergence of particle method — [Carrillo, Choi, Hauray 2013]

other numerical methods

- developed finite volume method — [Carrillo, Chertock, Huang 2014]
- convergence of finite difference method to measure solutions, 1D — [James, Vauchelet 2014]

Numerical Methods: Our Goal

- Develop new numerical method for multidimentional aggregation equation
- Allow singular and non-singular potentials
- Prove quantitative estimates on convergence to classical solutions
- Validate sharpness of estimates with numerical examples

Blob Method for the Aggregation Equation

Theorem (C., Bertozzi 2014)

Let $K(x)$ have power law growth $|x|^s$, $s \geq 2-d$
(for simplicity of notation $d \geq 3$, Newtonian potential admissible for $d = 2$).

Suppose $\rho : \mathbb{R}^d \times [0,T] \to \mathbb{R}^+$ is a smooth, compactly supported solution.

The blob method discretizes $\rho_0(x)$ on a mesh of size h and prescribes

- *approximate particle trajectories $\tilde{X}_i$,*
- *approximate density along particle trajectories $\tilde{\rho}_i$,*

so that for $\frac{1}{2} \leq q < 1$ and $m \geq 4$ (parameters specifying shape of blobs)

$$||X_i(t) - \tilde{X}_i(t)||_{L^p_h} \leq Ch^{mq} \quad ||\rho_i(t) - \tilde{\rho}_i(t)||_{W_h^{-1,p}} \leq Ch^{mq} \; ,$$

for $1 \leq p < \infty$.

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Euler vs Aggregation: Similarities

Vorticity formulation of Euler equations:

$$\textbf{(V)} \begin{cases} \omega_t + v \cdot \nabla \omega = \omega \cdot (\nabla v) \\ v = K_d * \omega \end{cases} \xrightarrow{\text{material derivative}} \begin{cases} D\omega/Dt = \omega \cdot (\nabla v) \\ v = K_d * \omega \end{cases}$$

Biot-Savart kernel: $K_2(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1)$, $K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}$.

$v = \nabla^\perp \Delta^{-1} \omega$

Aggregation equation:

$$\textbf{(A)} \begin{cases} \rho_t + \nabla \cdot (v\rho) = 0 \\ v = -\nabla K * \rho \end{cases} \xrightarrow{\text{material derivative}} \begin{cases} D\rho/Dt = -\rho(\nabla \cdot v) \\ v = -\nabla K * \rho \end{cases}$$

Newtonian potential: $K(x) = \frac{1}{d(d-2)\omega_d}|x|^{2-d}$ ($K(x) = -\log|x|/2\pi$ when $d = 2$).

$v = \nabla \Delta^{-1} \rho$

Euler vs Aggregation: Differences

Euler Equations

- velocity is divergence free
- Biot Savart kernel
- 2 and 3 dimensions

Aggregation Equation

- mass is conserved
- Newtonian, Riesz, and non-singular kernels (growth at infinity)
- $d \geq 1$

Aggregation equation: Lagrangian perspective

Particle trajectories:
$$\begin{cases} \frac{d}{dt}X(\alpha,t) & = -\nabla K * \rho(X(\alpha,t),t) \\ X(\alpha,0) & = \alpha\ . \end{cases}$$

Density along trajectories:
$$\begin{cases} \frac{d}{dt}\rho(X(\alpha,t),t) = (\Delta K * \rho(X(\alpha,t),t))\ \rho(X(\alpha,t),t) \\ \rho(X(\alpha,0),0) \quad = \rho_0(\alpha)\ . \end{cases}$$

By conservation of mass, $\rho(X(\beta,t),t)J(\beta,t) = \rho_0(\beta)$,

$$\begin{aligned} \int \nabla K(x-y)\rho(y,t)dy &= \int \nabla K(x - X(\beta,t))\rho(X(\beta,t),t)J(\beta,t)d\beta \\ &= \int \nabla K(x - X(\beta,t))\rho_0(\beta)d\beta\ . \end{aligned}$$

... and similarly for $\Delta K * \rho(X(\alpha,t),t)$.

Steps for blob method

1. Remove the singularity of K by convolution with a mollifier, $K_\delta = K * \psi_\delta$.
2. Replace ρ_0 with a particle approximation on the grid $h\mathbb{Z}^d$.

$$\rho_0^{\text{particle}}(y) = \sum_{j \in \mathbb{Z}^d} \delta(y - jh) {\rho_0}_j h^d$$

