Much Ado About Nothing, or All for Naught? Investigating the Zeros of Complex Functions

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1 Introduction

Locating the roots of a function, that is, where a function obtains the value 0, has a long and rich history, dating back hundreds of years. In the 9th century AD, for example, mathematician Al-Khwârizmî developed a method for solving all quadratic equations, providing a general solution for all equations of the form $ax^2 + bx + c = 0$. Roots are a necessary tool which help us describe functions and solve equations. Because of this, the roots of functions are vastly important and have practical uses in almost every field tangential to mathematics. Although locating the roots of a function might seem trivial, one of the most famous mathematical conjectures, unproved for over 150 years, deals with locating the roots of a seemingly simple equation.

In this paper, we explore some recent discoveries concerning the roots of what are known as complex-harmonic polynomials. We also provide a brief overview of the history and significance of the Riemann Hypothesis.

1.1 Motivating Examples

In high school, one typically learns through the Fundamental Theorem of Algebra that any polynomial of the form $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ has exactly n complex roots. However, by applying the complex conjugate to just one term in a polynomial, the Fundamental Theorem appears to go awry. Figure 1, for example, shows the case of a "degree three polynomial" with nine roots!

1.1.1 An Algebraic Explanation

To trace the origins of extra roots such as these, we consider two specific trinomials, namely, $f(z) = z^2 + cz - 1$ and $g(z) = z^2 + c\overline{z} - 1$ with $c \in \mathbb{R}$. Although the first of the following lemmas is evident from the Fundamental Theorem of Algebra, our intention is to demonstrate *why* we get the roots of f and g that we do.

Lemma 1.1
The polynomial $f(z) = z^2 + cz - 1$ has exactly 2 roots.

Proof. Suppose $f(z_0) = 0$ for some $z_0 = x + iy$ with $x, y \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= (x+iy)^2 + c(x+iy) - 1 \\ &= (x^2 - y^2 + cx - 1) + iy(2x + c). \end{aligned}$$

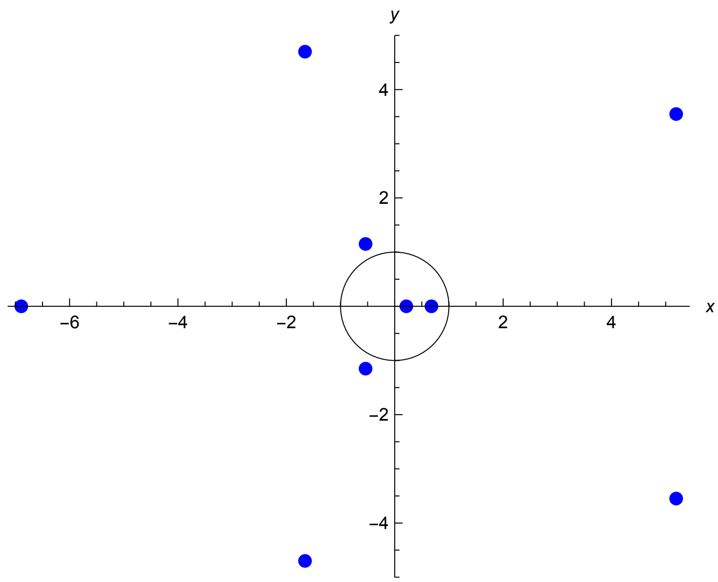

Figure 1: The nine roots of $z^3 + 6\bar{z}^2 - 6z + 1$

This requires $x^2 - y^2 + cx = 1$ and $y(2x + c) = 0$, so that either $y = 0$ or $x = \frac{-c}{2}$.

Suppose first that $y = 0$. Then $x^2 + cx = 1$, which has 2 real solutions as given by the quadratic formula, since $c \in \mathbb{R}$. Next, suppose $x = \frac{-c}{2}$. Then $y^2 = -(\frac{c^2}{4} + 1)$, which has no real solutions. Thus f has exactly 2 roots. □

Lemma 1.2
The polynomial $g(z) = z^2 + c\bar{z} - 1$ has up to 4 roots.

Proof. Suppose $g(z_0) = 0$ for some $z_0 = x + iy$ with $x, y \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= (x + iy)^2 + c(x - iy) - 1 \\ &= (x^2 - y^2 + cx - 1) + iy(2x - c). \end{aligned}$$

This requires $x^2 - y^2 + cx = 1$ and $y(2x - c) = 0$, so that either $y = 0$ or $x = \frac{c}{2}$.

As above, if $y = 0$ then $x^2 + cx = 1$ has 2 real solutions. Next, suppose $x = \frac{c}{2}$. Then $y^2 = \frac{3c^2}{4} - 1$ which has 2 real solutions if $|c| > \sqrt{\frac{4}{3}}$, 1 real solution if $|c| = \sqrt{\frac{4}{3}}$, and no solutions otherwise. Thus, g has up to 4 roots. □

Figure 2 illustrates Lemma 1.2 by depicting the roots of $z^2 + 3\bar{z} - 1$. In this example, we now expect there to be 4 such roots, since $c = 3 > \sqrt{\frac{4}{3}}$.

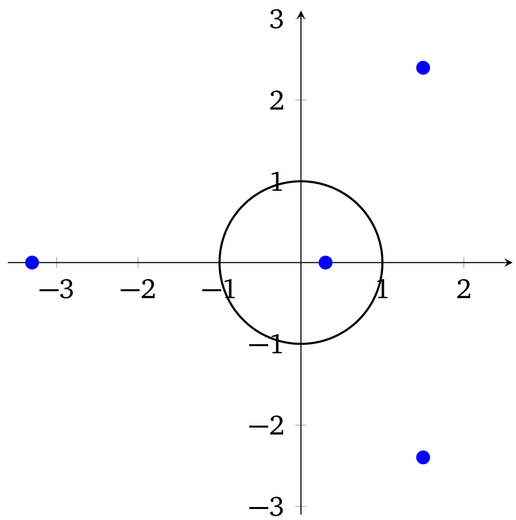

Figure 2: The four roots of $z^2 + 3\overline{z} - 1$

1.1.2 Level Curves

The above lemma and figure are useful for understanding the origin of these roots algebraically, but are not able to give us the full picture. Another approach to determining where these extra roots come from is to consider the level curves produced by those inputs whose outputs have 0 as either their imaginary component or their real component. Every intersection of these two curves is a root.

Figures 3, 4, 5, and 6 depict the level curves for $g(z) = z^7 + c\overline{z}^4 - 1$ for $c = 1, 1.5, 2$ and 3 respectively. Blue lines indicate the level curves for where the real component is 0, and red lines indicate the level curves where the imaginary component is 0. As c changes, the level curves move to form more intersections, sometimes causing up to 3 roots where in the analytic case there could only be one. (See Appendix A for Mathematica code used to produce these level curves.)

Rather than simply seeing the complex-harmonic case as an anomaly, this makes the Fundamental Theorem of Algebra more amazing: no matter how the level curves change due to coefficients in the analytic case, there are always exactly n intersections for a polynomial of degree n.

