Fusion Categories and Jellyfish Relations

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Monoidal Categories

A *monoidal category* $\mathcal{C}$ is a category equipped with a multiplication which is unital and associative, up to specified isomorphisms. This means we have a unit object $\mathbf{1}: \mathcal{C}$, and a multiplication functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Since the multiplication is not "strictly" unital and associative, we have to specify isomorphisms:

- The *left unitor* $\lambda_A: \mathbf{1} \otimes A \to A$.
- The *right unitor* $\rho_A: A \otimes \mathbf{1} \to A$.
- The *associator* $\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.

We don't want to be able to derive a map $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ other than the associator by composing associators and unitors. It suffices for the following diagrams to commute (indices on the associators and unitors omitted for space):

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha \otimes D"] \arrow[d, "\alpha"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "A \otimes \alpha"]
\end{tikzcd}$$

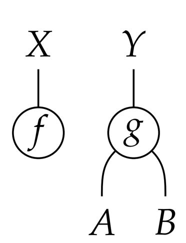
$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes B"'] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "A \otimes \lambda"] \\
& A \otimes B &
\end{tikzcd}$$

Some monoidal categories which are particularly relevant to us:

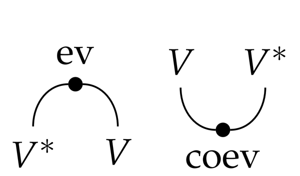
- **Vec**, the category of finite-dimensional complex vector spaces, with $\mathbf{1} = \mathbb{C}$ the one dimensional space, and $\otimes$ the tensor product.
- **Rep**(G), the category of finite-dimensional complex representations of the finite group G, again with $\mathbf{1} = \mathbb{C}$, the trivial representation, and $\otimes$ the tensor product of representations.
- **Vec**(G) the category of finite-dimensional G-graded vector spaces. The unit has one dimension, graded in the group identity, and the tensor product is given by $(V \otimes W)_g = \bigoplus_h V_h \otimes W_{h^{-1}g}$.

String Diagrams

We can visualize morphisms in a monoidal category in a nice way using *string diagrams*. An example string diagram is drawn on the left. Each object in the category is represented by a string, and horizontal juxtaposition of strings represents the monoidal product. The monoidal unit is sometimes written with a dotted string, but more often the strings representing unit objects are simply omitted. Similarly, we don't draw the associator explicitly. We represent morphisms with little labeled circles or boxes. So, the example diagram represents $f \otimes g: A \otimes B \to X \otimes Y$, where $f: \mathbf{1} \to X$ and $g: A \otimes B \to Y$.



Rigidity



In the category of finite-dimensional vector spaces, every vector space V has a *dual space*, written V^*, consisting of the linear functionals on V. Furthermore, we have a linear map $\mathrm{ev}: V^* \otimes V \to \mathbf{1}$ which evaluates a linear functional at a vector, and another map $\mathrm{coev}: \mathbf{1} \to V \otimes V^*$, which is dual to evaluation. These morphisms are drawn as string diagrams on the left. The evaluation and coevaluation maps satisfy the relations below. This makes **Vec** an example of a *rigid monoidal category*.

Simple Objects

Simple groups are common objects of study in group theory. Similarly, we can talk about simple objects in an arbitrary category, which are objects with no nontrivial proper quotients. The categories we care about on this poster are abelian categories enriched over **Vec**, which essentially means we have a direct sum operation on our objects, and the Hom sets form complex vector spaces with bilinear composition. In this context, an important theorem about simple objects is *Schur's lemma*:

Theorem 1. *If A and B are simple objects in an abelian category enriched over* **Vec**, *then either A and B are not isomorphic and* $\mathrm{Hom}(A, B) = 0$, *or A and B are isomorphic, and* $\mathrm{Hom}(A, B)$ *is one-dimensional. In particular,* $\mathrm{Hom}(A, A) = \mathbb{C}$.

An important concept related to simple objects is *semisimplicity*. A category is semisimple if every object can be decomposed into a finite direct sum of simple objects.

Fusion Categories

A fusion category is a rigid linear monoidal category which:

- is semisimple,
- has finitely many simple objects, up to isomorphism,
- and has a simple monoidal unit $\mathbf{1}$.

The *rank* of a fusion category is the number of simple objects it has, up to isomorphism. All the examples of monoidal categories to the left are in fact fusion categories:

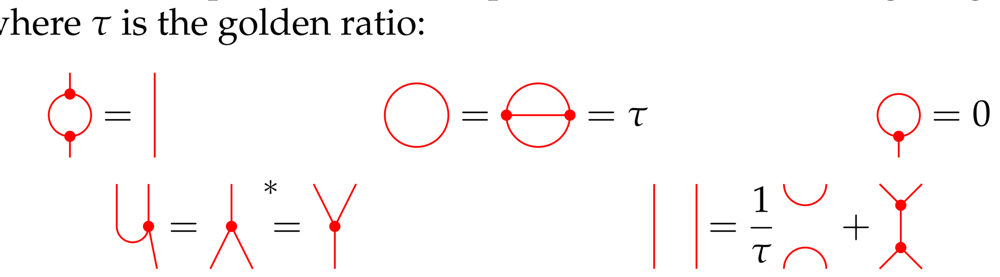
- In **Vec**, the only simple object is the one-dimensional vector space; every vector space can be decomposed into a direct sum of one-dimensional spaces. This is the only fusion category with exactly one simple object.
- In **Rep**(G), the simple objects are the irreducible representations.
- In **Vec**(G), there is a simple object for each element g of G, which is a one-dimensional vector space in the g-graded component and zero dimensional in all the other components.