Blob method for the aggregation equation

Exact Particle Trajectories: $\frac{d}{dt}X(\alpha,t) = -\int \nabla K(X(\alpha,t) - X(\beta,t))\rho_0(\beta)d\beta$

$$X(\alpha,0) = \alpha$$

Exact Density : $\frac{d}{dt}\rho(X(\alpha,t),t) = \rho(X(\alpha,t),t)\int \Delta K(X(\alpha,t) - X(\beta,t))\rho_0(\beta)d\beta$

$$\rho(\alpha,0) = \rho_0(\alpha)$$

Approx Particle Trajectories: $\frac{d}{dt}\tilde{X}_i(t) = -\sum_j \nabla K_\delta(\tilde{X}_i(t) - \tilde{X}_j(t))\rho_{0j}h^d$

$$\tilde{X}_i(0) = ih$$

Approx Density : $\frac{d}{dt}\tilde{\rho}_i(t) = \tilde{\rho}_i(t)\left(\sum_j \Delta K_\delta(\tilde{X}_i(t) - \tilde{X}_j(t))\rho_{0j}h^d\right)$

$$\tilde{\rho}_i(0) = \rho_0(ih)$$

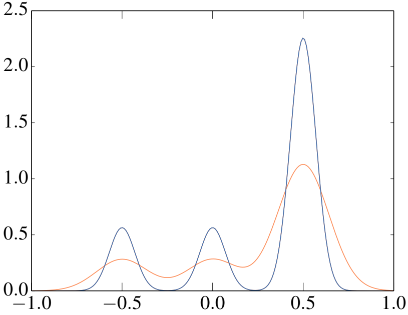
(For pure particle method, take $\delta = 0$.)

Heuristic interpretation of blob method

When K is the Newtonian potential, $v = -\nabla K * \rho$ implies $\rho = -\nabla \cdot v$.

Applying this to the approximate velocity $\tilde{v}$...

$$\tilde{\rho}^{alt}(x,t) = -\nabla \cdot \left(-\sum_j \nabla K_\delta(x - \tilde{X}_j(t)) \rho_{0_j} h^d \right) = \sum_j \psi_\delta(x - \tilde{X}_j(t)) \rho_{0_j} h^d$$



Advantages of blob method

- Avoids main source of numerical diffusion
- Only requires computational elements on support of density
- Inherently adaptive
- Accommodates singular kernels, up to and including the Newtonian potential
- Arbitrarily high order rates of convergence, depending on the accuracy of the mollifier and the widths of the blobs

Without regularization: fewer admissible potentials, slower rates of convergence.

These agree with the rate of $\mathcal{O}(h^{2-\epsilon})$ for the Euler equations [Goodman, Hou, Lowengrub 1990].

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Mollifier

Assumption

Assume ψ is radial, $\int \psi = 1$, and for some $m \geq 4$, $L \geq d+2$

1. *Accuracy:* $\int x^\gamma \psi(x)dx = 0$ *for* $1 \leq |\gamma| \leq m-1$
2. *Regularity:* $\psi \in C^L(\mathbb{R}^d)$
3. *Decay:* $|x|^n |\partial^\beta \psi(x)| \leq C$ *for all* $n \geq 0$

1 ensures convolution with ψ preserves polynomials of order $|\alpha| \leq m-1$,

$$\int (x-y)^\alpha \psi(y)dy = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^{\alpha-k} \int y^k \psi(y)dy = x^\alpha \int \psi(y)dy = x^\alpha.$$

2 and 3 ensure $\nabla K_\delta, \Delta K_\delta \in C^L(\mathbb{R}^d)$.

Mollifier

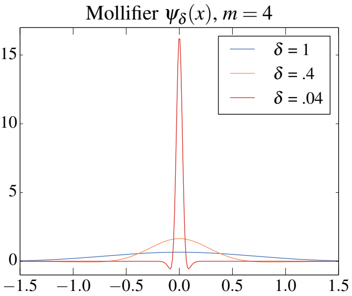
Assumption

Assume ψ *is radial,* $\int \psi = 1$*, and for some* $m \geq 4$*,* $L \geq d+2$

1. *Accuracy:* $\int x^\gamma \psi(x) dx = 0$ *for* $1 \leq |\gamma| \leq m-1$
2. *Regularity:* $\psi \in C^L(\mathbb{R}^d)$
3. *Decay:* $|x|^n |\partial^\beta \psi(x)| \leq C$ *for all* $n \geq 0$

Example: $d = 1$, $m = 4$, $L = +\infty$,

$$\psi(x) = \frac{4}{3\sqrt{\pi}} e^{-x^2} - \frac{1}{6\sqrt{\pi}} e^{-(x/2)^2}$$



Kernel

Assumption

Suppose that $K(x) = \sum_{n=1}^{N} K_n(x)$.

For each $K_n(x)$*, there exists* $S_n \geq 2-d$ *such that*

$$|\partial^\beta K_n(x)| \leq C|x|^{S_n - |\beta|},\ \forall x \in \mathbb{R}^d \setminus \{0\}, |\beta| \geq 0\ .$$

If $S_n = 2-d$*, we additionally require that* $K_n(x)$ *is a constant multiple of the Newtonian potential.*

Let $s = \min_n S_n$ *be the smallest power of the kernel.*

Example: $K(x) = |x|^a/a - |x|^b/b,\ \ 2-d \leq b < a$.