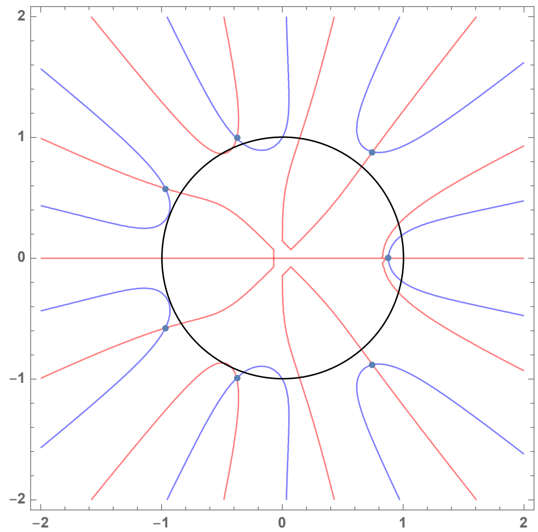

Figure 3: Level Curves of $z^7 + \bar{z}^4 - 1$

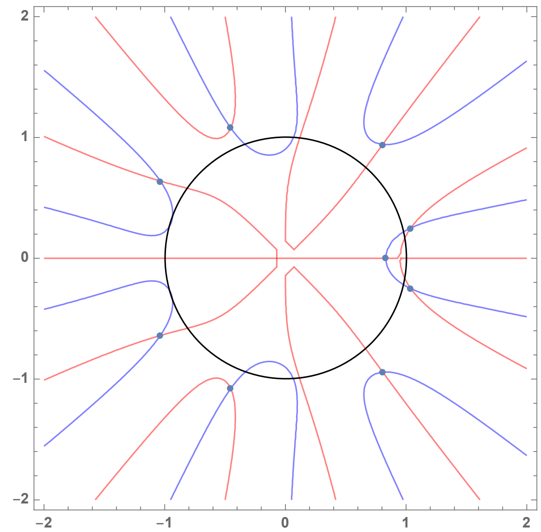

Figure 4: Level Curves of $z^7 + 1.5\bar{z}^4 - 1$

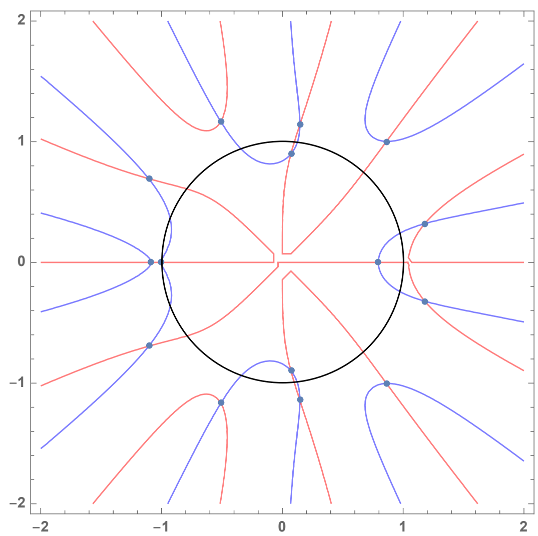

Figure 5: Level Curves of $z^7 + 2\bar{z}^4 - 1$

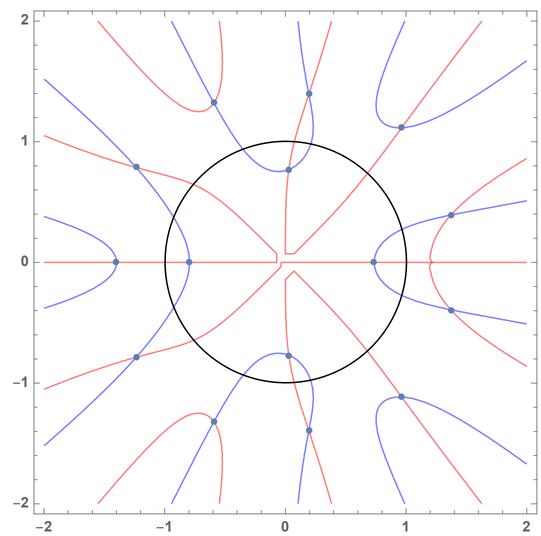

Figure 6: Level Curves of $z^7 + 3\bar{z}^4 - 1$

1.2 Problems for Investigation

The illustrations provided above in Section 1.1 seem to raise more questions than they answer about these complex-harmonic polynomials. There appear to be some conditions under which the Fundamental Theorem of Algebra applies, and some under which it does not.

Lemma 1.2 in particular demonstrates that the number of roots for complex-harmonic polynomials somehow depends on coefficients. Furthermore, what happens if we increase the degree of the polynomial? Does there exist some sort of Generalized Fundamental Theorem of Algebra which we might apply to complex-harmonic polynomials?

Ever since its fairly controversial acceptance in the world of mathematics, Complex Analysis has played a vital role in developing both theoretical and applied branches of mathematics. By creating simple solutions to otherwise impossible problems, and elegant proofs of some foundational mathematics (including the Fundamental Theorem of Algebra), Complex Analysis has certainly earned its place in the science. Given the relatively slow pace at which it has been progressing since its appearance, though, it might be easy to stereotype Complex Analysis as a stagnant or even closed field of study. The questions raised above, however, suggest otherwise.

For this reason, we present recent results and conjectures in the field of Complex Analysis throughout the remainder of the paper. In Section 2 we investigate the roots of complex-harmonic binomials, and in Section 3 we extend our investigations to include complex-harmonic trinomials. Section 4 highlights ongoing and future work in the area of complex-harmonic polynomials. We conclude in Section 5 with an overview of the formulation of the Riemann Hypothesis: a famous, unproved conjecture from 1859.

1.3 Preliminary Definitions and Theorems

Below we provide a few useful definitions to which we make reference throughout the rest of the paper.

Definition 1.1

A root z_0 of a function f is said to be an **interior root** if $|z_0| < 1$, an **exterior root** if $|z_0| > 1$, and a **unimodular root** if $|z_0| = 1$.

Definition 1.2

A function f is **complex-harmonic** if there exist analytic functions h and g such that $f = h + \overline{g}$.

Definition 1.3

The **dilatation** ω of a complex-harmonic function $f = h + \overline{g}$ at a point z_0 is $\omega(z_0) = \dfrac{g'(z_0)}{h'(z_0)}$.

Definition 1.4

> A complex-harmonic function $f = h + \overline{g}$ is **sense-preserving** at a point z_0 if the dilatation ω of f is analytic at z_0 (possibly with a removable singularity) with $h'(z_0) \not\equiv 0$, and $|\omega(z_0)| < 1$. Moreover, f is said to be **sense-reversing** at z_0 if $\overline{f}$ is sense-preserving there, which is to say, for the dilatation ω of f, we have $|\omega(z_0)| > 1$. A point is called **singular** if f is neither sense-preserving or sense-reversing there.

Note: Intuitively, a function f is sense-preserving along a curve C if, as you travel along C in one orientation (that is, clockwise or counter-clockwise), the image $f(C)$ is traced out with the same orientation. Likewise, a function is sense-reversing along a curve C if the image $f(C)$ is traced out with the opposite orientation. A function is singular along a curve C if the image $f(C)$ changes orientation along this curve.

Throughout this paper, we will make frequent use of the following extension of Rouché's Theorem for complex-harmonic functions, provided in [4]. This version of Rouché's Theorem is essentially the same as that for analytic functions, with a key distinction being that we require f and $f + g$ have no singular zeros in the domain.

Theorem 1.1 - Rouché's Theorem for Sense-Preserving Functions

If f and $f+g$ are complex-harmonic functions with no singular zeros in D, are continuous in $\overline{D}$, and if $|g(z)| < |f(z)|$ on the curve C containing D, then f and $f + g$ have the same number of zeros inside D, where the number of zeros is interpreted as the sum of the (positive or negative) orders of the zeros.

2 Complex-Harmonic Binomials

It doesn't take much complexity to see an example of when the Fundamental Theorem of Algebra appears to fail when it comes to complex-harmonic functions. In fact, with a function as simple as a binomial, things already seem to go "wrong."

Dr. Michael Dorff, president of the Mathematical Association of America (MAA), has posed a few conjectures concerning complex-harmonic polynomials. We begin by generalizing a conjecture posed for binomials, and proving our generalization. The proof, though simple in its solution, highlights some of the nuances of working with complex-harmonic polynomials, while providing a simple introduction to working with the complex-harmonic version of Rouché's Theorem (Theorem 1.1).

The original conjecture posed by Michael Dorff provides a suggestion for the number of roots for the binomial

$$b(z) = z^n + \frac{\overline{z}^k}{k}$$

for $n, k \in \mathbb{N}$. We generalize this conjecture to let the coefficient on the second term be any non-zero real number, and we exactly locate where these roots reside. The original conjecture also considers the case when $n = k$, though the proof of this case, and the case when $c = 0$, are both trivial.

