Math 164: Homework 2

Due Friday, April 10th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1* (Friedburg, *Linear Algebra*, "The Rank of a Matrix and Matrix Inverses" and "Systems of Linear Equations")

Consider the matrix and vector

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- What is the rank of A? Give a basis for the row space and a basis for the column space.
- Show that A has full row rank. (An $m \times n$ matrix A has *full row rank* if is has rank m.) This is a quick consequence of the previous part.
- If an $m \times n$ matrix A has full row rank, then for all vectors b of length m, there exists a solution to $Ax = b$. Find all solutions to $Ax = b$.
- Is it possible for an $m \times n$ matrix to have full row rank if $m > n$ (i.e. if the matrix is taller than it is wide)?

Question 2* (Similar to Textbook Problem 2.3.3)

Consider a feasible region S defined by a set of linear constraints

$$S = \{x : Ax \geq b\}.$$

(Note that we have $Ax \geq b$, **not** $Ax \leq b$.) Prove that for any $m \times n$ matrix A and any $b \in \mathbb{R}^m$, S is convex.

Question 3 (Textbook Problem 2.3.4)

Prove that a function f is concave if and only if $-f$ is convex.

Question 4 (Textbook Problem 2.3.13)

Let f be a convex function on the convex set S. Prove that the set

$$T = \{x \in S : f(x) \leq k\}$$

is convex for all real numbers k. Draw a picture of this set in the case $f(x) = x^2$ and $k = 1$.

Question 5* (Similar to Textbook Problem 2.3.18)

Express $(4, 4)^T$ as a convex combination of $(0, 0)^T$, $(2, 8)^T$, and $(6, 2)^T$.

Question 6 (Textbook Problem 2.3.8)

Let f be a function on a convex set $S \subseteq \mathbb{R}^d$. Prove that f is convex if and only if

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) \le \sum_{i=1}^{k} \alpha_i f(x_i)$$

for all $x_1, \ldots, x_k \in S$ and $0 \le \alpha_i \le 1$ satisfying $\sum_{i=1}^{k} \alpha_i = 1$.

Question 7 (Textbook Problem 2.3.9)

Prove the famous arithmetic-geometric mean inequality: for $x_1, \ldots, x_k \ge 0$,

$$(x_1 + \cdots + x_k)/k \ge (x_1 \cdot \cdots \cdot x_k)^{1/k} .$$

The left hand side is the *arithmetic mean* (or average) of $x_1, \ldots, x_k$. The right hand side is the *geometric mean* of $x_1, \ldots, x_k$. If you want to impress your friends, start using the phrase "geometric mean" in casual conversation. (Hint: show $f(x) = -\log(x)$ is convex on $S = (0, +\infty)$ and apply the previous problem. You may assume that $f \in C^2(R; R)$ is convex iff $f''(x) \ge 0$.)

Question 8* (Textbook Problem 2.2.7)

Let $f(x)$ be any function and let S be the set of integers. Prove that every point in S is a local minimizer of f.

Question 9* (Textbook Problem 2.3.1)

Prove that the intersection of two convex sets is also a convex set. Then cut this part: use this to prove that the intersection of a finite number of convex sets is also a convex set.

Question 10 (Textbook Problem 2.3.1)

Let $S_1 = \{x : x_1 + x_2 \le 1, x_1 \ge 0\}$ and $S_2 = \{x : x_1 - x_2 \ge 0, x_1 \le 1\}$, and let $S = S_1 \cup S_2$. Prove that S_1 and S_2 are both convex sets but S is not a convex set. This shows that the union of convex sets is not necessarily convex.