$s = b$

Discrete L^p norms

Definition

For $1 \le p \le \infty$,

$$||u_i||_{L_h^p} = \left(\sum_{i \in \mathbb{Z}^d} |u_i|^p h^d \right)^{1/p} \qquad (u_i, g_i)_h = \sum_{i \in \mathbb{Z}^d} u_i g_i h^d$$

$$||u_i||_{W_h^{1,p}} = \left(||u_i||_{L_h^p}^p + \sum_{j=1}^{d} ||D_j^+ u_i||_{L_h^p}^p \right)^{1/p} \qquad ||u_i||_{W_h^{-1,p}} = \sup_{\{g_i\} \in W_h^{1,p'}} \frac{|\langle u_i, g_i \rangle|}{||g_i||_{W_h^{1,p'}}}$$

D_j^+ is the forward difference operator in the j^{th} coordinate direction.

We measure the convergence of X and v in L_h^p and we measure the convergence of ρ in $W_h^{-1,p}$.

This is because, in the most singular case when K is the Newtonian potential, $v = -\nabla K * \rho \implies \rho = -\nabla \cdot v$.

Convergence

Theorem (C., Bertozzi 2014)

Suppose...

- $\psi \in C^L(\mathbb{R}^d)$ *for* $L > s + d$,
- $\rho : \mathbb{R}^d \times [0, T] \to \mathbb{R}^+$ *is a smooth, compactly supported solution,*
- $0 \leq h^q \leq \delta \leq 1/2$ *for some* $\frac{1}{2} < q < 1$.

Then for $1 \leq p < \infty$,

$$||X_i(t) - \tilde{X}_i(t)||_{L^p_h} \leq C(\delta^m + \delta^{-(L-s-d)} h^L)$$
$$||\rho_i(t) - \tilde{\rho}_i(t)||_{W^{-1,p}_h} \leq C(\delta^m + \delta^{-(L+1-s-d)} h^L) \ ,$$

provided that for some $\epsilon > 0$,

$$C(\delta^m + \delta^{-(L+1-s-d)} h^L) < \delta^2 h^{1+\epsilon}/2 \ .$$

Convergence of arbitrarily high order

Take $\delta = h^q$ for $\frac{1}{2} < q < 1$.
Then the technical condition $C(\delta^m + \delta^{-(L+1-s-d)} h^L) < \delta^2 h^{1+\epsilon}/2$ holds.

By the previous theorem

$$||X_i(t) - \tilde{X}_i(t)||_{L^p_h} \leq C(\delta^m + \delta^{-(L-s-d)} h^L) \quad \leq \quad C\delta^m$$

$$||\rho_i(t) - \tilde{\rho}_i(t)||_{W^{-1,p}_h} \leq C(\delta^m + \delta^{-(L+1-s-d)} h^L) \leq \underbrace{C\delta^m}_{\text{for } L \text{ sufficiently large}}$$

Theorem (C., Bertozzi 2014)

Let $\delta = h^q$. If L is sufficiently large, then for $1/2 \leq q < 1$, $m \geq 4$,

$$||X_i(t) - \tilde{X}_i(t)||_{L^p_h} \leq Ch^{mq}$$

$$||\rho_i(t) - \tilde{\rho}_i(t)||_{W^{-1,p}_h} \leq Ch^{mq}$$

Benefit of blob methods: arbitrarily high order of convergence.

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Sketch of proof: convergence of particle trajectories

Velocity		
Exact	$v(x,t)$	$= -\int \nabla K(x - X(\beta,t))\rho_0(\beta)d\beta$
Approx	$\tilde{v}(x,t)$	$= -\sum_j \nabla K_\delta(x - \tilde{X}_j(t))\rho_{0j}h^d$
Approx along exact traj.	$v^h(x,t)$	$= -\sum_j \nabla K_\delta(x - X_j(t))\rho_{0j}h^d$

Main Steps:

1. Control difference between exact and approximate velocity by separately estimating consistency and stability,

$$|v(x,t) - \tilde{v}(x,t)| \leq |v(x,t) - v^h(x,t)| + |v^h(x,t) - \tilde{v}(x,t)| .$$

2. Use Gronwall's inequality to deduce control of particle error.

Consistency

Proposition (Consistency) (C., Bertozzi 2014)

$$\|v - v^h\|_{L_h^\infty} \leq C\left(\delta^m + \delta^{-(L-s-d)}h^L\right) .$$

$$|v(x,t) - v^h(x,t)|$$

$$= |v(x,t) - \nabla K_\delta * \rho(x,t)| + |\nabla K_\delta * \rho(x,t) - v^h(x,t)|$$

$$= |\nabla K * \rho(x,t) - \nabla K_\delta * \rho(x,t)| + \left|\nabla K_\delta * \rho(x,t) - \sum_j \nabla K_\delta(x - X_j(t))\rho_{0j} h^d\right|$$

$$\leq \underbrace{|\nabla K * \rho(x,t) - \nabla K * \rho * \psi_\delta(x,t)|}_{\psi \text{ is accurate of order } m} + \underbrace{C\|\nabla K_\delta\|_{W^{1,L}(B_R)}h^L}_{\text{quadrature, kernel estimates}}$$

$$\leq C\delta^m + C\delta^{-(L-s-d)}h^L$$

Lemma (Regularized Kernel Estimates) (C., Bertozzi 2014)

For $L > s + d$, $\|\nabla K_\delta\|_{W^{1,L}(B_R)} \leq C\delta^{-(L-s-d)}$.