Theorem 2.1

Let $n, k \in \mathbb{N}$ with $n > k$ and let $c \in \mathbb{R} \setminus \{0\}$. Then the complex-harmonic binomial $b_c(z) = z^n + c\overline{z}^k$ has $n+k$ roots evenly spaced on the circle $|z| = |c|^{\frac{1}{n-k}}$ and a root of order k at the origin for a total of $n + 2k$ roots.

Proof. Suppose $b_c(z_0) = 0$ for some $z_0 = re^{i\theta} \neq 0$. Substituting this identity in to b_c, we obtain $r^n e^{in\theta} + cr^k e^{-ik\theta} = 0$. Since $r \neq 0$, dividing by r^k and rearranging yields the following:

$$-c = r^{n-k} e^{i(n+k)\theta}.$$

When $c > 0$, the right hand side of this equation is negative, so we must have $e^{i(k+n)\theta} = -1$. Thus, $\theta = \frac{(2m+1)\pi}{k+n}$ for all $m \in \mathbb{N}$, and there are exactly $n + k$ distinct nonzero roots corresponding to θ. Note that these $n + k$ nonzero roots are spaced evenly around a circle, whose radius is given by $r = c^{\frac{1}{n-k}}$ since $r^{n-k} = c$. The case when $c < 0$ yields the same roots reflected across the y-axis.

To see that b_c has a root of order k at the origin, consider the circle of radius $R = (\frac{|c|k}{n})^{\frac{1}{n-k}}$, which is exactly the location of the singular points of b_c, and let $0 < r < R$. Note that $b_c(z)$ has no singular roots in the disk $|z| < r$. Since $n > k$ it is trivial to see that $r^{n-k} < R^{n-k} < |c|$ for all $c \neq 0$, so that

$$|z^n| = r^n < |c|r^k = |c\overline{z}^k|.$$

Because $r > 0$ is arbitrarily small, by our extention of Rouché's Theorem we conclude that b_c has a root of multiplicity k at the origin. Therefore, b_c has $n+k+1$ distinct roots, with the root at the origin having multiplicity k for a total of $n + 2k$ roots. □

3 Complex-Harmonic Trinomials

Although the study of complex-harmonic binomials provides some area of interest, trinomials of the form $p_c(z) = z^n + \overline{z}^k - 1$ provide a broader view of complex-harmonic polynomials and their relation to the Fundamental Theorem of Algebra. In this section, motivated by work done in [1] and [5], we consider the problem of counting the roots of trinomials, as well as the extension of locating these roots of trinomials in relation to the unit circle.

In [1], the analytic trinomial $q(z) = z^n + z^k - 1$ is shown to have $2g$ unimodular roots when $6g$ divides $n+k$, where $g = \gcd(n,k)$. Furthermore, this work is extended in [5] to show that q has $2g\left\lceil \frac{n+k}{6g} \right\rceil - g$ interior and $n - 2g\left\lfloor \frac{n+k}{6g} \right\rfloor - g$ exterior roots. This result for analytic trinomials motivates our work on locating the roots of the complex-harmonic trinomial $p_1(z) = z^n + \overline{z}^k - 1$ in Sections 3.2 and 3.3.

Before attempting to locate all the roots of p_1, however, we first show in Section 3.1 that p_1 has n roots. As seen above, this claim is non-trivial, since we cannot simply apply the Fundamental Theorem of Algebra. We extend this work in Section 3.4 to count the number of roots for trinomials of the form $p_c(z) = z^n + c\overline{z}^k - 1$ for particular values of c.

3.1 Counting the Roots of p_1

Using our extension of Rouché's Theorem, we prove the following:

Theorem 3.1
The trinomial $p_1(z) = z^n + \overline{z}^k - 1$ has exactly n roots.

Proof. Let $f(z) = z^n - 1$ and $g(z) = \overline{z}^k$. It is evident that f is sense-preserving everywhere, and thus has no singular zeros. In order to show that we can apply the extension of Rouché's Theorem to our choice of f and g, we show that $p_1 = f + g$ has no singular zeros.

In what follows, we let $R = \left(\frac{k}{n}\right)^{\frac{1}{n-k}}$. Note that the dilatation of p_1 is given by

$$\omega(z) = \left(\frac{k}{n}\right)\frac{1}{z^{n-k}}.$$

It is evident then, that p_1 is sense-preserving if and only if $|z| > R$. Moreover, p_1 is sense-reversing if and only if $|z| < R$. Thus p_1 is singular only on the circle $|z| = R$. The following lemmas (first proved by Ethan Berkove of Lafayette College and Russell Howell of Westmont College, respectively) show that p_1 has no zeros on this circle.

Lemma 3.1
If $1 \leq k < n$, then $(n+k)^n k^k < (n+k)^k n^n$.

Proof. Note first that, letting $x = k/n$ and using common properties of logs,

$$\begin{aligned} (n+k)^n k^k < (n+k)^k n^n &\iff (1+k/n)^n < (1+n/k)^k \\ &\iff (1+x) < (1+1/x)^x \\ &\iff \ln(1+x) < \ln(1+1/x)^x \\ &\iff \ln(1+x) < x\ln(1+x) - x\ln x. \end{aligned}$$

It is sufficient to show that $h(x) = x\ln(1+x) - x\ln x > 0$ on the open interval $(0,1)$.

Begin by observing that

$$h''(x) = \frac{x-1}{x(x+1)^2}.$$

Thus, $h''(x) < 0$ on $(0,1)$ and h is concave down within the interval. Moreover, since $\lim_{x\to 0^+} h(x) = 0$ and $\lim_{x\to 1^-} h(x) = 0$, we see that $h(x) > 0$ for all $x \in (0,1)$. □

Lemma 3.2
$p_1(z_R) \neq 0$ for any $z_R \in \{z : |z| = R\}$ where $R = \left(\frac{k}{n}\right)^{1/(n-k)}$.

Proof. Suppose to the contrary that $p_1(z_R) = 0$ for some $z_R = Re^{i\theta}$. Equating the real components of this expression and rearranging, we see that $R^n \cos n\theta + R^k \cos k\theta = 1$. Thus,

$$R^{n-k}\cos n\theta + \cos k\theta = \frac{1}{R^k}.$$

Since $R^{n-k} = \frac{k}{n}$, substituting and multiplying by n yields

$$k\cos n\theta + n\cos k\theta = \frac{n}{R^k}.$$

However, $k\cos n\theta + n\cos k\theta < k + n$, and by Lemma 3.1, $k + n < \dfrac{n}{R^k}$, since

$$\begin{aligned} k+n < \frac{n}{R^k} &\iff (k+n)^{n-k} < \frac{n^{n-k}}{\left(\frac{k}{n}\right)^k} \\ &\iff k^k(k+n)^{n-k} < n^n \\ &\iff (n+k)^n k^k < (n+k)^k n^n. \end{aligned}$$

Thus $R^n \cos n\theta + R^k \cos k\theta < 1$, and so for any $z_R \in \{z : |z| = R\}$, we have $p_1(z_R) \neq 0$. □

Thus $p_1 = f + g$ has no singular roots. The remainder of the proof of Theorem 3.1 is a straightforward application of Rouché's Theorem:

Let $D = D_\rho(0)$ and $C = C_\rho(0)$ for $\rho > 2$, and let $z \in C$. Then $\rho^k < \rho^n - 1$, and

$$\begin{aligned} |g(z)| &= \rho^k \\ &< \rho^n - 1 \\ &\leq |\rho^n - 1| = |f(z)|. \end{aligned}$$

Since f has n roots inside C, and $\rho > 2$ is arbitrary, $p_1(z) = z^n + \overline{z}^k - 1$ has exactly n roots for all $1 \leq k < n$, and we have proved Theorem 3.1. □

3.2 Unimodular Roots of p_1

The result that p_1 has n roots is significant, and provides us with the confidence to attempt locating these roots. Following exactly the process in [1] with some slight modifications, we prove the following fact about the unimodular roots of p_1:

Theorem 3.2

Let $p_1(z) = z^n + \overline{z}^k - 1$, and let $g = \gcd(n, k)$. If $6g$ divides $n - k$, then p_1 has exactly $2g$ unimodular roots, conjugate pairs z_m and $\overline{z_m}$ determined by

$$z_m = \exp\left[i\left(\frac{\pi}{3g} + \frac{2\pi m}{g}\right)\right],$$

where $0 \leq m \leq g - 1$.

