Math 164: Practice Midterm 1

Question 1

Consider the following linear program:

$$\begin{aligned} \text{minimize } z &= 3x_1 + x_2 \ , \\ \text{subject to } x_1 + 2x_2 &\le 4 \ , \\ x_2 &\le 2 \ , \\ x_2 &\ge 0 \ , \\ x_1 &\text{ free.} \end{aligned}$$

In standard form, this becomes

$$\begin{aligned} \text{minimize } z &= 3x_1' - 3x_1'' + x_2 \ , \\ \text{subject to } x_1' - x_1'' + 2x_2 + x_3 &= 4 \ , \\ x_2 + x_4 &= 2 \ , \\ x_1', x_1'', x_2, x_3, x_4 &\ge 0 \ . \end{aligned}$$

(a) Graph the feasible region of the linear program in its original form.

(b) Find a basic feasible solution corresponding to the set of basic variables $\{x_1', x_4\}$.

(c) Clearly mark the point on your graph that this basic feasible solution corresponds to.

(d) In the standard form coordinates, show that $[0, 2, 0, 2, 0]$ is a direction of unboundedness.

(e) What direction does $[0, 2, 0, 2, 0]$ correspond to in your graph of the feasible region?

Question 2

Consider the following linear program:

$$\begin{aligned} \text{minimize } z &= f(x) \ , \\ \text{subject to } x_1 - 2x_2 + x_3 &\ge 1 \ , \\ x_2 - x_3 &\ge 0 \ , \\ 2x_1 + x_2 &\ge 2 \ , \\ x_1, x_2, x_3 &\ge 0 \ . \end{aligned}$$

(a) Show that $x = (2, 1, 1)^T$ is a feasible solution.

(b) Show that $p = (0, -2, -3)^T$ is a feasible direction at $x = (2, 1, 1)^T$.

(c) Find the minimum value of p_1 so that $p = (p_1, -2, -3)^t$ is a feasible direction at $x = (2, 1, 1)^T$.

(d) For x and p as above, determine the maximal step length α so that $x + \alpha p$ is feasible.

Question 3

(a) State the definition of a convex set.

(b) Prove that the feasible region of a linear program in standard form is convex, i.e. show that if A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then $S := \{x \in \mathbb{R}^n : Ax = b, x \ge 0\}$ is convex.

Question 4

Convert the following linear program to standard form:

$$\begin{aligned} \text{minimize } & z = -x_1 + 2x_2 \ , \\ \text{subject to } & 5x_1 + 2x_2 \geq 10 \ , \\ & 2x_1 + 3x_2 \leq 40 \ , \\ & x_1 \leq 15 \ , \\ & x_2 \leq 15 \ . \end{aligned}$$

Question 5

Solve the following linear program graphically, i.e. find a minimizer or show that none exists.

$$\begin{aligned} \text{minimize } & z = -x_1 - x_2 \ , \\ \text{subject to } & x_1 + 2x_2 \leq 8 \ , \\ & x_1 \geq -2 \ , \\ & x_2 \geq 0 \ . \end{aligned}$$

Question 6

Consider the following linear program:

$$\begin{aligned} \text{minimize } & z = x_1 + x_2 \ , \\ \text{subject to } & x_1 - 2x_2 \leq 1 \ , \\ & -x_1 + x_2 \leq 1 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

Solve the linear program using the simplex method, starting at the basic feasible solution corresponding to point $x_1 = 0, x_2 = 1$.

At each step of the simplex method, be sure to indicate...

(i) the current basic feasible solution

(ii) the dictionary corresponding to the basic feasible solution (i.e. express the basic variables in terms of the nonbasic variables)

(iii) why you choose to move to another basic feasible solution/why you choose to stop because the current solution is optimal.