Stability

Proposition (Stability) (C., Bertozzi 2014)

If $||X(t) - \tilde{X}(t)||_{L_h^\infty} \leq \delta$, *then for* $1 < p < \infty$

$$||v^h(t) - \tilde{v}(t)||_{L_h^p} \leq C||X(t) - \tilde{X}(t)||_{L_h^p} \; .$$

$$\begin{aligned} v_i^h - \tilde{v}_i &= \sum_j \nabla K_\delta(X_i - \tilde{X}_j)\rho_{0j}h^d - \sum_j \nabla K_\delta(X_i - X_j)\rho_{0j}h^d \\ &\quad + \sum_j \nabla K_\delta(\tilde{X}_i - \tilde{X}_j)\rho_{0j}h^d - \sum_j \nabla K_\delta(X_i - \tilde{X}_j)\rho_{0j}h^d \\ &= \sum_j D^2K_\delta(X_i - X_j + y_{ij}^{(1)})(X_j - \tilde{X}_j)\rho_{0j}h^d \\ &\quad + (\tilde{X}_i - X_i)\sum_j D^2K_\delta(X_i - X_j + y_{ij}^{(2)})\rho_{0j}h^d \end{aligned}$$

use mean value theorem to pull out $X - \tilde{X}$

discrete → continuous convolution and $L_h^p \to L^p$

$$\begin{aligned} \|v^h - \tilde{v}\|_{L_h^p} &\leq C||D^2K_\delta * [(X - \tilde{X})\rho]||_{L^p} + C||(X - \tilde{X})(D^2K_\delta * \rho)||_{L^p} \\ &\leq C||X - \tilde{X}||_{L^p} \\ &\leq C||X - \tilde{X}||_{L_h^p} \end{aligned}$$

apply Calderón-Zygmund or Young

Convergence

Therefore, for $||X(t) - \tilde{X}(t)||_{L_h^\infty} \leq \delta$,

$$\begin{aligned}\|v(t) - \tilde{v}(t)\|_{L_h^p} &\leq \|v(t) - v^h(t)\|_{L_h^p} \qquad\qquad + \|v^h(t) - \tilde{v}(t)\|_{L_h^p} \\ &\leq C(\delta^m + \delta^{-(L-s-d)}h^L) + C\|X(t) - \tilde{X}(t)\|_{L_h^p(B_{R_0})}\end{aligned}$$

With Gronwall's inequality and a bootstrap argument, we obtain the result:

$$||X(t) - \tilde{X}(t)||_{L_h^p} \leq C(\delta^m + \delta^{-(L-s-d)}h^L)\ .$$

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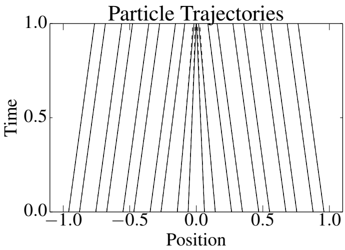
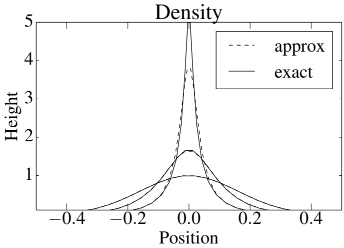
Newtonian potential, one dimension