Lemma 3.3

If p_1 has unimodular roots, then $n - k = 0 \mod 6$

Proof. Assume $p_1(z) = 0$ for some $z = e^{i\theta}$. Then $e^{in\theta} + e^{-ik\theta} = 1$. As in [1], we see that $e^{in\theta}$ and $e^{-ik\theta}$ are guaranteed to be conjugates satisfying $e^{in\theta} = e^{ik\theta} = \frac{1}{2} \pm i\sqrt{\frac{3}{2}} = e^{\pm i\pi/3}$.

Thus there exist integers α and β such that

$$n\theta = \pm\frac{\pi}{3} + 2\pi\alpha, \tag{1}$$

$$k\theta = \pm\frac{\pi}{3} + 2\pi\beta, \tag{2}$$

Solving for θ yields

$$\pm\frac{\pi}{3n} + \frac{2\pi\alpha}{n} = \pm\frac{\pi}{3k} + \frac{2\pi\beta}{k},$$

which simplifies to

$$n - k = \pm 6(\alpha k - \beta n), \tag{3}$$

which completes the proof. □

Lemma 3.4

Let $n, k \in \mathbb{N}$, with $1 \leq k \leq n-1$ and let $g = \gcd(n,k)$. Then $p_1(z) = z^n + \overline{z}^k - 1$ has unimodular roots iff $q(z) = z^{\frac{n}{g}} + \overline{z}^{\frac{k}{g}} - 1$ has unimodular roots.

Proof. The proof of this fact is exactly that in [1]. We include it here for completeness.

Assume that λ is unimodular with $p(\lambda) = 0$. Then λ^g is also unimodular and

$$q(\lambda^g) = (\lambda^g)^{\frac{n}{g}} + \overline{(\lambda^g)}^{\frac{k}{g}} - 1 = \lambda^n + \overline{\lambda}^k - 1 = p_1(\lambda) = 0.$$

Similarly, suppose that γ is unimodular and that $q(\gamma) = 0$. Let $\omega^g = \gamma$ so that ω is a unimodular gth root of γ. Then

$$p_1(\omega) = (\omega)^n + (\overline{\omega})^k - 1 = (\omega^g)^{\frac{n}{g}} + (\overline{\omega}^g)^{\frac{k}{g}} - 1 = q(\gamma) = 0,$$

which completes the proof. □

Lemma 3.5

If $n - k = 0 \mod 6$ and $\gcd(n,k) = 1$, then $e^{\pm i\pi/3}$ are the only unimodular roots of $p_1(z) = z^n + \overline{z}^k - 1$.

Proof. By the hypotheses on n and k, neither is divisible by 2 or 3. Thus, both integers are congruent to $\pm 1 \mod 6$. That is, there exist nonnegative integers s and t such that $n = 6s \pm 1$ and $k = 6t \pm 1$. We assume $n = 6s + 1$ and $k = 6t + 1$, the other case being similar. It is evident that, since

$$p_1(z) = z^{6s+1} + \overline{z}^{6t+1} - 1 = z^{6s}z + \overline{z}^{6t}(\overline{z}) - 1,$$

we have $p_1(e^{\pm i\pi/3}) = 0$ by direct calculation.

Now suppose that $p_1(e^{i\theta}) = 0$. Then from (3), we get two equations of α and β, namely:

$$6\beta k - 6\alpha n = n - k \tag{4}$$

$$6\beta k - 6\alpha n = k - n. \tag{5}$$

Since $\gcd(n,k) = 1$, it follows that $\gcd(6n, 6k) = 6$, and since we assumed that $n - k = 0 \mod 6$, the "classical result from the theory of linear Diophantine equations" (hereafter "The Diophantine Theorem") referenced in [1] holds.

Recalling that $n = 6s + 1$ and $k = 6t + 1$, we note that $\alpha = t$, $\beta = s$ is a solution for equation (4), and that $\alpha = -t$, $\beta = -s$ is a solution for equation (5). By the Diophantine Theorem then, the complete set of integer solutions for equation (4) is given by

$$\alpha = t + m \cdot n, \quad \beta = s + m \cdot n \tag{6}$$

for $m \in \mathbb{Z}$. Similarly, for equation (5) the complete solution set is given by

$$\alpha = -t - m \cdot n, \quad \beta = -s - m \cdot n \tag{7}$$

for $m \in \mathbb{Z}$. Subtracting equation (2) from equation (1), substituting $n = 6s + 1$ and $k = 6t + 1$, and solving for θ yields

$$\theta = \frac{\pi}{3} \cdot \frac{\alpha - \beta}{s - t} \tag{8}$$

Substituting the possible values for α and β from (6) and (7) into equation (8) gives $\theta = \pm\frac{\pi}{3} + 2m\pi = \pm\frac{\pi}{3} \mod 2\pi$ as required. □

Combining these lemmas completes the proof of Theorem 3.2.

3.3 Interior and Exterior Roots of p_1

3.3.1 Introduction and Motivation

Together, [1] and [5] completely categorize the location of roots relative to the unit circle for analytic trinomials of the form $p(z) = z^n + z^k - 1$. The following conjectures motivate a search for analogous results for complex-harmonic trinomials of the form $p_1(z) = z^n + \overline{z}^k - 1$. In the following conjectures, we let $g = \gcd(n, k)$.

Conjecture 3.1
The number of interior roots of p_1 is $2g\left\lceil \frac{n-k}{6g} \right\rceil - g$.

Conjecture 3.2
The number of exterior roots of p_1 is $n - 2g\left\lfloor \frac{n-k}{6g} \right\rfloor - g$.

We adopt the same process and terminology as in [5] in order to provide an upper bound for the number of interior and exterior roots of p_1. As the following sketch of this process shows, however, this approach does not yield the expected results and overestimates the total number of roots.

For the purposes of the following sections, we suppose that n and k are relatively prime, though we suspect that (as in [5]) this can be generalized with similar results.

3.3.2 Native Zones for Interior Roots

Suppose $|z_0| < 1$ for some $z_0 \in \mathbb{C}$ with $p_1(z_0) = 0$. Then since $\overline{z} = \frac{|z|^2}{z}$ for all $z \neq 0$, we see that $|z_0|^k \left| \frac{z_0^{n+k}}{|z_0|^{2k}} + 1 \right| = |z_0^n + \overline{z_0}^k| = 1$ so that, since $|z_0|^k < 1$, we have $\left| \frac{z_0^{n+k}}{|z_0|^{2k}} + 1 \right| > 1$. As in [5], this leads us to see that $\text{Arg}\,(z_0^{n+k}) \in \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right)$, so that z_0 itself resides in one of $n+k$ disjoint regions which we dub *native zones*:

$$N_m = \left\{ re^{i\theta} : \theta \in \left(-\frac{2\pi}{3(n+k)} + m\frac{2\pi}{n+k}, \frac{2\pi}{3(n+k)} + m\frac{2\pi}{n+k} \right) \right\},$$

where $0 < r < 1$, and $m \in \mathbb{Z}$.

3.3.3 Echo Zones for Interior Roots

Let $q(z) = -z^n(p(1/z))$, and note that $p(z_0) = 0$ if and only if $q(1/\overline{z_0}) = 0$. Following the proof in [5], we see that z_0 must also reside in one of k disjoint rejoins which are dubbed *echo zones*:

$$E_j = \left\{ re^{i\theta} : \theta \in \left(-\frac{\pi}{3k} + j\frac{2\pi}{k}, \frac{\pi}{3k} + j\frac{2\pi}{k} \right) \right\},$$

where $0 < r < 1$ and $j \in \mathbb{Z}$.

