Tannaka Duality

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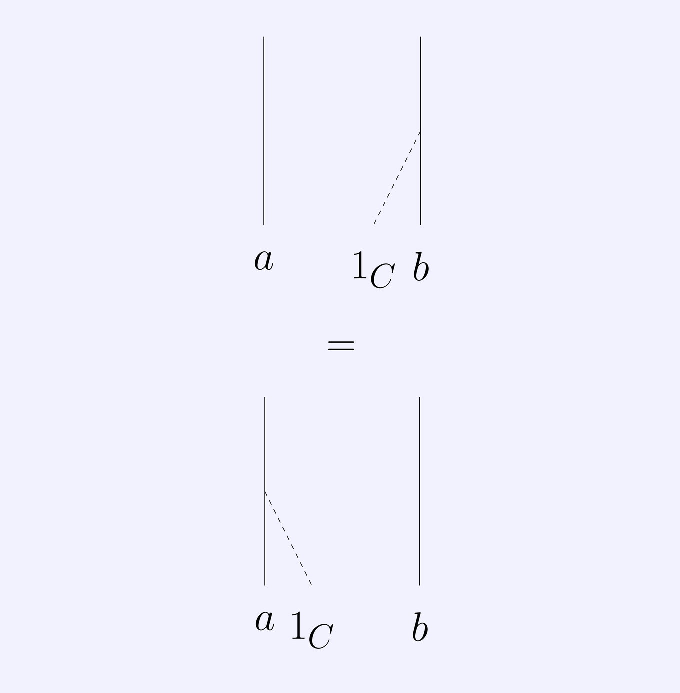
Monoidal Category

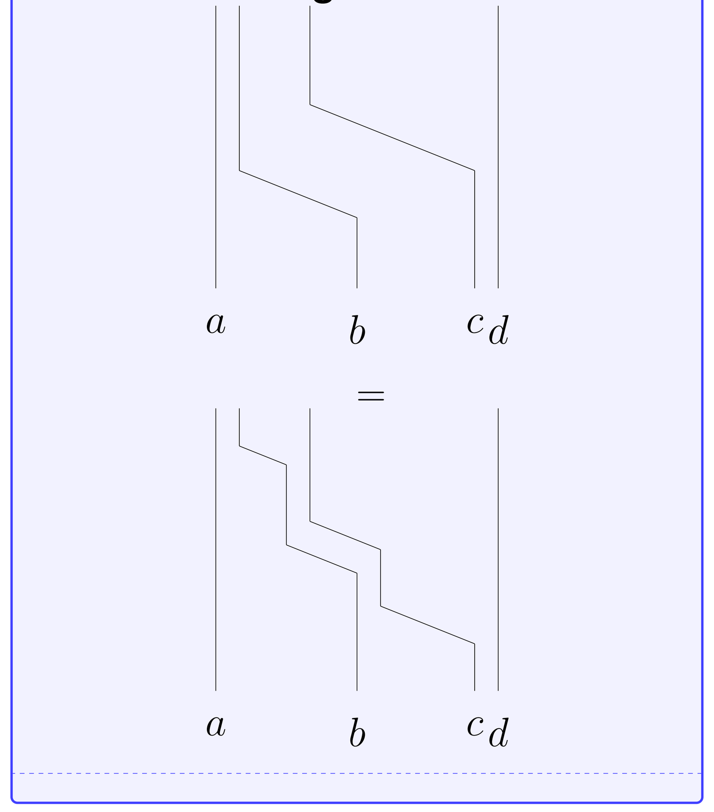
Definition 1.1 A Monoidal category is a tuple $(C, \otimes, 1_C, \alpha_{a,b,c}, \lambda_a, \rho_a)$ where $-\otimes- : C\times C \to C$ is a bifunctor called the **tensor product**, 1_C is an object called the **unit object** in C, $\alpha_{(a,b,c)} : (a\otimes b)\otimes c \to a\otimes(b\otimes c)$ is a natural isomorphism called the **associator**, $\lambda_a : 1_C\otimes a \to a$ and $\rho_a : a\otimes 1_C \to a$ are natural isomorphism called **unitors**.

This data must satisfy the following two axioms:

Triangle axiom



Pentagon axiom



Examples:

- ($G-$graded vector spaces) Let G be a finite group and Vec(G) be the category of all finite dimensional $G-$graded complex vector space $V = \bigoplus_{g\in G} V_g$ with grading-preserving linear map, i.e. , if $T : V \to W$, then $T(V_g) \subseteq W_g$. We endow Vec(G) with the structure of tensor product as follows:

$$(V\otimes W)_g := \bigoplus_{hk=g} V_h \otimes W_k.$$

  The associator moves the parentheses, and the unit object is the field $\mathbb{C}$. Under this definition, Vec(G) is a monoidal category.

