Hopf Algebra and Representation Theory

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Abstract

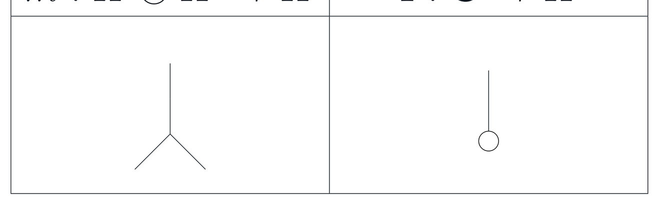
This poster will explore various algebraic structures and, in particular, Hopf algebras, a special algebraic structure that plays an important role in a variety of fields due to its algebraic properties. We will also discuss its representation theory, and study these fields with examples. A website containing more information and complementary code can be accessed through the QR code at the end of the poster.

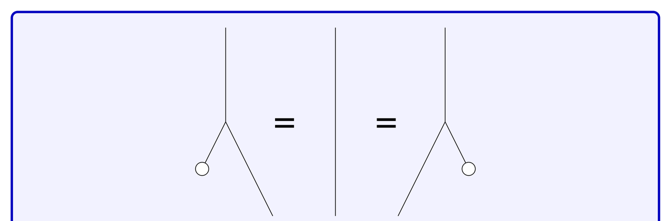
Unital Associative Algebra

We start with **unital associative algebra**, a mathematical structure where elements interact with a binary operation, *multiplication*, and contains a multiplicative identity element, the *unit*. Multiplication is an operation used in many other mathematical structures that takes two elements to construct a new element in the algebra.

We can write these algebraic axioms diagrammatically, as seen in the right column. This diagrammatic notation will follow us throughout the poster, and **all operations are assumed to be linear**. Studying such algebraic structures can serve as powerful tools to uncover underlying patterns and abstracted relationships in various mathematical systems.

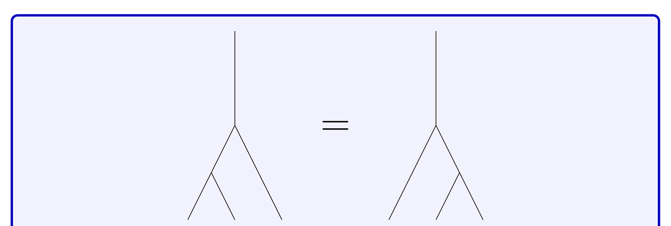
multiplication	unit
$m : H \otimes H \to H$	$1 : \mathbb{C} \to H$





unital axiom

$$1 \cdot h = h = h \cdot 1$$



associativity axiom

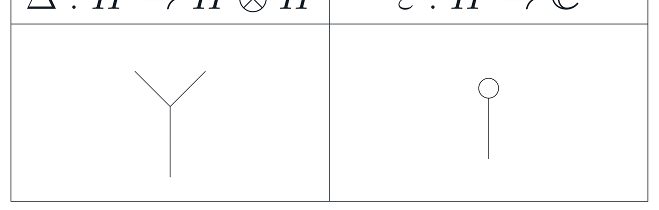
$$(g \cdot h) \cdot k = g \cdot (h \cdot k)$$

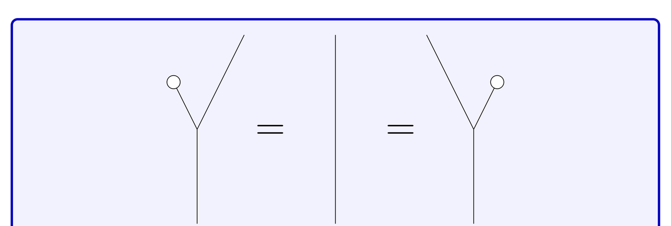
Counital Coassociative Coalgebra

Another interesting structure is the dual of unital associative algebras, **counital coassociative coalgebra**. The "co-" prefix indicates swapping the domain and codomain of our operations, namely the multiplication and unit of the algebra.

To motivate the concept of coalgebra, note that defining a tensor product of modules requires a method to distribute scalar multiplication over each tensor component. Comultiplication, defined as $\Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)}$, provides a natural mechanism for achieving such distribution. This will also be further discussed in the representation theory section.