3.3.4 Interior Regions for Interior Roots

Because of the work above in Section 3.1, as in [5] we can apply Rouché's Theorem to develop the "Rouché Sectors" described in [1]. Namely, we use the fact that each root of p_1 exists in a region R_a, and that each region R_a contains exactly one root of p_1, where

$$R_a = \left\{ re^{i\theta} : \theta \in \left(\frac{2a\pi}{n} - \frac{\pi}{2n}, \frac{2a\pi}{n} + \frac{\pi}{2n} \right) \right\},$$

with $0 < r < 2$, and $a \in 1, 2, \ldots, n$.

Using this construction of Rouche Sectors, we can continue to follow the proof as in [5] to show that every "interior region" (i.e., every intersection between a native zone and an echo zone) will only intersect at most one Rouché Sector, so that each interior region contains at most one root. Thus, counting the number of interior regions will provide us with an upper bound on the number of interior roots.

3.3.5 Upper Bounds for Roots

Considering the three cases presented in [5], and proceeding with the same techniques for counting interior regions, we see that, since we are assuming $\gcd(n,k) = 1$, we have an upper bound of

$$\left\lceil \frac{k-n}{6} \right\rceil + \left\lceil \frac{n+3k}{6} \right\rceil - 1 \tag{9}$$

interior regions that fall under case 1 (i.e. the right border of an echo zone belongs to a native zone), and an upper bound of

$$-\left\lceil \frac{k-n}{6} \right\rceil + \left\lceil \frac{n+3k}{6} \right\rceil \tag{10}$$

interior regions that fall under cases 2 and 3 (i.e. the right border of a native zone belongs to an echo zone, or the borders co-align, respectively). Adding together expressions (9) and (10) yields an upper bound on the total number of interior regions, and thus on the number of interior roots:

$$2\left\lceil \frac{n+3k}{6} \right\rceil - 1. \tag{11}$$

Applying the same techniques to produce exterior regions, the number of exterior roots is bounded above by

$$\left\lceil \frac{3k-n}{6} \right\rceil + 2\left\lceil \frac{n-k}{6} \right\rceil + \left\lceil \frac{n+3k}{6} \right\rceil - 1. \tag{12}$$

3.3.6 Analysis and Results

Compare equations (11) and (12) to the values proposed in Conjectures 3.1 and 3.2. It is easy to see that these results significantly over-count the conjectured number of roots. To see where these extra regions are coming from, we take a look at the native zones for interior roots.

In [5], analytic trinomials produce a total of $n-k$ interior native zones. In the complex-harmonic case, we instead have $n+k$ such zones. Although Conjecture 3.1 proposes *fewer* interior roots than in the analytic case, the increase in the number of native zones creates more interior regions, and thus provides an upper bound that is not sharp on the number of interior roots.

Totalling the number of interior and exterior roots, we notice something of interest. Namely, summing expressions (11) and (12) suggests that the total number of roots is bounded above by

$$\left\lceil \frac{3k-n}{6} \right\rceil + 2\left\lceil \frac{n-k}{6} \right\rceil + 3\left\lceil \frac{n+3k}{6} \right\rceil - 2 \le \frac{3k-n+2(n-k)+3(n+3k)}{6} - 2$$
$$< \frac{4n+10k}{6}$$
$$< n+2k.$$

This bound obviously over-counts the number of roots of p_1 since, as shown in Section 3.1, p_1 always has a total of n roots. However, when paired with the following results of Section 3.4, the bound $n+2k$ takes on some significance.

3.4 Counting the Roots of p_c

A generalized problem of that presented in Section 3.1 is to determine the number of roots for a polynomial p_c with a coefficient attached to the second term. Namely, we consider $p_c(z) = z^n + c\overline{z}^k - 1$ for $c \in \mathbb{R}^+$. The dilatation ω of p_c satisfies

$$|\omega(z)| = \frac{ck}{n|z|^{n-k}}.$$

Set $R = \left(\frac{ck}{n}\right)^{\frac{1}{n-k}}$ and note that p_c is sense-preserving if $|z| > R$, sense-reversing if $|z| < R$, and singular wherever $|z| = R$.

We show that p_c has between n and $n+2k$ roots, and that under certain conditions on c, p_c attains these bounds. (Note that this bound of $n+2k$ roots coincides with the bound suggested by our work at the end of Section 3.3.6.)

In what follows, we let $c_1 = \frac{n}{n+k}\left(\frac{n+k}{k}\right)^{\frac{k}{n}}$, $c_a = \frac{n}{n-k}\left(\frac{n-k}{k}\right)^{\frac{k}{n}}$, and $D_R = \overline{D(0,R)}$.

Theorem 3.3

When $c < c_1$, the trinomial $p_c(z) = z^n + c\overline{z}^k$ has exactly n roots outside the disk D_R. When $c > c_a$, p_c has exactly k roots inside D_R and $n+k$ roots outside D_R for a total of $n+2k$ roots.

The following lemmas combine to provide a complete proof of Theorem 3.3.

Lemma 3.6

If $c < c_1$, then p_c has no roots in the disk D_R.

Proof. Note that R is a function of c. We will suppress the dependence on c. For $|z| \leq R$, we have

$$\begin{aligned} |p_c(z)| &\geq 1 - c|\overline{z}^k| - |z^n| \\ &\geq 1 - cR^k - R^n. \end{aligned}$$

However, $c < c_1$ implies

$$\left(\frac{n}{k}\right)^{\frac{k}{n-k}} \left(\frac{n}{n+k}\right) > c^{\frac{n}{n-k}},$$

so that

$$\begin{aligned} 1 &> c^{\frac{n}{n-k}} \left(\frac{k}{n}\right)^{\frac{k}{n-k}} \left(\frac{n+k}{n}\right) \\ &= c^{\frac{n}{n-k}} \left(\frac{k}{n}\right)^{\frac{k}{n-k}} \left(1 + \frac{k}{n}\right) \\ &= c^{\frac{n}{n-k}} \left[\left(\frac{k}{n}\right)^{\frac{k}{n-k}} + \left(\frac{k}{n}\right)^{\frac{n}{n-k}}\right] \\ &= c\left(\frac{ck}{n}\right)^{\frac{k}{n-k}} + \left(\frac{ck}{n}\right)^{\frac{n}{n-k}} \\ &= cR^k + R^n. \end{aligned}$$

□

Thus $1 - cR^k - R^n > 0$ and p_c has no roots inside D_R. Note as a corollary that p_c has no singular roots.

Lemma 3.7

If $c < c_1$, then p_c has exactly n roots outside the disk D_R.

Proof. Since p_c was just shown to have no roots on the circle of radius R, p_c has no singular roots and we can apply the harmonic extension of Rouché's Theorem.

Let $f(z) = z^n - 1$ and $g(z) = c\overline{z}^k$, and see that $p_c = f + g$. Note that $c < c_1$ implies $R < 1$. Consider a circle given by $|z| = M > 1$, and note that f has exactly n roots inside M. Since $n > k$, we see that

$$\lim_{M\to\infty} \frac{M^n - 1}{cM^k} = \infty$$

so that there exists $M_0 > 1$ for which $M \geq M_0$ implies

$$\frac{M^n - 1}{cM^k} > 1,$$

or, equivalently,

$$|g(z)| = cM^k < M^n - 1 \leq |f(z)|.$$

Thus by Rouché's Theorem, p_c has n roots, and combined with Lemma 3.6, we see that all these roots lie outside the disk D_R. □

We have succeeded in categorizing the roots of p_c when $c < c_1$. The following lemma considers the case when $c > c_a$.

Lemma 3.8

If $c > c_a$, then p_c has exactly k roots inside the disk D_R and exactly $n+k$ roots outside the disk D_R.