- (Representation for G) Let G be a finite group. **Rep**$_k(G)$ is a category of all representations of G over complex field $\mathbb{C}$. The objects in this category are pairs (V, π_V), where $\pi_V : G \to GL(V)$ is the homomorphism for the representation V. We define the tensor product of pairs as:

$$(V, \pi_V)\otimes(W, \pi_W) = (V\otimes W, \pi_{V\otimes W}), \pi_{V\otimes W}(g) := \pi_V(g)\otimes \pi_W(g).$$

  The unit is the trivial representation $1 = k$, and the associator moves parentheses. Then **Rep**$_k(G)$ is a monoidal category. A similar statement holds for the category Rep$_k(G)$ of all finite-dimensional representations of G.

Theorem 1.2 (Coherence) Any monoidal category is monoidally equivalent to a strict monoidal category.

Tensor Category

Definition 2.1 Rigidity is the generalization of duality. The dual of an object $X \in C$ is an object $X^\vee$ with evaluation and coevaluation morphisms $\mathrm{ev}_c : c^\vee \otimes c \to 1_c$ and $\mathrm{coev}_c : 1_c \to c\otimes c^\vee$ satisfying the following relation

$$(\mathrm{id}_X \otimes \mathrm{ev}_X)\circ(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X, \qquad (\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee})\circ(\mathrm{id}_{X^\vee}\otimes \mathrm{coev}_X) = \mathrm{id}_{X^\vee}.$$

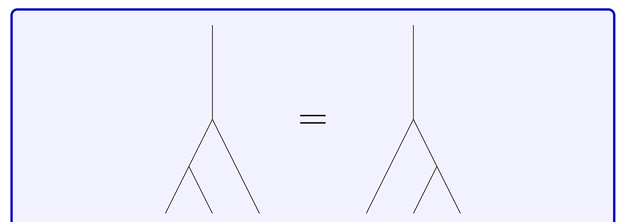
Definition 2.2 A **finite tensor category** is a locally finite k-linear abelian rigid monoidal category such that there are enough projective covers of simple objects, finitely many isomorphism classes of simple objects, and 1_C is simple.

Hopf Algebras

Definition 3.1 A Hopf algebra is a sextuple $(H, \nabla, \eta, \Delta, \varepsilon, S)$ where $\nabla : H\otimes H \to H$ is **multiplication**, $\eta : \mathbb{F} \to H$ is **unit**, $\Delta : H \to H\otimes H$ is **comultiplication**, $\varepsilon : H \to \mathbb{F}$ is **counit**, and $S : H \to H$ is **antipode**. All those maps are linear.

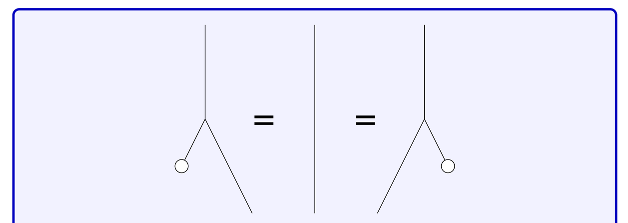
This data must satisfy the following axioms:

1 **Associativity axiom**;



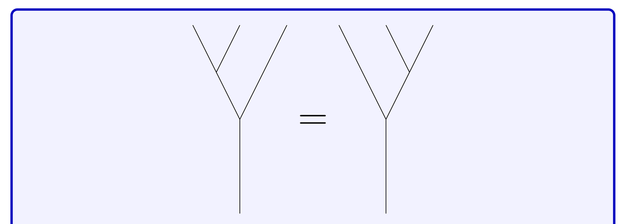
$\nabla\circ(\nabla\otimes \mathrm{id}_H) = \nabla\circ(\mathrm{id}_H \otimes\nabla)$

2 **Unital axiom** ;



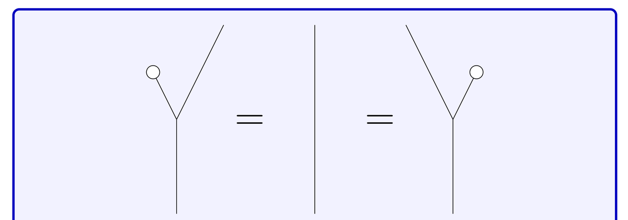
$\nabla\circ(\eta\otimes \mathrm{id}_H) = \nabla\circ(\mathrm{id}_H \otimes\eta)$

3 **Coassociativity axiom**;