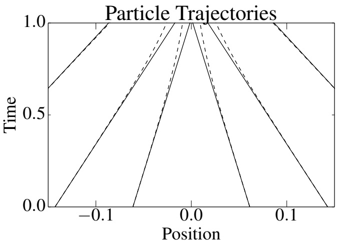
$h = 0.04$

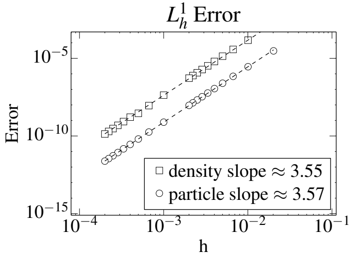
$q = 0.9$

$m = 4$

$\rho_0(x) = (1 - x^2)_+^{20}$

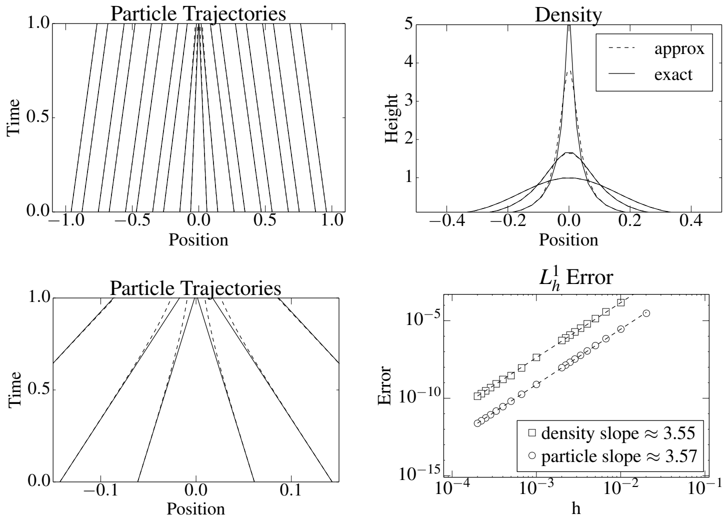
blowup: $t = 1$





- approximate particle trajectories bend to avoid collision
- convergence of method agrees with theoretically predicted $3.6 = m \cdot q$

Newtonian potential, one dimension



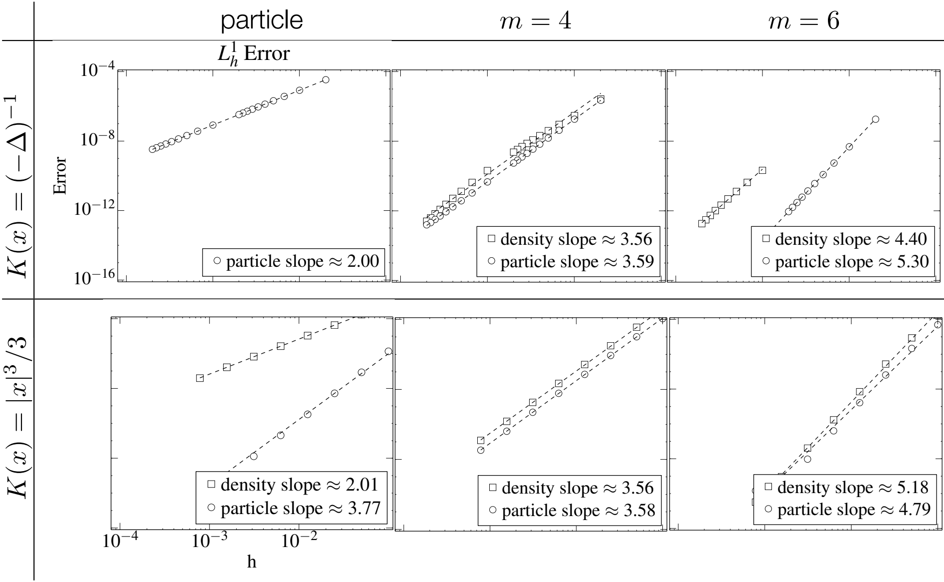
$h = 0.04$
$q = 0.9$
$m = 4$

$\rho_0(x) = (1 - x^2)_+^{20}$

blowup: $t = 1$

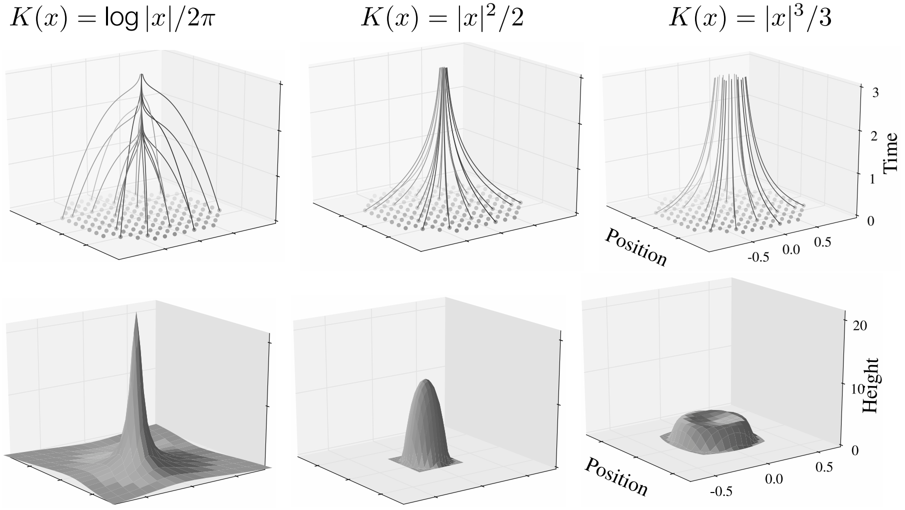
- approximate particle trajectories bend to avoid collision
- convergence of method agrees with theoretically predicted $3.6 = m \cdot q$