Proof. First note that $c > c_a$ is equivalent to

$$c^{\frac{n}{n-k}} > \frac{n}{n-k}\left(\frac{n}{k}\right)^{\frac{k}{n-k}}.$$

We rewrite this as

$$\left(\frac{ck}{n}\right)^{\frac{n}{n-k}} + 1 < c\left(\frac{ck}{n}\right)^{\frac{k}{n-k}},$$

which is exactly $cR^k - R^n - 1 > 0$, or equivalently, $1 < cR^k - R^n$.

Using a similar approach to our proof of Lemma 3.6, we see that whenever $|z| \leq R$, we have $|p_c(z)| \geq cR^k - R^n - 1 > 0$ so that p_c has no singular roots.

For our application of Rouché's Theorem, we appeal to Theorem 2.1 to see that the binomial $b_c(z) = z^n + c\bar{z}^k$ has $n+k$ roots on a circle of radius $c^{\frac{1}{n-k}}$ and k roots at the origin. Note that since $k < n$, we have $c^{\frac{1}{n-k}} > R$, so that b_c has exactly k roots inside the disk D_R.

Furthermore, applying Rouché's Theorem to $f(z) = b_c(z)$ and $g(z) = -1$, we see that

$$1 < cR^k - R^n \leq |c\bar{z}^k - z^n| = |b_c(z)|.$$

Thus since b_c and p_c are non-singular within D_R, by Rouché's Theorem, p_c has k roots in the disk D_R.

Moreover, using Rouché's Theorem, we show that there exists M_0 such that, for all $M > M_0$, p_c has $n+k$ zeros in $\{z : R < |z| < M_0\}$, so that p_c has $n+k$ roots outside the disk D_R.

Consider the case on the circle $|z| = M$ for $M > c^{\frac{1}{n-k}}$, and note that

$$|b_c(z)| \geq M^n - cM^k.$$

Since $k < n$, we have $c^{\frac{1}{n-k}} > R$, and the $n+k$ non-zero roots of b_c are inside the annulus $A = \{z : R < |z| < M\}$. Furthermore,

$$\lim_{M\to\infty} M^n - cM^k = \infty.$$

Thus there exists $M_0 > c^{\frac{1}{n-k}}$ so that for all $M > M_0$, we have $M^n - cM^k > 1$. Thus on any circle of such a radius M,

$$1 < M^n - cM^k \leq |b_c(z)|.$$

From this, we appeal to Rouché's Theorem to see that p_c has a total of $n+2k$ roots, but since k of these roots are inside the disk D_R, there are $n+k$ roots outside the disk D_R. □

Combining the previous lemmas concludes our proof of Theorem 3.3. □

3.5 Further Results

3.5.1 Roots of $z^n + c\overline{z}^n - 1$

While a Fundamental Theorem of Algebra for all complex-harmonic polynomials would be an ideal generalization of the problem domain, there are a variety of specialized cases in trinomials alone that indicate formulating such a statement (let alone providing a proof of it) would be a significant challenge. For example, the above theorems suppose that $k < n$, but for obvious reasons, when $k = n$ the value $c_a = \frac{n}{n-k}\left(\frac{n-k}{k}\right)^{\frac{k}{n}}$ is not defined. The following theorem considers this specific case.

Theorem 3.4

Let $p_c^*(z) = z^n + c\overline{z}^n - 1$ for $n \in \mathbb{N}$ and $c > -1$, $c \neq 1$. Then p_c^* has n zeros, all of which lie evenly spaced on the circle $|z| = \frac{1}{(c+1)^{1/n}}$.

Proof. First suppose that $c > 1$. Since, for the dilatation ω of p_c^*, we have $|\omega(z)| = |\frac{1}{c}| < 1$, we see that p_c^* is sense-preserving everywhere and we can apply Rouché's Theorem. Let $g(z) = z^n$ and $f(z) = c\overline{z}^n - 1$. Notice that for any $R > \frac{1}{(c-1)^{1/n}}$ we have $R^n < cR^n - 1$ so that, using any circle of such a radius R,

$$\begin{aligned} |g(z)| &= |z^n| \\ &= R^n \\ &< cR^n - 1 \\ &\leq |c\overline{z}^n - 1| \\ &= |f(z)|. \end{aligned}$$

Next, suppose that $-1 < c < 1$ and note that p_c^* is sense-reversing in this case. For this situation, let $g(z) = c\overline{z}^n$ and $f(z) = z^n - 1$. Again, notice that for any $R > \frac{1}{(1-|c|)^{1/n}}$ we have $|c|R^n < R^n - 1$ so that, applying Rouché's Theorem on a circle with such a radius,

$$\begin{aligned}|g(z)| &= |c\overline{z}^n| \\ &= |c|R^n \\ &< R^n - 1 \\ &\leq |z^n - 1| \\ &= |f(z)|.\end{aligned}$$

Thus, since R is arbitrarily large in each case, $p_c^*(z) = f(z) + g(z)$ has exactly n roots for $c > -1$, $c \neq 1$.

Furthermore, it is trivial to compute that $z_0 = \frac{1}{(c+1)^{1/n}}$ is a root of p_c^*. Let $z_k = z_0 e^{i2k\pi/n}$ for $k = 0, 1, 2, \dots, n-1$. Then each z_k is a distinct root of p_c^*, since

$$\begin{aligned}p_c^*(z_k) &= (z_0 \cdot e^{i2k\pi/n})^n + c(\overline{z_0 \cdot e^{i2k\pi/n}})^n - 1 \\ &= z_0^n e^{-i2k\pi} + c\overline{z_0}^n e^{i2k\pi} - 1 \\ &= z_0^n + c\overline{z_0}^n - 1 \\ &= p_c^*(z_0) \\ &= 0.\end{aligned}$$

Thus, the n roots of p_c^* lie evenly spaced around the circle $|z| = \frac{1}{(c+1)^{1/n}}$. □

Putting this in the context of the work done in Section 3.4, we consider $c_1 = \frac{n}{n+k}\left(\frac{n+k}{k}\right)^{\frac{k}{n}}$. Letting $k = n$, then, we see that $c_1 = 1$. If we were to consider the case for p_c^* with $c = c_1$, we see that p_c^* becomes singular on the entire complex plane, and we would not be able to appeal to Rouché's Theorem.

3.5.2 Locating Roots of p_c

We can also extend the work in Section 3.3 having considered the questions raised by Section 3.4. In particular, we can construct native and echo zones for trinomials of the form $p_c(z) = z^n + c\overline{z}^k - 1$ by considering the case when $c < 1$ and the case when $c > 1$. In these situations, however, we cannot rely on each Rouché Sector to contain exactly one root of p_c, and so the construction ends with the native and echo zones.

First, we consider the interior roots. That is, we suppose that $p_c(z_0) = 0$ for some $|z_0| < 1$. Then $\left|\frac{z_0^{n+k}}{|z_0|^{2n}} + c\right| > 1$. Moreover, using $q(z) = -z^n p_c(1/z)$ and letting $w_0 = 1/\overline{z_0}$, we get the expression $\left|w_0^k - c\right| < 1$.

First, suppose $c < 1$. Then we obtain the native zones

$$N_{m;c<1} = \left\{ re^{i\theta} : \theta \in \left(-\frac{\arccos(c/2)}{n+k} + m\frac{2\pi}{n+k}, \frac{\arccos(c/2)}{n+k} + m\frac{2\pi}{n+k} \right) \right\},$$

and the echo zones

$$E_{j;c<1} = \left\{ re^{i\theta} : \theta \in \left(-\frac{\arccos(c/2)}{k} + j\frac{2\pi}{k}, \frac{\arccos(c/2)}{k} + j\frac{2\pi}{k} \right) \right\}.$$

Next, suppose that $c > 1$. Then we obtain the native zones

$$N_{m;c>1} = \left\{ re^{i\theta} : \theta \in \left(-\frac{2\pi}{3(n+k)} + m\frac{2\pi}{n+k}, \frac{2\pi}{3(n+k)} + m\frac{2\pi}{n+k} \right) \right\},$$

and the echo zones

$$E_{j;c>1} = \left\{ re^{i\theta} : \theta \in \left(-\frac{\arcsin(1/c)}{k} + j\frac{2\pi}{k}, \frac{\arcsin(1/c)}{k} + j\frac{2\pi}{k} \right) \right\}.$$

Note that the zones of $E_{j;c>1}$ are strict when $c > \sqrt{2}$, but for $1 < c < \sqrt{2}$, they can be further restricted by finding the argument of the points at which the circles of radius 1 around the origin and around the point $(c, 0)$ intersect.

