Math 164: Homework 4

(Due Friday, April 24th)

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1 (Textbook Problem 4.3.1)

Consider the system of linear constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \ . \end{aligned}$$

(a) Write this system of constraints in standard form, and determine all the basic solutions (feasible and infeasible).

(b) Determine the extreme points of the feasible region (corresponding to both the standard form of the constraints, as well as the original version).

Question 2* (Similar to Textbook Problem 4.3.4)

Consider the problem

$$\begin{aligned} \text{minimize } & z = -5x_1 - 7x_2 \ , \\ \text{subject to } & -3x_1 + 2x_2 \leq 30 \\ & -2x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

(a) Draw a graph of the feasible region.

(b) Determine the extreme points of the feasible region.

(c) Determine two linearly independent directions of unboundedness.

(d) Represent the point $x = (6, 12)^T$ as a convex combination of extreme points plus, if applicable, a direction of unboundedness.

(e) Show that this problem has no minimizer.

(f) Convert the linear program to standard form and determine the basic feasible solutions and two linearly independent directions of unboundedness for this version of the problem. Verify that the directions of unboundedness satisfy $Ad = 0$ and $d \geq 0$.

Question 3* (Textbook Problem 4.3.5)

Consider a linear program with the constraints in standard form

$$Ax = b \text{ and } x \geq 0 \ .$$

Prove that if $d \neq 0$ satisfies

$$Ad = 0 \text{ and } d \geq 0 \ ,$$

then d is a direction of unboundedness.

Question 4* (Textbook Problem 4.3.9)

Consider a linear program with the following constraints:

$$\begin{aligned} 4x_1 + 7x_2 + 2x_3 - 3x_4 + x_5 + 4x_6 &= 4 \\ -x_1 - 2x_2 + x_3 + x_4 - x_6 &= -1 \\ x_2 - 3x_3 - x_4 - x_5 + 2x_6 &= 0 \\ x_i \geq 0 \ , \quad i = 1, \ldots, 6 \ . \end{aligned}$$

Determine every basis that corresponds to the basic feasible solution $(0, 1, 0, 1, 0, 0)^T$.

Question 5 (Textbook Problem 4.1.1)

Let x be a feasible point for the constraints $Ax = b, x \geq 0$ that is not an extreme point. Prove that there exists a vector $p \neq 0$ satisfying

$$Ap = 0 \ , \quad p_i = 0 \text{ if } x_i = 0 \ .$$

Question 6 (Textbook Problem 4.4.4)

Let p be a direction of unboundedness for the constraints

$$Ax = b \ , \quad x \geq 0 \ .$$

Prove that $-p$ cannot be a **nonzero** direction of unboundedness for these constraints.

Question 7 (Similar to Textbook Problem 4.4.5)

Let $\{d_1, \ldots, d_k\}$ be directions of unboundedness for the constraints $Ax = b$, $x \geq 0$. Prove that

$$d = \sum_{i=1}^{k} \alpha_i d_i \text{ with } \alpha_i \geq 0$$

is also a direction of unboundedness for these constraints.

Question 8 (Similar to Textbook Problem 4.4.6)

Consider the linear program

$$\begin{aligned} \text{minimize } z = 2x_1 - 3x_2 \ , \\ \text{subject to } 6x_1 + 8x_2 \leq 24, \\ x_2 - 2x_1 \leq 2 \ , \\ x_1, x_2 \geq 0 \ . \end{aligned}$$

Represent the point $x = (1, 1)^T$ as a convex combination of extreme points, plus, if applicable, a direction of unboundedness. Find two different representations.

Question 9*

Consider the linear program in standard form:

$$\begin{aligned} \text{minimize } & z = c^T x \ , \\ \text{subject to } & Ax = b \ , \\ & x \geq 0 \ . \end{aligned}$$

Suppose that for some x in the feasible region and direction of unboundedness d, the point $x + d$ is a minimizer. Show that we must have $c^T d = 0$. (Hint: show that if $c^T d < 0$, the objective function is unbounded below and if $c^T d > 0$, $x + d$ could not have been the minimizer.)

Question 10* (inspired by Prof. Yinye Ye's online lecture notes)

In class, we discussed an algorithm by which any linear program can be put into standard form. In a sense, the original linear program is *equivalent* to the linear program written in standard form because the manipulations we performed did not change the underlying *meaning* of the linear program. In general, we will say that two linear programs are *equivalent* if the manipulations that you use to go from the first LP to the second LP can be "unwrapped" or "reversed" to go back from the second to the first. In this problem, you will prove this equivalence for a few of the manipulations we used.

(a) Suppose $S \subseteq \mathbb{R}^n$. Show that $x = (x_1, \ldots, x_n)^t$ is a solution to the first mathematical program if and only if x is a solution to the second mathematical program.

$$\begin{aligned} \text{minimize } & z = f(x) \\ \text{subject to } & x \in S \end{aligned} \qquad\qquad \begin{aligned} \text{maximize } & z = -f(x) \\ \text{subject to } & x \in S \end{aligned}$$

(b) Suppose $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$ and $S \subseteq \mathbb{R}^n$. Show that $x = (x_1, \ldots, x_n)^t$ is a solution to the first mathematical program if and only if $x' = (x_1, x_2, \ldots, x_n, s)$ is a solution to the second mathematical program for some $s \geq 0$.

$$\begin{aligned} \text{minimize } & z = f(x) \\ \text{subject to } & a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b \\ & x \in S \end{aligned} \qquad \begin{aligned} \text{minimize } & z = f(x) \\ \text{subject to } & a_1x_1 + a_2x_2 + \cdots + a_nx_n - s = b \\ & x \in S \\ & s \geq 0 \end{aligned}$$

(c) Suppose $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$ and $S \subseteq \mathbb{R}^n$. Show that $x = (x_1, \ldots, x_n)^t$ is a solution to the first mathematical program if and only if $x' = (x_1, x_2, \ldots, x_n, s)$ is a solution to the second mathematical program for some $s \geq 0$.

$$\begin{aligned} \text{minimize } & z = f(x) \\ \text{subject to } & a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b \\ & x \in S \end{aligned} \qquad \begin{aligned} \text{minimize } & z = f(x) \\ \text{subject to } & a_1x_1 + a_2x_2 + \cdots + a_nx_n + s = b \\ & x \in S \\ & s \geq 0 \end{aligned}$$

(d) Suppose $S \subseteq \mathbb{R}^n$. Show that $x = (x_1, \ldots, x_n)^t$ is a solution to the first mathematical program if and only if $x' = (x_1', x_2, \ldots, x_n, s)$ is a solution to the second mathematical program, where $x_1' = x_1 - 2$.

$$\begin{aligned} \text{minimize } & z = f(x_1, x_2, \ldots, x_n) \\ \text{subject to } & (x_1, x_2, \ldots, x_n)^t \in S \\ & x_1 \geq 2 \end{aligned} \qquad \begin{aligned} \text{minimize } & z = f(x_1' + 2, x_2, \ldots, x_n) \\ \text{subject to } & (x_1' + 2, x_2, \ldots, x_n)^t \in S \\ & x_1' \geq 0 \end{aligned}$$