Various kernels, one dimension: blob vs particle



- Blob method is more beneficial for more singular kernels

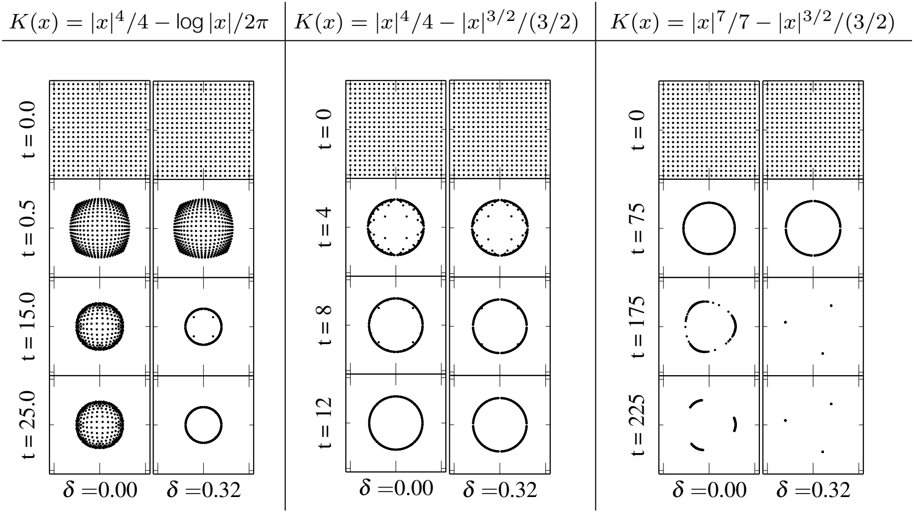
two dimensions, aggregation



- finite vs infinite time collapse
- delta function vs delta ring

$h = 0.04$, $q = 0.9$, $m = 4$

two dimensions: repulsive-attractive kernels



- large δ affects steady state behavior
- illustrates role of kernel's regularity in dimensionality of steady states [Balagué, Carrillo, Laurent, Raoul 2013]

Future Work

- Keller-Segel equation [Yao, Bertozzi 2013]
- Interplay between particle methods and theoretical results
  - Finite time blowup, confinement [Carrillo, DiFrancesco, Figalli, Laurent, Slepčev 2010, 2011]
  - Existence of weak measure solutions [Lin, Zhang 2000]
- ongoing work with Ihsan Topaloglu (Fields Institute): Γ-convergence of regularized interaction energy; convergence of blob method to steady states

$$E_\delta(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x) K_\delta(x-y) \rho(y) dx dy.$$

- ongoing work with Andrea Bertozzi: long time error estimates for repulsive attractive kernels?

Thank you!

Backup

Associated particle system and gradient flow

Given the blob method particle trajectories

$$\begin{cases} \frac{d}{dt}\tilde{X}_i(t) = -\sum_j \nabla K_\delta(\tilde{X}_i(t) - \tilde{X}_j(t))\rho_{0j}h^d \\ \tilde{X}_i(0) \quad = ih \ , \end{cases}$$

we may define the corresponding particle measure

$$\hat{\rho}(x,t) = \sum_j \delta(x - \tilde{X}_j(t))\rho_{0j}h^d \ .$$

This is

- energy decreasing
- formally Wasserstein gradient flow

for the regularized energy functional

$$E_\delta(\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)K_\delta(x-y)\rho(y)dxdy.$$

(For pure particle method, take $\delta = 0$.)

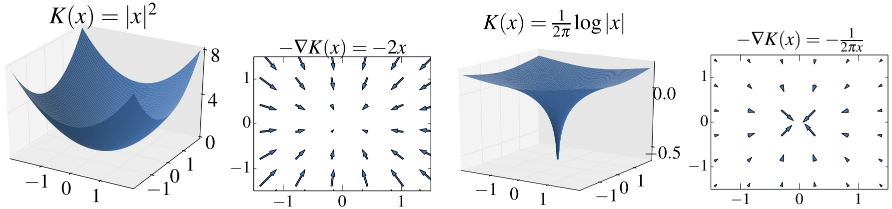
Blowup

For which kernels does finite time blowup occur?

Intuition from particle approximation: $\rho(x,t) = \sum_{j=1}^{N} \delta(x - X_j(t)) m_j$

$$v(x,t) = -\int \nabla K(x-y)\rho(y,t)dy = -\sum_{j=1}^{N} \nabla K(x - X_j(t)) m_j$$

$$\frac{d}{dt} X_i(t) = -\sum_{j=1}^{N} \nabla K(X_i(t) - X_j(t)) m_j$$



Osgood Condition

Simple case: particle moving toward minimum of attractive potential: $K(x) = k(|x|)$

$$\frac{d}{dt}X(t) = -\nabla K(X(t)) \quad X(0) = x_0$$
$$\frac{d}{dt}r(t) = -k'(r(t)) \quad r(0) = R_0$$

To move a distance dr, it takes time $\frac{dr}{|k'(r)|}$.