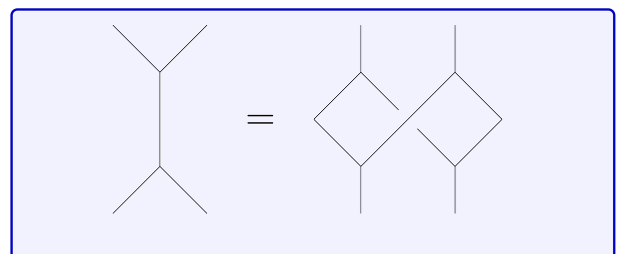
$(\Delta\otimes \mathrm{id}_H)\circ\Delta = (\mathrm{id}_H \otimes\Delta)\circ\Delta$

4 **Counital**;

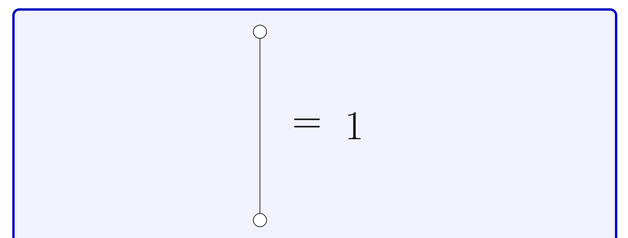


$(\varepsilon\otimes \mathrm{id}_H)\circ\Delta = \mathrm{id}_H = (\mathrm{id}_H \otimes\varepsilon)\circ\Delta$

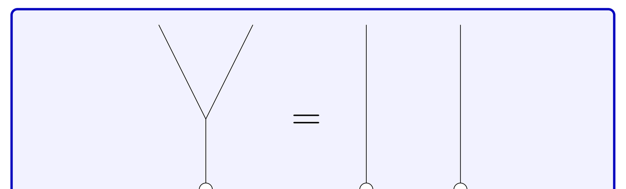
5. Δ and ϵ are both unital algebra homomorphisms for all $h_1, h_2 \in H$ (bialgebra)



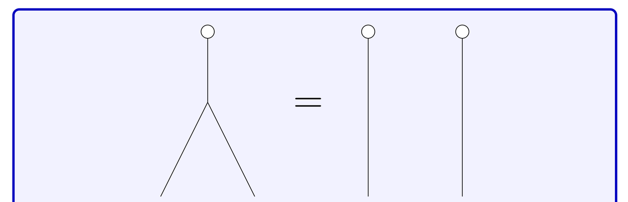
$\Delta\circ\nabla = (\nabla\otimes\nabla)\circ(\mathrm{id}_H \otimes\beta_{H,H}\otimes \mathrm{id}_H)\circ(\Delta\otimes\Delta)$



$\varepsilon(1) = 1$

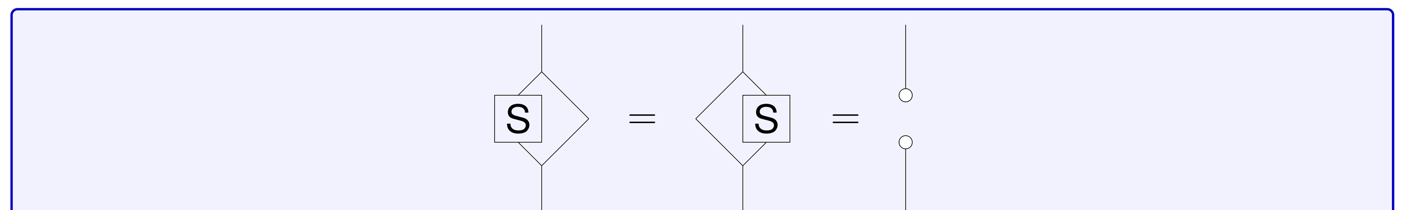


$\Delta(\eta(1)) = \eta(1)\otimes\eta(1)$



$\varepsilon\circ\nabla = \varepsilon\otimes\varepsilon,$

6 **Antipode axiom**.



$\nabla\circ(\mathrm{id}_H \otimes S)\circ\Delta = \eta\circ\varepsilon = \nabla\circ(S\otimes \mathrm{id}_H)\circ\Delta$

Examples:

- (Trivial Hopf algebra) Let $\mathbb{F}$ be a field. It is also a Hopf algebra with multiplication and unit of the field, comultiplication $\Delta(a) = a\otimes 1_{\mathbb{F}}$, counit $\varepsilon(a) = a$, and antipode $S(a) = a$

- (Group Algebra) Let G be any group and $\mathbb{F}$ be a field. Considering the group ring $F[G]$. We define the multiplication $\nabla : F[G]\otimes F[G] \to F[G]$ by extending the usual group multiplication $g\circ h = g *_G h$ for $g, h \in G$. $e \in G$ is the unit object under this multiplication.