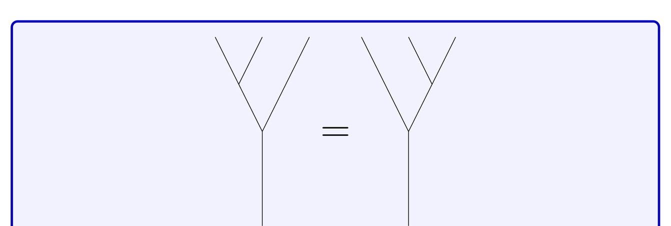
comultiplication	counit
$\Delta : H \to H \otimes H$	$\varepsilon : H \to \mathbb{C}$





counital axiom

$$\sum_i \varepsilon(h_i^{(1)}) h_i^{(2)} = h = \sum_i h_i^{(1)} \varepsilon(h_i^{(2)})$$

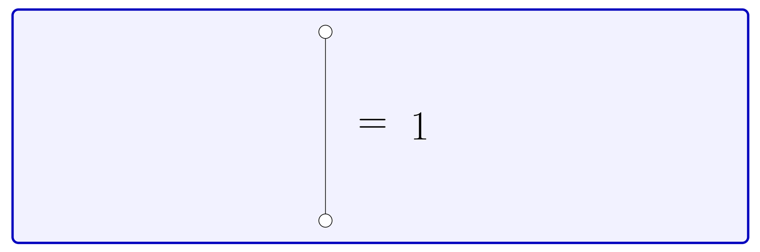


coassociativity axiom

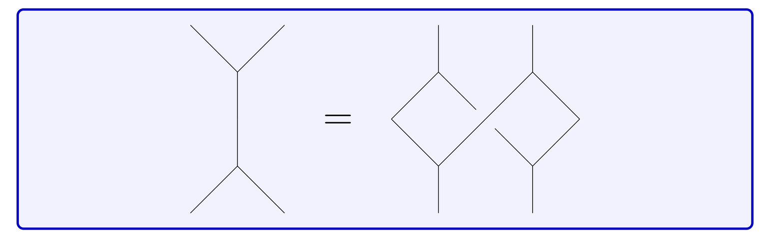
$$\sum_i \Delta(h_i^{(1)}) \otimes h_i^{(2)} = \sum_i h_i^{(1)} \otimes \Delta(h_i^{(2)})$$

Bialgebra

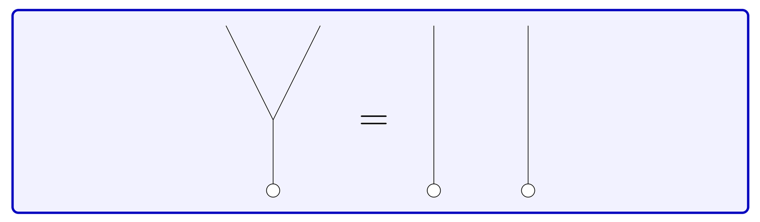
Combining these definitions of two important algebraic building blocks, we can create a **bialgebra**, a mathematical structure that is a unital associative algebra and a counital coassociative coalgebra with compatible operations. This compatability is encoded in the diagrams below.



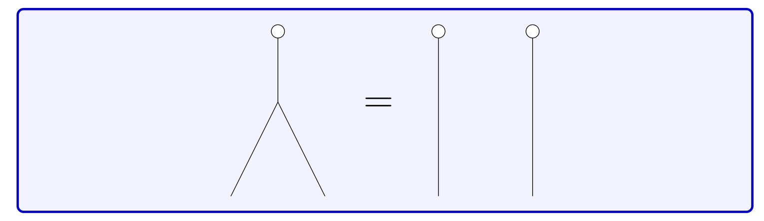
$\varepsilon(1) = 1$



$\Delta(hk) = \Delta(h)\Delta(k)$ Braid:



$\Delta(1) = 1 \otimes 1$

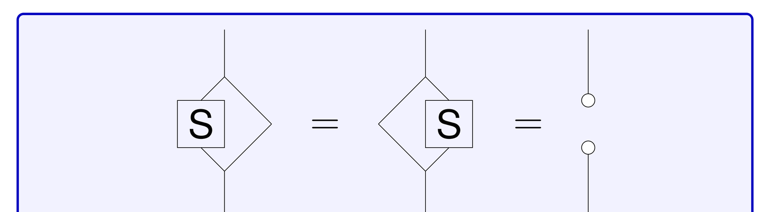


$\varepsilon(hk) = \varepsilon(h)\varepsilon(k)$

Hopf Algebras

We now have built a foundation to define Hopf algebras. A **Hopf algebra** is a structure that is a bialgebra, and it contains an additional linear operation – the *antipode* $S : H \to H$.

The antipode operation acts similar to the inverse operation of groups, except that it is linear and adapts to the counit and comultiplication operations.



antipode axiom

$$\sum_i h_i^{(1)} S(h_i^{(2)}) = \sum_i S(h_i^{(1)}) h_i^{(2)} = \varepsilon(h)1$$

Given a group G, we can define a **group Hopf algebra** $\mathbb{C}[G]$, a complex vector space with basis G and multiplication induced by the group multiplication. $\mathbb{C}[G]$ is always *cocommutative*. Cocommutativity is analagous to commutivity, but with comultiplication instead. On group elements g, comultiplication Δ is defined as $\Delta(g) = g \otimes g$, counit ε is defined as $\varepsilon(g) = 1$, and antipode S is defined as $S(g) = g^{-1}$. These operations can be computed on arbitrary elements by linearity.