Thus, the particle reaches the origin at time

$$T = \int_0^{R_0} \frac{dr}{k'(r)} \ .$$

Osgood Condition

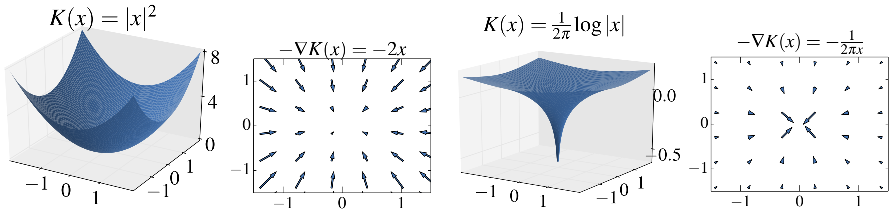
Theorem (Osgood Condition) (Bertozzi, Carrillo, Laurent 2009)

A kernel K satisfies the Osgood condition in case

$$\int_0^{R_0} \frac{dr}{k'(r)} < \infty \,.$$

This is a necessary and sufficient condition for finite time blowup.

$K(x) = |x|^\alpha$: $\alpha \geq 2 \implies$ no finite time blowup, $\alpha < 2 \implies$ finite time blowup



K = Newtonian potential

Rewriting the aggregation equation in terms of the material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$,

$$\begin{cases} \rho_t + \nabla \cdot (v\rho) = 0 \\ v = -\nabla K * \rho \, . \end{cases} \xrightarrow{\text{material derivative}} \begin{cases} \frac{D\rho}{Dt} = -\rho(\nabla \cdot v) \\ v = -\nabla K * \rho \, . \end{cases}$$

When K is the Newtonian potential, $v = -\nabla K * \rho$ implies $\rho = -\nabla \cdot v$, so

$$\frac{D\rho}{Dt} = \rho^2$$

If $X(\alpha, t)$ denotes the particle trajectories induced by the velocity field v,

$$\frac{d}{dt}\rho(X(\alpha,t),t) = \rho(X(\alpha,t),t)^2 \, .$$

Hence,

$$\rho(X(\alpha,t),t) = \begin{cases} \left(\frac{1}{\rho_0(\alpha)} - t\right)^{-1} & \text{if } \rho_0(\alpha) \neq 0 \\ 0 & \text{if } \rho_0(\alpha) = 0. \end{cases}$$

K = Newtonian potential

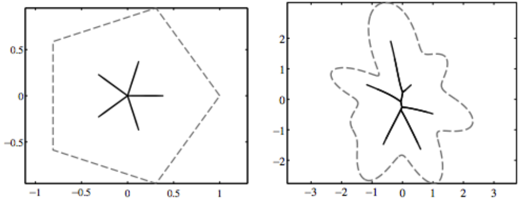
$$\rho(X(\alpha,t),t) = \begin{cases} \left(\frac{1}{\rho_0(\alpha)} - t\right)^{-1} & \text{if } \rho_0(\alpha) \neq 0 \\ 0 & \text{if } \rho_0(\alpha) = 0 \end{cases}$$

blowup: If $\rho_0(\alpha) > 0$ for any α, the first blowup occurs at time $t = ||\rho_0||_{L^\infty}^{-1}$.

patch solutions: If $\rho_0(\alpha) = 1_\Omega(\alpha)$, for $\Omega_t := X^t(\Omega)$,

$$\rho(X(\alpha,t),t) = (1-t)^{-1} 1_\Omega(\alpha) = (1-t)^{-1} 1_{\Omega_t}(X(\alpha,t)) \ .$$

Patch solutions collapse onto a set of Lebesgue measure zero at $t = 1$.



[Bertozzi, Laurent, Léger 2012]

2D Euler equations: Lagrangian perspective

For simplicity of notation, write $K = K_2$.

Particle trajectories: $\begin{cases} \frac{d}{dt}X(\alpha,t) &= K * \omega(X(\alpha,t),t) \\ X(\alpha,0) &= \alpha\ . \end{cases}$

Since the velocity field is divergence free and $\omega(X(\beta,t),t) = \omega_0(\beta)$,

$$\begin{aligned} \int K(x-y)\omega(y,t)dy &= \int K(x - X(\beta,t))\omega(X(\beta,t),t)d\beta \\ &= \int K(x - X(\beta,t))\omega_0(\beta)d\beta\ . \end{aligned}$$

Thus the particle trajectories evolve according to

$$\begin{cases} \frac{d}{dt}X(\alpha,t) &= \int K(X(\alpha,t) - X(\beta,t))\omega_0(\beta)d\beta \\ X(\alpha,0) &= \alpha\ . \end{cases}$$

Blob method for the 2D Euler equations

Steps for blob method:

1. Remove the singularity of K by convolution with a mollifier. Write $K_\delta = K * \psi_\delta$.
2. Replace ω_0 with a particle approximation on the grid $h\mathbb{Z}^d$.