  The comultiplication $\Delta : F[G] \to F[G]\otimes F[G]$ is generated by $g \to g\otimes g$ for $g\in G$. By the counital axiom, $\varepsilon(g)g = g$, we could conclude that $\varepsilon(g) = 1_F$ for all $g \in G$. Antipode is defined as $S(g) = g^{-1}$ for all $g\in G$. All together makes $F[G]$ a Hopf algebra.

Tannaka Reconstruction

Definition 4.1 A **Fiber functor** is a exact faithful linear functor

$$F : C \to \mathrm{Vec}$$

preserving identity. It is also equipped with a natural isomorphism:

$$J_{X,Y} : F(X)\otimes F(Y) \to F(X\otimes Y),\ \text{for } X, Y \in \mathrm{Obj}(C)$$

such that $J : F(\mathbb{K}_C) \to \mathbb{C}$

Theorem 4.2 Every finite tensor category equipped with a fiber functor is realized as the category of finite-dimensional representations of finite-dimensional Hopf algebras over the field $\mathbb{C}$. In particular,

$$(C, F) \to H := \mathrm{End}(F), H \to (\mathrm{Rep}(H), F)$$

are mutually bijections up to tensor equivalence.

Proof: Generally speaking, the proof is just checking that End(F) is a Hopf alegbra, (Rep(H), F) becomes a finite tensor category, and Rep(End(F)) $\cong C$

End(F) is equipped with an algebra structure. The crucial point for the End(F) is to construct the comultiplication, counit, and antipode. Comultiplication and axiom 5 make it into a bialgebra, while the Antipode and axiom 6 make it into a Hopf algebra. Comultiplication is a homomorphism from End(F) to End(F) $\otimes$ End(F). Consider the Delign's tensor product of End(F) with itself, there are canonical isomorphism α between End(F) $\otimes$ End(F) and End($F\boxtimes F$). For $X, Y \in C$, we have that $(F\boxtimes F)(X\boxtimes Y) = F(X)\otimes F(Y)$. Hence, for a natural transformation $a = \{a_Z : F(Z) \to F(Z)|\text{for all } Z \in C\} \in \mathrm{End}(F)$, we define a new natural transformation

$$\tilde{\Delta}_{X\boxtimes Y} = J^{-1}_{X\boxtimes Y}\circ\phi_{X\boxtimes Y}\circ J_{X\boxtimes Y};\ (F\boxtimes F)(X\boxtimes Y) \to (F\boxtimes F)(X\boxtimes Y)$$

Pulling $\tilde{\Delta}$ back through α^{-1} lands in End(F) $\otimes$ End(F) and is defined as $\Delta(a)$. Coassociativity and the algebra map property of Δ then follow from the monoidal functor axioms for F and the multiplicativity of α. We then define the counit ε : End(F) $\to \mathbb{C}$ by $\varepsilon(a) = a$ where $a_1 \in \mathrm{End}(F(1_C)) = \mathrm{End}(1) = \mathbb{C}$. Overall, we have that H is a bialgebra. Transporting the evaluation and coevaluation maps through the fibre functor F will obtain the usual linear duality maps on the vector spaces. One could define the antipode as $S(a_X) = a^*_{X^*}$. Hence, it makes H into a Hopf algebra.

On the other hand, $C = \mathrm{Rep}_{\mathbb{C}}(H)$ is a category whose objects are all finite representations of H over the field $\mathbb{C}$. The tensor product of H-modules X, Y is defined as the usual tensor product of vector spaces $X\otimes Y$. And the $H-$ action is defined as

$$h\cdot(x\otimes y) = \Delta(h)(x\otimes y) = (h^{(1)}\cdot x)\otimes(h^{(2)}\cdot y)$$

By the coassociativity and counital axioms, the tensor product here is associative and unital up to isomorphism, and the unit object is the trivial representation. Pentagon and triangular axioms hold because the underlying category is Vec. Every object in C is dualizable by the antipode axiom. The remaining axioms for a Finite Tensor Category are easy to verify.

And intuitively, H = End(F) contains every way that the objects of C can act on themselves inside vector spaces. And those actions could become modules that recreate all of C, giving the desired isomorphism of categories Rep(End(F)) $\cong C$

Acknowledgements and References

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[1] Tensor categories by Etingof, Gelaki, Nikshych and Ostrik [2] https://people.math.osu.edu/penneys.2/8800//Notes/FusionCats.pdf
[3] https://arxiv.org/pdf/math/0401246