Module Categories Over Categories of Modules Over Hopf Algebras

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Category Theory

Category theory is a branch of mathematics that studies mathematical objects by focusing on the maps between them. The most basic notion in category theory is a **category**, which consists of the following:

- A class of **objects**
- A class of **morphisms**, each of which goes between two objects
- An associative **composition** operation $\circ$ that takes a morphism $a \to b$ and a morphism $b \to c$ to obtain a morphism $a \to c$.
- Identity morphisms id_c on every object c satisfying $f \circ \mathrm{id}_c = \mathrm{id}_d \circ f = f$ for all morphisms $f : c \to d$.

For example, in the **category of sets**, objects are sets, and morphisms are functions between them. Composition of morphisms is just composition of functions.

An **isomorphism** is a morphism that has a left and right inverse, i.e. a morphism f such that there exists a morphism g such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. If there exists an isomorphism $f : a \to b$, we say a and b are **isomorphic**. For the purposes of category theory, isomorphic objects are the same in every way that we care about.

A **functor** $F : \mathcal{C} \to \mathcal{D}$ is a map that sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and sends each morphism $f : a \to b$ in $\mathcal{C}$ to a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{D}$. In order to be a functor, it should preserve composition, meaning $F(f \circ g) = F(f) \circ F(g)$.

If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, a **natural transformation** η from F to G is a collection of morphisms $\eta_a : F(a) \to G(a)$ such that for all morphisms $f : a \to b$, the following diagram commutes:

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"] & G(b). \end{tikzcd}$$

A diagram **commutes** if any two paths with matching start and end points that give you the same morphism after composing. In this case, the statement that the above diagram commutes is saying that $G(f) \circ \eta_a = \eta_b \circ F(f)$.

If each morphism η_a is an isomorphism, then we say that η is a **natural isomorphism** between F and G.

Monoidal Categories and Module Categories

To understand monoidal categories, consider the Cartesian product in the category of sets. It's almost associative, but not quite: technically $A \times (B \times C)$ contains tuples of the form $(a, (b, c))$ whereas $(A \times B) \times C$ contains tuples of the form $((a, b), c)$. While these aren't equal, they are naturally isomorphic as sets, i.e. $\times$ is associative up to natural isomorphism. This idea leads to the following definition:

A **monoidal category** is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $1_\mathcal{C} \in \mathcal{C}$, an **associator** natural isomorphism $\alpha_{a,b,c} : a \otimes (b \otimes c) \xrightarrow{\cong} (a \otimes b) \otimes c$, and **unitor** natural isomorphisms $\lambda_a : 1_\mathcal{C} \otimes a \xrightarrow{\cong} a$ and $\rho_a : a \otimes 1_\mathcal{C} \xrightarrow{\cong} a$ such that the following pentagon and triangle diagrams commute for all $a, b, c, d \in \mathcal{C}$:

$$\begin{tikzcd} a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha_{a,b,c\otimes d}"] \arrow[dd, "id_a \otimes \alpha_{b,c,d}"] & (a \otimes b) \otimes (c \otimes d) \arrow[d, "\alpha_{a\otimes b,c,d}"] \\ & ((a \otimes b) \otimes c) \otimes d \\ a \otimes ((b \otimes c) \otimes d) \arrow[r, "\alpha_{a,b\otimes c,d}"] & (a \otimes (b \otimes c)) \otimes d \arrow[u, "\alpha_{a,b,c} \otimes id_d"] \end{tikzcd}$$

$$\begin{tikzcd} a \otimes (1_\mathcal{C} \otimes b) \arrow[rr, "id_a \otimes \lambda_b"] \arrow[dr, "\alpha_{a,1_\mathcal{C},b}"'] & & a \otimes b \\ & (a \otimes 1_\mathcal{C}) \otimes b \arrow[ur, "\rho_a \otimes id_b"'] & \end{tikzcd}$$

A monoidal category is **strict** if $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ and $1_\mathcal{C} \otimes a = a = a \otimes 1_\mathcal{C}$ and $\alpha_{a,b,c}$, λ_a, and ρ_a are all the identity.

A **linear category** is a category where for all objects a and b, the set of morphisms from a to b is a finite-dimensional complex vector space and composition is a bilinear map.