As it turns out, the exterior native zones correspond to the interior echo zones, and the exterior echo zones correspond to the interior native zones. In each of these exterior zones, the arguments are simply reflected across the imaginary axis from their Interior counterpart. Further work must be done on the Rouché Sectors to give much meaning to any intersection of these zones.

4 Current Investigations and Future Work

The work above lends itself to multiple avenues of future research, much of which is suitable for undergraduate research. In this section we provide a few conjectures worth pursuing, as well as comment on our current work.

In what follows, we assume $k < n$, and consider the polynomial $p_c(z) = z^n + c\bar{z}^k - 1$ for $c > 0$. Moreover, let $R_0 = (ck/n)^{1/(n-k)}$ and $R_1 = (cn/k)^{1/(n-k)}$. Note that $R_0 < R_1$, and recall $R_0 = R$ from the proofs in Section 3.4.

Conjecture 4.1

We have already proved that the n roots of p_1 lie outside the disk $D_{R_0} = \overline{D(0, R_0)}$. We conjecture that these n roots lie strictly in the annulus $A = (0, R_0, R_1)$.

Conjecture 4.2

For c sufficiently large, we have already proved that p_c has $n+2k$ zeros, k of which have modulus less than R_0. We conjecture the remaining $n+k$ roots lie strictly in the annulus $A=(0,R_0,R_1)$.

Conjecture 4.3

For c sufficiently large, we conjecture that the k roots of p_c inside the disk of radius R_0 approach the k^{th} roots of unity.

Conjecture 4.4

For c sufficiently large, the arguments of the $n+k$ roots outside the disk of radius R_0 clearly become uniformly distributed. We conjecture that these roots approximate the $n+k$ roots of $z^{n+k}-c^{\frac{1}{n-k}}$.

Along with the Conjectures 3.1 and 3.2 for locating the roots of p_1, and the analogous work started in Section 3.5.2 for p_c, our current work includes experimenting and hypothesizing concerning the relationships between complex-harmonic polynomials and "analytic" polynomials.

As another point of interest, we notice in running numerical experiments that the analytic polynomial $z^7+\frac{7}{3}\left(\frac{3}{4}\right)^{\frac{4}{7}}z^4-1$ has a root of multiplicity two. Notice that the coefficient of the second term is equivalent to c_a. We suspect there is some connection between the value of c_a and the coefficients which cause roots of analytic polynomials to obtain multiplicity greater than 1. Note that there are also several examples where this result does not hold (e.g. $n=9$ and $k=5$), so that whatever connection there might be appears rather weak.

In Section 3.4 we introduce the coefficients c_1 and c_a, and demonstrate that for $c<c_1$, p_c has exactly n roots, and for $c>c_a$, p_c has exactly $n+2k$ roots. Our current work includes finding a sequence of coefficiencts $c_1<c_2<c_3<\cdots<c_{a-1}<c_a$ such that the number of roots of p_c increase incrementally whenever c increases beyond each cutoff. Moreover, it is conjectured that there are $k-1$ such cutoffs.

5 The Riemann Hypothesis

We now consider one of the most famous mathematical conjectures in the past couple of centuries, the Riemann Hypothesis. While the Riemann Hypothesis is not correlated in any known way to the theory of complex-harmonic functions, it too deals with locating the roots of a seemingly innocent function.

This hypothesis is notorious for appearing simple, yet evading proof (or disproof) for over 150 years. At its foundation, the hypothesis appears to be centered on Complex Analysis, but a tour through the history of vain attempts to prove (or disprove) the hypothesis shows that it eventually finds its way into almost every branch of mathematics.

5.1 The Hypothesis

In 1859, Bernhard Riemann published the paper "On the Number of Prime Numbers Less Than a Given Quantity," in which he addressed the question of how the prime numbers are distributed. In particular, the Prime Number Theorem (the PNT) is given as follows:

Theorem 5.1 - The Prime Number Theorem [3, p. 45]
Let $N \in \mathbb{N}$. Then $\pi(N) \sim \frac{N}{\log(N)}$, where $\pi(N)$ counts the number of prime numbers less than or equal to N.

In this paper, Riemann explained how $\pi(N)$ (or, equivalently, $\pi(x)$ for $x \in \mathbb{R}^+$) can be written as an expression dependent on the roots of a function now known as "The Riemann Zeta Function," which is an analytic continuation of the real-valued function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. The first step in this analytic continuation gives us a function which converges for all $s \in \mathbb{C} \setminus \{1\}$ with $\operatorname{Re} s > 0$, and allows us to further analytically continue the function to the half plane with a negative real component. For the sake of completeness, we include Riemann's Zeta function below:

Definition 5.1

The **Riemann Zeta Function** $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is defined by

$$\zeta(1-s) = \begin{cases} s^{1-s}\pi^{-s}\sin\left(\frac{1-s}{2}\pi\right)\Gamma(s-1)\zeta^*(s), & \operatorname{Re}(1-s) \leq 0 \\ \zeta^*(1-s), & \operatorname{Re}(1-s) > 0 \end{cases}$$

where $\zeta^*(s) = \frac{\eta(s)}{1-\Gamma(1/2^{s-1})}$, with $\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}(n^{-s})$, and where Γ is the standard Gamma function.

In his paper, Riemann shows how $\pi(x)$ can be transformed into a step function which can then be written in terms of the roots of Riemann's Zeta Function. This allows one to express the distribution of prime numbers as a formula of the roots of Riemann's Zeta Function.

There are some roots which are called "trivial roots," namely, those roots which are negative even integers. The exact location of the remaining "non-trivial" roots of the Riemann Zeta Function is the core of the Riemann Hypothesis, which we supply below:

Conjecture 5.1 - The Riemann Hypothesis Version 1

For any non-trivial root s of ζ, Re $(s) = 1/2$.

The hypothesis itself is so brief that if you're skimming this paper, you might miss it. Yet its significance and brilliance are not to be missed; after all, 150 years of mathematics—now with immense computing power at its disposal—has yet to either prove or disprove this hypothesis. Riemann himself did not attempt any proof of the hypothesis, mostly because he did not consider it relevant to the goal of his paper. In fact, he claims that he has "put aside such search for a proof after some fleeting vain attempts" [3, p. 151].

Perhaps one of the most remarkable things about this hypothesis is that Riemann made it without the use of any of the computational aids we have at our disposal today. Riemann was, as Derbyshire explains, an "intuitive mathematician" [3, p. 152], so that when a hypothesis such as this appears in his work, it is not logically clear how he came to the conclusion that he did. Nevertheless, his intuitions appear to be well founded.

5.2 The Distribution of Prime Numbers

In 1737, Leonard Euler published a paper in which he showed that

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1},$$

where the p values are the primes. This insight from Euler suggests that ζ is intimately related to the prime numbers, and this fact became instrumental in Riemann's original paper and the development of the Riemann Hypothesis.

The book *Prime Numbers and the Riemann Hypothesis* [6] further unpacks (as the title would suggest) the relationship between prime numbers and the Riemann Hypothesis. In particular, they show how the non-trivial roots of the Riemann Zeta function can apparently be derived from the prime numbers, and how the inverse seems to be true as well. In this section, we summarize the approach taken in [6] to connect "The Staircase of Primes" with "The Riemann Spectrum" and develop the relationship to the Riemann Hypothesis.

