The sphere is the simplest example of a compact manifold, and it has positive (sectional) curvature when regarded as a Riemannian manifold. It is therefore natural to study positive curvature, and doing so has fostered the development of Riemannian geometry throughout its history. In view of this, it is striking that the simplest example, the sphere, remains the only compact, simply connected manifold with odd dimension above 24 known to admit positive curvature. The shortage of examples extends to even dimensions, where only the complex and quaternionic projective spaces must be added to the list. On the other hand, there are very few known obstructions to a manifold admitting positive curvature (see [24] for a recent survey).

About two decades ago, Karsten Grove proposed a program to classify manifolds admitting positively curved metrics with a large degree of symmetry. This flexible program has greatly increased our understanding of positive curvature. In fact, a partial classification recently led to the discovery of a new 7–manifold that admits positive curvature, the first such discovery to be published since 1996 (see [6, 11]).

This program has also spurred the development of new methods that draw on analysis, topology, and the theory of transformation groups. My work has introduced further topological tools and applied them to better our understanding of positive curvature. These tools include Steenrod squares and cohomology operations in general. My collaborators and I have also had success applying in new ways results from rational homotopy theory, surgery theory, and group cohomology. I look forward to communicating with experts in all of these areas to examine conjectures my collaborators and I have formulated, both in topology and geometry.

I focus here on four aspects of my research program. The first two involve computing invariants of manifolds admitting positive curvature and symmetry. This was the focus of my thesis, but the work continues today, partly in collaboration with Manuel Amann (KIT, Germany). The third fits into the broader context of classifying positively curved manifolds and is based on a generalization of positive curvature I developed with William Wylie (Syracuse University). The final aspect discusses joint work with Zhixu Su (Indiana University) on a topological realization problem.

1. Euler characteristics

In the 1930s, Hopf conjectured that a compact, even-dimensional Riemannian manifold with positive sectional curvature (a local property) has positive Euler characteristic (a global property). This conjecture holds in dimensions two and four by classical theorems of Gauss–Bonnet, Synge, or Bonnet–Myers, but it remains open in general.

In the context of the Grove program, there are multiple verifications of the Hopf conjecture under sufficiently strong symmetry assumptions. Usually, a torus T^r is assumed to act (effectively) by isometries on the manifold M^n, and the rank r is assumed to be at least a linear function of n (e.g., see [16, 18]). In my thesis, as refined and published in *Geometry & Topology* (see [13]), I prove that a logarithmic function of n also suffices when n is divisible by four.

Theorem 1.1. *Let M^{4k} be a closed Riemannian manifold with positive sectional curvature. If M admits an isometric T^r–action with $r \geq 2\log_2(4k)$, then $\chi(M) > 0$.*

In order to replace a linear assumption by a logarithmic one, new tools had to be developed. One of them builds upon a remarkable result of Burkhard Wilking called the *connectedness lemma* (see [22]), which imposes strong topological restrictions in the presence of positive curvature and symmetry. In special situations, it implies that the manifold in question has *periodic cohomology.*

In my thesis, I analyzed manifolds with periodic cohomology from the point of view of homotopy theory. The main contribution is to introduce the action of the Steenrod algebra and use it to refine this periodicity. The main result generalizes a theorem of J. Adem [2]. I state it here in the special case of simply connected, compact manifolds:

Theorem 1.2. *Let M^n be a compact, simply connected manifold. If $H^*(M;\mathbb{Z}_2)$ is k–periodic for some $k \leq \frac{n}{2}$, then the minimal such k is a power of 2.*

Here, k–periodic cohomology means that the (finite) sequence of cohomology groups is k–periodic. More, it means that a fixed element in degree k induces these isomorphisms via multiplication. The basic examples of periodic cohomology rings are those generated by a single element, and this special case of Theorem 1.2 is Adem's result.

I conjecture that Theorem 1.2 could be improved to state that $k \in \{2, 4, 8\}$. This would imply that Theorem 1.1 holds in all even dimensions. It also might be of interest in its own right. Indeed, it would strengthen Theorem 1.2 in the same way that J.F. Adams' theorem on singly generated cohomology rings generalized Adem's (see [1]). Adams' result had many applications in topology, the most famous of which being the resolution of the Hopf invariant one problem. Moreover, Adams's proof required him to develop the theory of secondary cohomology operations. Just as my result and Adem's used Steenrod operations, I believe that my conjecture, like Adams' theorem, might be proven using these higher operations. This project is ongoing, and I will continue to advertise this conjecture because of my interest in it and its applications.

Theorem 1.2 has seen further applications to the Grove program. In [14], which will appear in *Commentarii Mathematici Helvetici*, we prove estimates on Betti numbers.

