Math 164: Practice Midterm

Question 1 (28 points)

(a) (6 points) Prove the following version of the weak duality theorem for a linear program in canonical form. Make clear where you use $x \geq 0$ and $y \geq 0$.

Theorem: *Given a linear program in canonical form, if x if feasible for the primal linear program and y is feasible for the dual, then $c^t x \geq b^t y$.*

A corollary of this theorem is the following:

Corollary: *If the primal is unbounded, then the dual is infeasible. If the dual is unbounded, then the primal is infeasible.*

You do not need to prove this corollary.

(b) (4 points) Consider the following linear program.

$$\begin{aligned} \text{minimize } & z = -x_1 + 3x_2 \ , \\ \text{subject to } & -x_1 + x_2 \geq 2 \ , \\ & -2x_1 + x_2 \geq -2 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

Show that $d = [0, 1]^T$ is a direction of unboundedness.

(c) (4 points) What is the dual linear program?

(d) (6 points) Does the dual linear program have a finite optimal solution? Name any theorems you use.

(e) (8 points) True or false: if a linear program in canonical form has a nonzero direction of unboundedness, then the dual linear program does not have a finite optimal solution. If true, prove it. If false, give a counterexample.

Question 2 (26 points)

Consider the following primal and dual linear programs.

$$\begin{aligned} \text{minimize } & z = -x_2 \ , \\ \text{subject to } & x_1 - 2x_2 + x_3 = 2 \ , \\ & x_1 - x_2 + x_4 = 3 \ , \\ & x_2 + x_5 = 3 \ , \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \ . \end{aligned}$$

$$\begin{aligned} \text{maximize } & w = 2y_1 + 3y_2 + 3y_3 \ , \\ \text{subject to } & y_1 + y_2 \leq 0 \ , \\ & -2y_1 - y_2 + y_3 \leq -1 \ , \\ & y_1 \leq 0 \ , \\ & y_2 \leq 0 \ , \\ & y_3 \leq 0 \ . \end{aligned}$$

Note that $x_a = [0, 3, 8, 6, 0]^T$ is feasible for the primal and $y_a = [0, 0, -1]^T$ is feasible for the dual.

(a) (6 points) Show that x_a is optimal for the primal and y_a is optimal for the dual. Name any theorems you use.

(b) (6 points) Show $x_b = [6, 3, 2, 0, 0]^T$ is also a basic feasible solution for the primal.

(c) (6 points) Using reduced costs, show that x_b is an optimal basic feasible solution. (Note: you do not *need* to use the formula $c_N^t - c_B^t B^{-1} N$ for the reduced costs unless you want to.)

(d) (8 points) Use parts (a) and (d) to show there are infinitely many optimal solutions to the primal problem. Name any theorems you use.

Question 3 (22 points)

Consider the following linear program:

$$\begin{aligned} \text{minimize } z &= x_1 + x_2 \ , \\ \text{subject to } x_1 - 2x_2 &\le 1 \ , \\ -x_1 + x_2 &\le 1 \ , \\ x_1, x_2 &\ge 0 \ . \end{aligned}$$

Solve the linear program using the simplex method, starting at the basic feasible solution corresponding to point $x_1 = 0, x_2 = 1$.

At each step of the simplex method, be sure to indicate...

(i) the current basic feasible solution

(ii) the dictionary corresponding to the basic feasible solution (i.e. express the basic variables in terms of the nonbasic variables)

(iii) why you choose to move to another basic feasible solution/why you choose to stop because the current solution is optimal.

Question 4 (8 points)

Consider the following primal and dual linear programs.

$$\begin{aligned} \text{minimize } z &= -3x_1 - x_2 \ , \\ \text{subject to } x_1 + x_2 + x_3 &= 6 \ , \\ 4x_1 + x_2 + x_4 &= 12 \ , \\ x_1, x_2, x_3, x_4 &\ge 0 \ . \end{aligned}$$

$$\begin{aligned} \text{maximize } w &= 6y_1 + 12y_2 \ , \\ \text{subject to } y_1 + 4y_2 &\le -3 \ , \\ y_1 + y_2 &\le -1 \ , \\ y_1 &\le 0 \ , \\ y_2 &\le 0 \ . \end{aligned}$$

The optimal solution to the primal is $x^* = [2, 4, 0, 0]^t$. Find the optimal solution to the dual using complementary slackness.

Question 5 (16 points)

Consider the following linear program

$$\begin{aligned} \text{minimize } z &= -ax_1 + 4x_2 + 5x_3 - 3x_4 \ , \\ \text{subject to } 2x_1 + bx_2 - 7x_3 - x_4 &= c, \\ x_1, x_2, x_3, x_4 &\ge 0 \ . \end{aligned}$$

(a) (4 points) Let $b = 1, c = 2$. Final all the values of the parameter a such that the following linear program has a finite optimal solution. Name any theorems you use. (Hint: there are many ways to solve this problem, but one way is to use duality.)

(b) (4 points) Let b and c be as in part (a). For all values of the parameter a so that a finite optimal solution exists, what is the optimal value of the objective function? (Hint: the objective function has different optimal values for different values of a.)

(c) (4 points) Now let $b = 1, c = -2$. For what values of a is there a finite optimal solution? What is the optimal value of the objective function?

(d) (4 points) Now let $b = 0, c = 2$. For what values of a is there a finite optimal solution? What is the optimal value of the objective function?

Question 5