A **linear monoidal category** is a linear category with a monoidal structure such that $\otimes$ is a bilinear map.

A **module category** over a linear monoidal category $\mathcal{C}$ is a linear category $\mathcal{A}$ equipped with the following additional data:

- A bilinear functor $\oslash : \mathcal{C} \times \mathcal{A} \to A$,
- A natural isomorphism $\lambda_a : 1 \oslash a \to a$,
- A natural isomorphism $\alpha_{c,d,a} : c \oslash (d \oslash a) \to (c \otimes d) \oslash a$.

Module categories must satisfy pentagon and triangle rules similar to those in the definition of a monoidal category.

Hopf Algebras

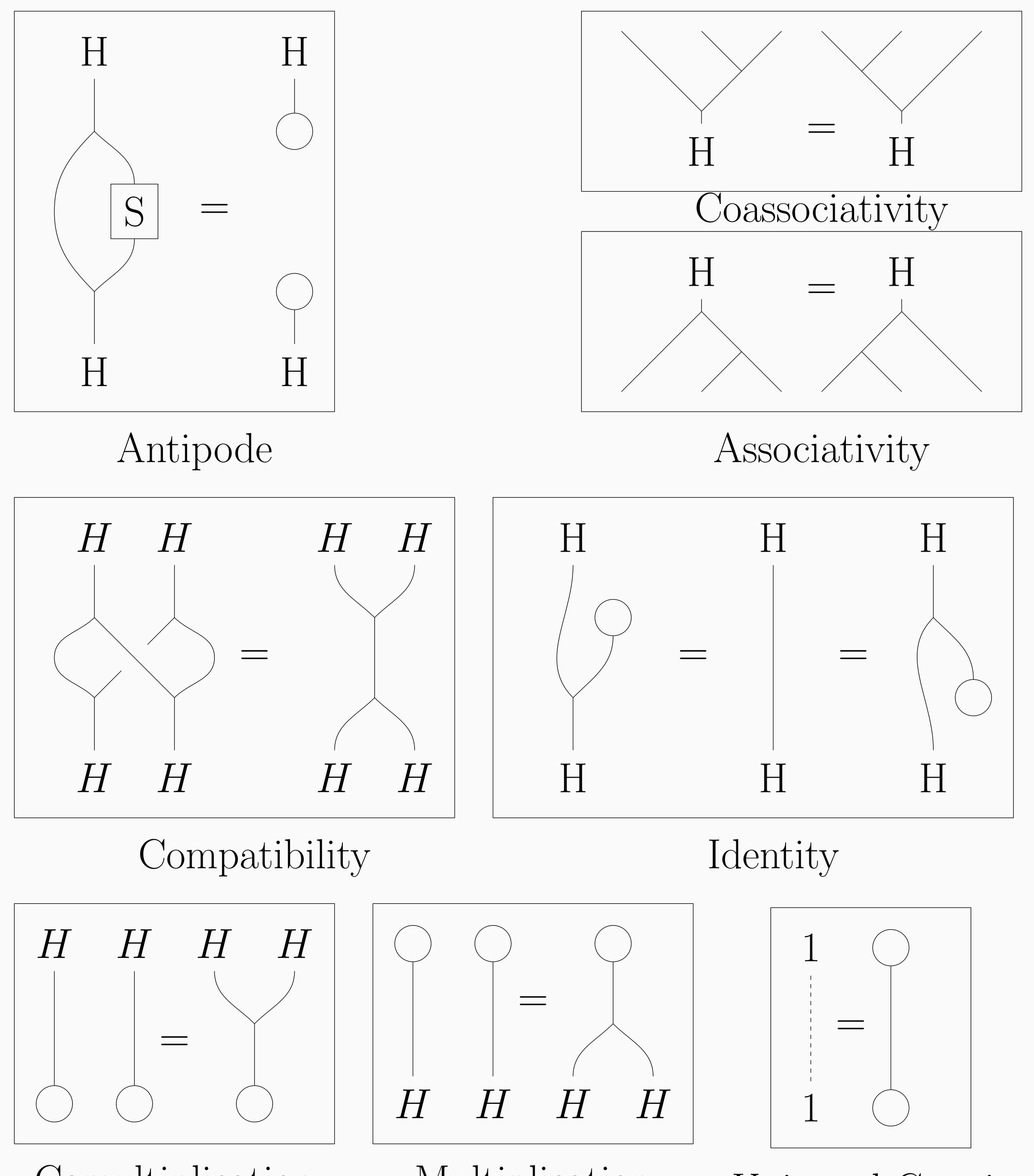
For two vector spaces V and W, their **tensor product** is the unique vector space $V \otimes W$ equipped with a bilinear map $\otimes : V \times W \to V \otimes W$ such that for any bilinear map $f : V \times W \to U$ there is a unique linear map $\tilde{f} : V \otimes W \to U$ satisfying $f(v, w) = \tilde{f}(v \otimes w)$ for all $(v, w) \in V \times W$.