Examples of small fusion categories

The simplest nontrivial examples of fusion categories we can come up with are ones with only two simple objects, one of which is the monoidal unit.

- One such category is **Vec**$(\mathbb{Z}/2)$, the category of (finite-dimensional) $\mathbb{Z}/2$-graded vector spaces. The two simple objects are $\mathbf{1}$, which is one-dimensional in the 0-graded component, and T, which is one-dimensional in the 1-graded component. We have that $T \otimes T$ must be a direct sum of simple objects, in this case $T \otimes T = \mathbf{1}$. Representing T by a blue strand, we can derive the following string diagram relations:

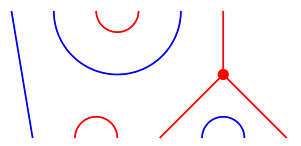
- Another small fusion category is **Fib**, which has simple objects $\mathbf{1}$ and A. In **Fib**, the decomposition of $A \otimes A$ is $\mathbf{1} \oplus A$. Because the tensor product distributes over direct sums, we can use this to show that the number of summands isomorphic to A in $A^{\otimes n}$ will be the n-th fibonacci number, hence the name **Fib**. Having an isomorphism $A \otimes A \to \mathbf{1} \oplus A$ means that we have maps $A \otimes A \to A$ and $A \to A \otimes A$. We represent A as a red strand, and we represent these maps as trivalent vertices, giving the following relations, where τ is the golden ratio:



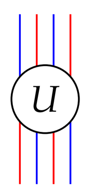
We can take linear combinations of morphisms like this because our categories are linear, thus the Hom-sets are vector spaces.

Free Product of Fusion Categories

We are interested in using our simple examples of fusion categories to build more complicated ones. A useful construction is the *free product* of fusion categories. The objects in the free product are formal direct sums of tensor products of objects from the factors. This means that the simple objects in a free product will be products of simple objects of the factors. For example, in the case of **Vec**$(\mathbb{Z}/2) *$ **Fib**, the simple objects will be of the form $A, T, A \otimes T, T \otimes A, A \otimes T \otimes A, \cdots$. A string diagram in this category would look like the following:



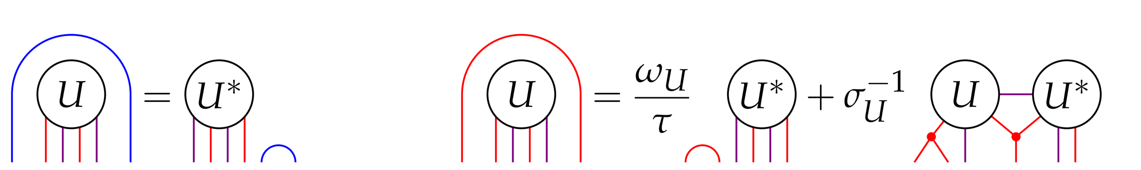
We have a blue diagram from **Vec**$(\mathbb{Z}/2)$ overlaid on a red diagram from **Fib**, such that the strands from one don't overlap the strands from the other. We can apply the relations we derived for the red and blue strings, as long as the strands don't overlap. This free product is still a semisimple rigid monoidal category, but it's no longer a fusion category: it has infinitely many nonisomorphic simple objects. We want to introduce some kind of relation which makes some of these simple objects isomorphic. In fact, in the case of **Vec**$(\mathbb{Z}/2)$ and **Fib**, it must be of the form $U: (A \otimes T)^{\otimes n} \to (T \otimes A)^{\otimes n}$. On the right, we see the case $n = 2$.



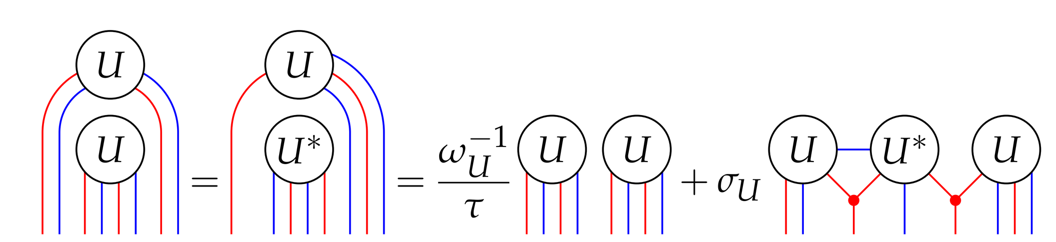
Jellyfish Relations

Once we introduce our isomorphism U, we want the resulting category to be fusion, and so we can use the properties of fusion categories in order to derive relations on the isomorphism U. However, we want to check that these relations are "consistent." We might find that by introducing the isomorphism U, we may have introduced more relations on the simple objects than we intended to.

In order to check this, we derive "jellyfish relations" for our category, pictured below. These relations apply for any n, so we represent $T \otimes (A \otimes T)^{\otimes n-1}$ with a purple strand. The number ω_U is some $2n$-th root of unity, and σ_U is a square root of ω_U.



Then, we use the jellyfish relations in order to "normalize" a particular diagram in two different ways, drawn below in the case $n = 1$.



We move the inner U out leftwards until we get a linear combination of diagrams to which no jellyfish relations apply, as shown above. We also move it rightwards to get another such linear combination. Then, we can set these two expressions equal to each other, and see if there are values for ω_U and σ_U which solve the equation. If we can solve the equation without introducing new linear dependencies, then we will know the isomorphism U we added hasn't introduced any more unwanted relations between simple objects.

Sources and Acknowledgements

- M. Izumi, S. Morrison, D. Penneys. Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$. *https://arxiv.org/abs/1308.5723*

I'd also like to thank Quinn Kolt for giving me the opportunity to work on this project and teaching me a lot of math, and the Directed Reading Program for organizing all this.