Math 164: Homework 5

Due Friday, May 8th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1* (Similar to Textbook Problem 5.2.1)

Consider the following linear program:

$$\begin{aligned} \text{maximize } & z = 8x_1 + 6x_2 \ , \\ \text{subject to } & x_1 + 4x_2 \leq 100, \\ & x_1 + x_2 \leq 80 \ , \\ & x_2 \leq 40 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

Solve the linear program using the simplex method, starting with the basic feasible solution corresponding to the point $[0,0]^t$. Along the way, do the following things...

(a) Graph the feasible region and outline the progress of the algorithm.

(b) Verify that the dictionary corresponding to the first basic feasible solution is given by the formula

$$x_B = B^{-1}(b - Nx_N) \ .$$

(c) Verify that, at the last basic feasible solution, the objective function may be expressed in terms of nonbasic variables according to the formula

$$z = c_B^t B^{-1} b + (c_N^t - c_B^t B^{-1} N) x_N \ .$$

Show that the reduced cost vector satisfies $(c_N^t - c_B^t B^{-1} N) \geq 0$, i.e. the optimality criterion is satisfied.

Question 2* (Similar to Textbook Problem 5.2.1)

Solve the following linear program using the simplex method, starting with the basic feasible solution corresponding to the point $[0,0]^t$. Graph the feasible region and outline the progress of the algorithm.

$$\begin{aligned} \text{minimize } & z = x_2 - 2x_1 \ , \\ \text{subject to } & x_1 - x_2 \leq 1, \\ & -2x_1 + x_2 \leq 2 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

Question 3 (Textbook Problem 5.2.1 (iv))

Solve the following linear program using the simplex method, starting with the basic feasible solution of your choice.

$$\begin{aligned} \text{minimize } & z = 3x_1 - 2x_2 - 4x_3 \ , \\ \text{subject to } & 4x_1 + 5x_2 - 2x_3 \leq 22, \\ & x_1 - 2x_2 + x_3 \leq 30 \ , \\ & x_1, x_2, x_3 \geq 0 \ . \end{aligned}$$

Question 4 (Textbook Problem 5.2.4)

Final all the values of the parameter a such that the following linear program has a finite optimal solution:

$$\begin{aligned} \text{minimize } & z = -ax_1 + 4x_2 + 5x_3 - 3x_4 \ , \\ \text{subject to } & 2x_1 + x_2 - 7x_3 - x_4 = 2, \\ & x_1, x_2, x_3, x_4 \geq 0 \ . \end{aligned}$$

Question 5*

The following enhancement of the simplex method is known as **Bland's rule**.

Suppose the variables of a linear program have a natural ordering $\{x_1, x_2, \ldots, x_n\}$. *At each iteration of the simplex method, choose the entering variable as the first variable from this list for which the reduced cost is strictly negative. Then, among all the potential leaving variables that give the minimum ratio in the ratio test, choose the one that appears first in this list.*

Consider the linear program

$$\begin{aligned} \text{minimize } & z = -x_1 - x_2 \ , \\ \text{subject to } & x_1 \leq 2, \\ & x_1 + x_2 \leq 2 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

(a) Put the linear program in standard form by adding slack variables $\{x_3, x_4\}$.

(b) Consider the set of basic variables given by the slack variables. What is the corresponding basic feasible solution?

(c) Write the objective function in terms of the nonbasic variables. Which variable should enter the basis, according to Bland's rule? What is the corresponding adjacent basic feasible solution?

(d) According to Bland's rule, which variable leaves the set of basic variables?

Question 6 (Textbook Problem 6.1.1)

Find the dual of

$$\begin{aligned} \text{minimize } & z = 3x_1 - 9x_2 + 5x_3 - 6x_4 \ , \\ \text{subject to } & 4x_1 + 3x_2 + 5x_3 + 8x_4 \geq 24, \\ & 2x_1 - 7x_2 - 4x_3 - 6x_4 \geq 17 \ , \\ & x_1, x_2, x_3, x_4 \geq 0 \ . \end{aligned}$$

Question 7* (Similar to Textbook Problem 6.1.3)

Find the dual of

$$\begin{aligned}\text{maximize } & z = -3x_1 + 6x_2 + 5x_3 - 2x_4 \ ,\\ \text{subject to } & 3x_1 + 4x_2 + 7x_3 - 8x_4 = 11,\\ & 2x_1 + 3x_2 + 6x_3 + 7x_4 \geq 23 \ ,\\ & 4x_1 + 7x_2 + 2x_3 + 3x_4 \leq 12 \ ,\\ & x_1, x_2 \geq 0 \ , x_3 \text{ free}, x_4 \leq 0 \ .\end{aligned}$$

Question 8 (Textbook Problem 6.1.6)

Find the dual to the problem

$$\begin{aligned}\text{minimize } & z = c^T x \ ,\\ \text{subject to } & b_1 \leq Ax \leq b_2 \ ,\\ & x \geq 0 \ .\end{aligned}$$