An **associative algebra** over $\mathbb{C}$ is a $\mathbb{C}$-vector space A with a multiplication operation $\cdot : A \times A \to A$ such that A forms a ring under this multiplication, and $\lambda \cdot (ab) = (\lambda a) \cdot b = a \cdot (\lambda b)$ for all $\lambda \in \mathbb{C}$ and $a, b \in A$.

A **Hopf algebra** consists of a $\mathbb{C}$-vector space H with the following additional linear maps:

- A **multiplication** map $\nabla : H \otimes H \to H$ such that $g \cdot h := \nabla(g \otimes h)$ makes H into an associative algebra
- A **comultiplication** map $\Delta : H \to H \otimes H$
- An **antipode** map $S : H \to H$
- **Unit** and **counit** maps $\delta : \mathbb{C} \to H$ and $\varepsilon : H \to \mathbb{C}$ respectively

In order for these structures to form a Hopf algebra, we also require that they satisfy the equations expressed by the following string diagrams (and their mirror images):



Hopf Algebras continued

To interpret these string diagrams, one should read bottom to top. Each vertical strand represents the identity on H, and any two things next to each other indicate taking monoidal products. A Y-shaped branch indicates comultiplication and an upside-down Y-shape indicates multiplication. Finally, the circles represent the unit and counit respectively.

An important example of a family of Hopf algebras is the **generalized Taft algebras** $H_{n,d}(\zeta)$, parametrized by $n, d \in \mathbb{N}$ with d dividing n and $\zeta \in \mathbb{C}$ a primitive dth root of unity. As an algebra, $H_{n,d}(\zeta)$ is generated by two elements $x, g \in H_{n,d}(\zeta)$ that commute up to multiplication by ζ. As a complex vector space, $H_{n,d}(\zeta)$ is generated by the $n \cdot d$ elements $g^i h^j$ for $0 \leq i < n$ and $0 \leq j < d$, so in particular it is nd-dimensional.

Since every Hopf algebra is also an algebra and therefore a ring, we can talk about **modules** over a Hopf algebra in the usual way. Importantly, given two modules M and N over a Hopf algebra H, the tensor product of M and N as $\mathbb{C}$-vector spaces can be given the structure of an H-module by defining $h \cdot (m \otimes n) = \sum_{i=1}^n h_i^{(1)} m \otimes h_i^{(2)} n$ where $\Delta(h) = \sum_{i=1}^k h_i^{(1)} \otimes h_i^{(2)}$. With this $\otimes$ operation, the modules over H form an example of a monoidal category $\mathrm{Rep}(H)$.

An **algebra** in the category $\mathrm{Rep}(H)$ consists of an H-module A with a linear map $m : A \otimes A \to A$ that is H-linear (where $A \otimes A$ has the H-module structure given above), as well as a map $\mathbb{C} \to A$, satisfying some string diagrams similar to the ones to the left. If A is an algebra in $\mathrm{Rep}(H)$, a (right) **module** over A consists of an H-module M with a multiplication $M \otimes A \to M$ satisfying commutative diagrams similar to the ones for associativity and identity in the previous section.

The right modules over A, along with A-linear maps between them, form an example of a module category over $\mathrm{Rep}(H)$. Furthermore, a theorem by Ostrik states that *all* module categories over $\mathrm{Rep}(H)$ are of this form. Because of this, a large part of our research has been trying to understand what the various algebras in $\mathrm{Rep}(H)$ are like, particularly for the case where H is a Taft algebra.

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