Let $G = D_3$, then $\mathbb{C}[D_3]$ has basis $\{1_G, s, r, r^2, sr, sr^2\}$. The multiplication is the same as the group operation:

$$s^2 = r^3 = (sr)^3 = 1_G,$$

and comultiplication, counit, and antipode are defined as follows:

$$\Delta(r) = r \otimes r, \quad \varepsilon(r) = 1,$$
$$S(r) = r^{-1}.$$

The *cocommutativity* of $\mathbb{C}[G]$ can be shown as

$$\Delta(r) = r \otimes r = \Delta^{op}(r).$$

Sweedler's Hopf algebra, denoted as H_4, is the smallest Hopf algebra that is both *noncommutative* and *noncocommutative* - meaning that elements neither commute nor cocommute. For example, $\Delta(\theta) \neq \Delta^{op}(\theta)$. Studying Hopf algebras gives a deeper understanding for how such properties behave and gain insight to representations of quantum groups and other important structures.

The basis of H_4 consists of $\{1, K, \theta, K\theta\}$. With this basis, multiplication is subject to the following constraints:

$$K^2 = 1, \qquad K\theta = -\theta K, \qquad \theta^2 = 0.$$

We define comultiplication, counit, and the antipode identity as follows (note these are all linear maps):

$$\Delta(K) = K \otimes K, \qquad \Delta(\theta) = K \otimes \theta + \theta \otimes 1,$$
$$\epsilon(K) = 1, \qquad \epsilon(\theta) = 0,$$
$$S(K) = K, \qquad S(\theta) = \theta K.$$

Representation Theory

Representation theory studies **abstract algebraic structures** by representing their **elements** as **linear transformations** of vector spaces. Motivating with representations of groups, we will see how the representations of Hopf algebras are defined analogously.

Definitions

A **representation** for a group G is equivalent to a G-module. A **G-module** M is a vector space with a map defined as

$$\rho : G \times M \to M,$$
$$(g, m) \mapsto g \cdot m,$$

such that as ρ is linear in M and

$$(g_1 \cdot g_2) \cdot m = g_1 \cdot (g_2 \cdot m),$$
$$1_G \cdot m = m,$$

for $g_1, g_2 \in G$, and $m \in M$.

A **representation** for a Hopf algebra H is equivalent to an H-module. An **H-module** M is a vector space with a linear map defined as

$$t : H \otimes M \to M,$$
$$h \otimes m \mapsto h \cdot m,$$

such that

$$(h \cdot k) \cdot m = h \cdot (k \cdot m),$$
$$1_H \cdot m = m,$$

for $h, k \in H$, and $m \in M$.

Example

An example of a 2-D D_3-module is the vector space $\mathbb{C}^2_{rot}$ with multiplication

$$s \cdot v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot v,$$
$$r \cdot v = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} \cdot v,$$

for $v \in \mathbb{C}^2_{rot}$.
An example of a 1-D D_3-module is the vector space $\mathbb{C}_{sign}$ with multiplication

$$s \cdot \lambda = -\lambda, \qquad r \cdot \lambda = \lambda,$$

for $\lambda \in \mathbb{C}_{sign}$.

An example of a 1-D H_4-module is the vector space $\mathbb{C}_\varepsilon$ with multiplication

$$K \cdot \lambda = \varepsilon(K)\lambda = \lambda,$$
$$\theta \cdot \lambda = \varepsilon(\theta)\lambda = 0,$$

for $\lambda \in \mathbb{C}_\varepsilon$.

Another example of a 1-D H_4-module is the vector space $\mathbb{C}_K$ with multiplication

$$1 \cdot \lambda = \lambda, \qquad K \cdot \lambda = -\lambda,$$
$$\theta \cdot \lambda = 0, \qquad K\theta \cdot \lambda = 0,$$

for $\lambda \in \mathbb{C}_K$.

Tensor Product

Given M, N are G-modules, we can make $M \otimes N$ a G-module by

$$g \cdot (m \otimes n) = (gm) \otimes (gn).$$

Given M, N are H-modules, we can make $M \otimes N$ an H-module by

$$h \cdot (m \otimes n) = \Delta(h)(m \otimes n).$$

Example

Using the D_3-module examples above, $\mathbb{C}_{sign} \otimes \mathbb{C}^2_{rot} \cong \mathbb{C}^2_{rot}$ as D_3-modules.

Using the H_4-module examples above, $\mathbb{C}_K \otimes \mathbb{C}_K \cong \mathbb{C}_\varepsilon$ as H_4-modules.

Acknowledgements and References

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[1] https://ncatlab.org/nlab/show/coalgebra
[2] https://en.wikipedia.org/wiki/Hopf_algebra
[3] https://en.wikipedia.org/wiki/Representation_theory