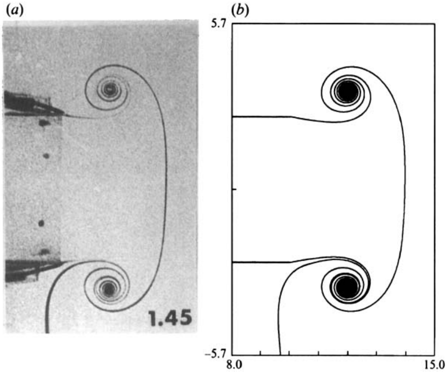
$$\omega_0^{\text{particle}}(y) = \sum_{j \in \mathbb{Z}^d} \delta(y - jh)\omega_{0j} h^d$$

Exact Particle Trajectories: $$\begin{cases} \frac{d}{dt} X(\alpha, t) & = \int K(X(\alpha, t) - X(\beta, t))\omega_0(\beta) d\beta \\ X(\alpha, 0) & = \alpha \ . \end{cases}$$

Approx Particle Trajectories: $$\begin{cases} \frac{d}{dt} \tilde{X}_i(t) & = \sum_j K_\delta(\tilde{X}_i(t) - \tilde{X}_j(t))\omega_{0j} h^d \\ \tilde{X}_i(0) & = ih \ . \end{cases}$$

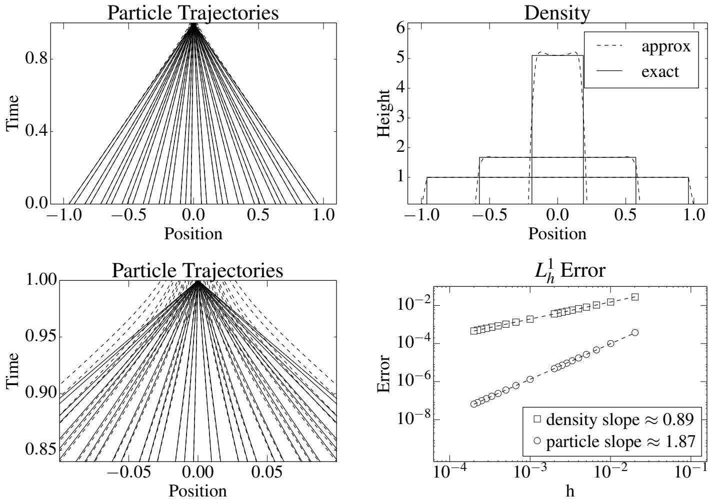
Blob Method for the 2D Euler Equations

- First used by [Chorin,1973]
- [Hald, Del Prete 1978] proved 2D convergence
- [Hald, 1979] proved second order convergence in 2D for arbitrary time intervals $[0, T]$
- [Beale, Majda 1982] proved convergence in 2D and 3D with arbitrarily high-order accuracy
- [Cottet, Raviart 1984] simplified 2D and 3D convergence proofs
- [Anderson, Greengard 1985] modified the 3D blob method, considered time discretization



Comparison at t = 1.45. (a) Experiment, from Didden (1979). (b) Simulation, $\delta = 0.2$. [Nitsche, Krasny 1994]

Newtonian potential, one dimension: patch initial data



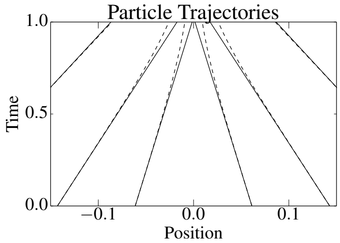
$h = 0.04$
$q = 0.9$
$m = 4$

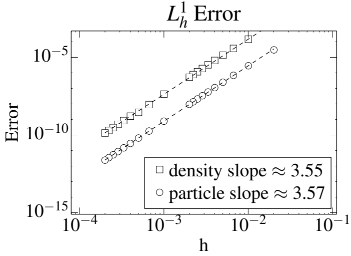
$\rho_0(x) = 1_{[-1,1]}$

blowup: $t = 1$

- particle trajectories bend, densities round
- lower order accuracy (≈ 0.9) compared to regular initial data (≈ 3.6)

Newtonian potential, one dimension: blob vs. particle

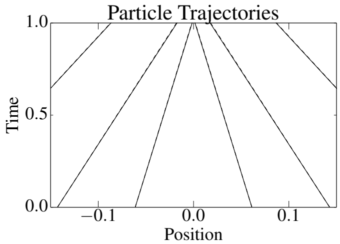


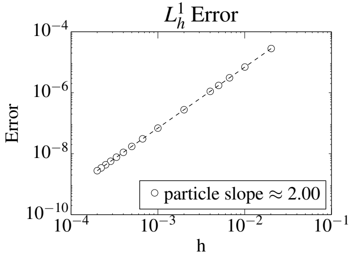


$h = 0.04$
$q = 0.9$
$m = 4$

$\rho_0(x) = (1 - x^2)_+^{20}$

blowup: $t = 1$





- blob has higher order accuracy (≈ 3.6) compared to particle (≈ 2)
- trajectories computed by pure particle method collide at blowup time