Theorem 1.3. *Under the conditions of Theorem 1.1, if the rank $r \geq 2\log_2(4k) + C$ for some constant $C \geq 1$, then the Betti numbers satisfy:*

(1) $b_i(M) \leq b_4(M) \leq 1$ *for all even* $i \leq 2C + 2$.
(2) $b_i(M) = 0$ *for all odd* $i \leq 2C + 1$.

Note that both conclusions are sharp, since the sphere $\mathbb{S}^{4k}$, complex projective space $\mathbb{CP}^{2k}$, and quaternionic projective space $\mathbb{HP}^k$ admit metrics as in the theorem.

Theorem 1.3 has many consequences. For example, if the manifold M is an irreducible symmetric space and $C \geq 7$, then M has rank at most three (see [14]). This supports a generalized conjecture of Hopf that the only symmetric spaces admitting positive curvature are those with rank one. For a similar application that obstructs the existence of positively curved metrics on products $N \times N$, please see [5].

Early versions of Theorems 1.1 and 1.3 are contained in my thesis. Subsequent work has addressed the task of improving these *estimates* on $\chi(M)$ to *calculations*. Based on the three just-mentioned examples, we have the following:

Conjecture 1.4. *If M^n is a closed, simply connected Riemannian manifold with positive sectional curvature and an isometric torus action of rank exceeding $3\log_2 n$, then the Euler characteristic of M equals that of $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, or $\mathbb{HP}^{n/4}$.*

The motivation for this was an inspiring conversation with Wilking at a meeting in Oberwolfach. The goal seemed reasonble given the machinery from his work and mine. It seemed more reasonable given a clever technical trick he suggested.

In the summer of 2012, I began collaborating with Manuel Amann (KIT). In our first paper [3], which has been published in *Geometric and Functional Analysis*, we further refine the applications of the periodicity results in this context, develop Wilking's trick, and prove the following (see also [4], where we apply rational homotopy theory to obtain better results in the special case of rationally elliptic manifolds).

Theorem 1.5. *If M^n is a closed Riemannian manifold with positive sectional curvature and an isometric torus action of rank $r \geq 3\log_2 n$, then $\chi(M) < 2^{3(\log_2 n)^2}$.*

Note that the bound is sub-exponential in n. This suggests a striking contrast with manifolds of non-negative curvature, where there exist examples such as products of spheres with Euler characteristics that grow exponentially in the dimension.

On the other hand, Theorem 1.5 is far from a calculation. While Conjecture 1.4 remains a goal, two examples of related, future projects with Amann are the following:

- (Small dimensions project) To build intuition for Conjecture 1.4, we have calculated the Euler characteristic and other invariants in small dimensions. Our philosophy is similar to Dessai's work in dimension 8 (see [7]). We hope to continue this work up to dimension 24, the last known to support a simply connected, positively curved manifold that is not $\mathbb{S}^n$, $\mathbb{CP}^n$, or $\mathbb{HP}^n$.
- (The case $b_4(M) = 0$) In [5], which has been submitted for publication, Amann and I prove Conjecture 1.4 in the special case where $b_4(M) = 0$. Our proof actually provides more information about M, and it suggests further classification problems, e.g., the computation of signature or the classification of cohomogeneity one manifolds that satisfy the assumptions of Conjecture 1.4.

2. Fundamental groups

Spherical space forms provide many examples of fundamental groups of positively curved manifolds. All of these groups share the property that every abelian subgroup is cyclic. S.-S. Chern asked in 1965 whether this property holds for all positively curved manifolds. Thirty years later, Krishnan Shankar discovered examples that answered Chern's question in the negative. In general, very little is known about fundamental group restrictions beyond the classical results of Synge and Bonnet–Myers. I discuss here two modifications of the Chern question with torus symmetry (see [15] and references therein for a more complete survey).

First, if the torus has rank $r > \frac{n+1}{6}$, where $n = \dim(M)$, then every abelian subgroup of $\pi_1(M)$ is indeed cyclic (see [9, 17]). In [15], I improve this result in large dimensions of the form $4k+1$ by showing that this conclusion holds if the torus has rank $r \geq \frac{n+1}{12} + 3$. The proof of this result applies in new ways tools from topology, including secondary cohomology operations and results from group cohomology and surgery theory. I expect that more progress could be made on this conjecture by communicating with experts in these areas.

Another natural modification of the Chern question is closely related to the classification of spherical space forms. In [15, Example 2.5], I construct examples which prove that the following is sharp with respect to the symmetry assumption:

Conjecture 2.1. *Let M^{2m-1} be a closed Riemannian manifold with positive sectional curvature. Denote the smallest prime divisor of m by p. If T^r acts effectively by isometries on M with $r \geq \frac{m}{p} + 1$, then $\pi_1(M)$ is cyclic.*

Wang stated an equivalent conjecture in [20] and proved it when $\tilde{M}$ is the round sphere. In [15], we prove the conjecture when $\tilde{M}$ is a homotopy sphere.

The conjecture also holds in dimensions for which $p \in \{2, 3\}$ (see [22, 9]). In [15], we prove the conjecture in a third infinite family of dimensions, namely, those of the form $49 + 60k$ with $k \geq 6$. I hope to extend this calculation to cover values of $k < 6$, and I believe this can be done by a brute-force approach, possibly using computers.

In addition, I hope to prove a weak version of Conjecture 2.1 for all primes p. The weak version has the conclusion that $\pi_1(M)$ contains a normal, cyclic subgroup of odd index less than p. The motivation is simply that we can prove it when $p = 5$, and it presents a more realistic task for $p > 5$.

3. Positive sectional curvature for manifolds with density

A positive function on a Riemannian manifold is called a density, as it can be interpreted as a mass density or probability distribution. Notions of scalar and Ricci curvature (introduced by Perelman and Lichnerowicz) for manifolds with density have been intensely studied, and have been applied with great success in optimal transport and Ricci flow. Recently William Wylie (Syracuse University) refined these notions by defining sectional curvature for manifolds with density and showing that many classical results in Riemannian geometry generalize to this setting (see [23]).

Last summer, Wylie and I formulated a notion of positive sectional curvature for manifolds with density. We call this condition *postive weighted sectional curvature* (PWSC), and it can be viewed as a generalization of positive sectional curvature (see [12]). We prove that the conditions of positive sectional curvature and PWSC share many properties. For example, Synge-type arguments using the second variation of energy formulas carry over to the case of PWSC. As a consequence, we have extended to the weighted case a large number of known obstructions to positive sectional curvature. Adding symmetry to the mix, we prove further results, including the following extension of the main result in Grove and Searle [10]:

Theorem 3.1 (Maximal symmetry rank)**.** *Let (M, g) be a Riemannian manifold with isometry group of rank r. If (M, g) admits PWSC, then $r \leq \lfloor \frac{n+1}{2} \rfloor$. Moreover, if M is simply connected, then equality holds only if M is homeomorphic to $\mathbb{S}^n$ or $\mathbb{CP}^{n/2}$.*

Looking forward, two directions we wish to pursue are the following.

- (Advancing the theory of PWSC) We are working to generalize Toponogov's triangle comparison theorem and the resulting convexity properties of distance functions. If we accomplish this, we should be able to generalize results which, in the classical case, follow from this convexity (e.g., diffeomorphism rigidity in the Grove–Searle result mentioned above). Our program here is analogous to the successful work of Wei and Wylie [21].
- (Examples of PWSC) In [12], Wylie and I construct examples of Riemannian manifolds with PWSC but not positive sectional curvature. These examples show that lower sectional curvature bounds can be "improved" by introducing a density. A natural question is the following: Given a manifold M that is not known to admit positive curvature, but which does admit a metric g with non-negative curvature, does M admit a density giving it PWSC? This question illustrates the potential power of the flexibility provided by the choice of density. We plan to study this question by studying examples with large symmetry or those which already have positive curvature almost everywhere.

4. Topological realization problems

A common theme in algebraic topology is the study of realization problems. A very basic example is, given a coefficient ring R and a graded algebra H^* over R, when is H^* realized as the cohomology $H^*(X; R)$ of a space X?

When R is the integers modulo two, Theorem 1.2 provides an answer for a new class of graded algebras H^*, and the conjecture following Theorem 1.2 would refine this answer. There are natural analogues of this result and conjecture for coefficients mod p, where p is an odd prime.

In this section, I describe joint work with Zhixu Su (Indiana University) on an analogue for rational cohomology in the category of smooth manifolds. The rational surgery realization theorem of Barge and Sullivan provides a general answer to this question. However, even in the simple special case of the polynomial ring $\mathbb{Q}[x]/(x^3)$, it is not always clear whether the rings can be realized as the rational cohomology of a smooth manifold. This case is studied in Su [19] and Fowler–Su [8], where they are called rational projective planes ($\mathbb{Q}$PP).

In [19], Su reformulates the Barge–Sullivan theorem to make it suitable for her problem. After restricting attention to the case of a $\mathbb{Q}$PP, Su proves that the question of their existence in some dimension is equivalent to the existence of a solution to a system of Diophantine equations. Applying these simplifications, Su proves the existence of a compact, simply connected smooth manifold M^{32} such that $H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^3)$. Note that there is no integral analogue by the work of Adams' [1].

In our joint work, we have further simplified the existence problem by replacing the system of Diophantine equations by a single equation. Unfortunately, this equation involves the mysterious Bernoulli numbers, so a full classification of dimensions supporting a $\mathbb{Q}$PP might be too difficult. While this remains our long-term goal, we hope to at least prove the existence of infinitely many dimensions that support a $\mathbb{Q}$PP. This would be an exciting application of our simplifications, and it would stand in sharp contrast to the analogous problem with integral or $\mathbb{Z}_p$ coefficients, where only finitely many dimensions support a cohomology projective plane.

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