5.2.1 The Riemann Spectrum of Primes

First, we deal with some preliminary terminology.

Definition 5.2

Given $\{s_k\}$ strictly increasing in $\mathbb{R}$, the **spectrum** of a **trignometric series** $F(\theta) = \sum_{k=1}^{\infty} a_k \cos(s_k \cdot \theta)$ is the sequence $\{s_k\}$.

Definition 5.3

> The trignometric series $F(\theta) = \lim_{C\to\infty} F(\theta, C)$ for $F(\theta, C) = \sum_{s_k \le C} a_k \cos(s_k \cdot C)$ is said to have a **spike** at $\theta = \tau \in \mathbb{R}$ if $|F(\tau, C)|$ is unbounded for all $C \in \mathbb{R}$.

In [6], Mazur and Stein take the function commonly known as the "Staircase of Primes"—that is, the step function $\pi(x)$—and apply a series of reversible transformations to it, including taking powers, logarithms, and Fourier Transforms. This sequence of transformations allows us to construct from the prime numbers the trignometric series

$$\hat{\Phi}(\theta) = 2\sum_{p^n} p^{-n/2}\log(p)\cos(\log(p^n)\theta),$$

where p^n range over all the powers of primes. We give the set of spikes of this function a special name.

Definition 5.4

> The increasing sequence of inputs (θ_k) at which $\hat{\Phi}(\theta)$ has a spike is called **The Riemann Spectrum**.

It is not readily apparent why this set should be called a spectrum in general, nor is it obvious why we call the set The *Riemann* Spectrum in particular. Enumerating this set, however, may provide some insight for those familiar with the Riemann Hypothesis. Namely, we have the following approximations for the first few values of the Riemann Spectrum:

$$\theta_1 \approx 14.134725\ldots$$
$$\theta_2 \approx 21.022039\ldots$$
$$\theta_3 \approx 25.010857\ldots$$
$$\theta_4 \approx 30.424876\ldots$$
$$\theta_5 \approx 32.935061\ldots$$
$$\theta_6 \approx 37.586178\ldots$$

Given this set, we provide a variation of the Riemann Hypothesis which gives us a more clear connection between the prime numbers and the non-trivial roots of the Riemann-Zeta function:

Conjecture 5.2 - The Riemann Hypothesis Version 2

The set of nontrivial zeros of $\zeta(s)$ is the set $\left\{\frac{1}{2} + i\theta_k : k \in \mathbb{N}\right\}$, where the sequence (θ_k) is The Riemann Spectrum.

Moreover, we call this sequence (θ_k) a spectrum because, by using the sequence as a spectrum for another trignometric series, we appear to obtain spikes at all the powers of prime numbers. In fact, the Riemann Hypothesis implies that one can go from The Riemann Spectrum *back* to the prime numbers using such a trignometric series. The Riemann Hypothesis states that the question "can we reconstruct the Staircase of Primes $\pi(x)$ only using Riemann Spectrum" has an affirmative answer (see [6]).

5.2.2 Approximating $\pi(x)$

Riemann developed such a function for approximating $\pi(x)$ using The Riemann Spectrum, and his function appears to model $\pi(x)$ almost exactly! This function requires the use of the following two functions:

Definition 5.5

The **Logarithmic Integral Function** is $\mathrm{li}(x) = \displaystyle\int_0^x \frac{dt}{\log t}$.

Definition 5.6

The **Möbius Function** is

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors,} \\ 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime factors.} \end{cases}$$

An equivalent formulation of the Riemann Hypothesis is stated in terms of the Logarithmic Integral Function:

Conjecture 5.3 - The Riemann Hypothesis Version 3

$\mathrm{li}(x)$ is essentially a square root accurate approximation for $\pi(x)$.

The approximation developed by Riemann, however, appears to completely dwarf that of $\mathrm{li}(x)$. This approximation, defined by $R(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{\frac{1}{n}})$, appears to coincide almost exactly with $\pi(x)$. We direct readers to [6] for striking illustrations of this claim, and for further discussion of the approximation $R(x)$.

5.3 Implications and Connections

What is so fascinating about Riemann's work is that his Zeta function seems to encode a profound sense of regularity among the prime numbers that otherwise appears hidden.

His simple hypothesis has deep implications for our understanding (or lack thereof) of the prime numbers and their distribution among the natural numbers.

While [6] dives into the details of the mathematics of the Riemann Hypothesis, especially focusing on the relationship between the Hypothesis and the distriubtion of prime numbers, the book *Prime Obsession* [3] provides more of a historical and "popular" overview of the Hypothesis and its implications. In this section, we briefly touch on a selection of these topics.

5.3.1 Quantum-Dynamics

One of the less obvious applications of the Riemann Hypothesis may be found in modern physics. A special set of $N \times N$ random matrices called Gaussian-random Hermitian matrices have eigenvalues which find use in modeling the energy levels in experiments on certain quantum-dynamical systems.

To construct a Gaussian-random Hermitian matrix, one uses a Gaussian-normal distribution to select real numbers for the diagonal of the matrix and to select real and imaginary components of complex numbers on the upper half triangle of the matrix. The lower half triangle of the matrix is the complex conjugate of the upper half, reflected across the diagonal.

The eigenvalues of such random matrices, however, actually demonstrate a striking amount of less-than-random structure. In particular, the eigenvalues appear to conform to some level of uniform spacing—what we call the "repulsion effect"—so that very few pairs of eigenvalues are very close together.

What does all this have to do with the Riemann Hypothesis? It turns out that the Riemann Spectrum demonstrates a striking amount of repulsion as well! So there appears to be some correlation between the Riemann Spectrum and the behavior of subatomic particles. For a more in-depth look at the formation of these results and the (bizarre!) history of how this correlation was first noticed, we direct the reader to [3, 280-295].

5.3.2 ERH and Cryptography

In [3], it is explained that the proof of a variety of theorems are awaiting the proof of the Riemann Hypothesis. In particular, a proof might begin "suppose the Riemann Hypothesis," and then produce a conclusion from that assumption. Such results have led to plenty of "myths" concerning the Riemann Hypothesis.

One such popular myth claims that a proof of the Hypothesis will break modern cryptography. Because algorithms such as RSA utilize the difficulty of factoring large composite numbers into their large prime components, and because the Riemann Hypothesis suggests some sort of regularity about the prime numbers, it would seem that the proof of the Hypothesis would lead to a method for cracking such cryptography systems.

We call this claim a "myth," however, because there seems to be a fundamental misunderstanding of what the Riemann Hypothesis will imply concerning our understanding of prime numbers. For example, in [7], a proof is provided that an algorithm for "prime factorization" would take polynomial time under what is called the "Extended Riemann Hypothesis" (or *ERH*). What is important to note, however, is that this result is only good for testing whether a number is prime, *not* for factoring large composite numbers.

Because algorithms such as RSA rely on the difficulty of factoring large composite numbers, the proof of the (Extended) Riemann Hypothesis does not necessarily break cryptography systems. In fact, some cryptography systems might assume the truth of the Extended Riemann Hypothesis in their implementation with an added benefit.

Suppose we have a cryptographic system C which tests for primality assuming ERH, such as that described in [7]. In other words, for some computational result C, suppose $ERH \implies C$. If the Extended Riemann Hypothesis is in fact true, then we benefit from the computational result of C, and we can continue to use C without any worry. On the other hand, if applying the computational result ever produces an error, then since $\neg C \implies \neg ERH$, we would have disproved ERH—a tremendous breakthrough! As [2] puts it, this sort of error by C would result in "fame, if you will, by modus tollens."

6 Conclusion

Throughout this paper, we have explored a variety of complex functions which, though seemingly simple, hide a great deal of complexity behind the covers. The topics addressed above indicate that the field of Complex Analysis is perhaps not quite as well understood as some would like to suppose. In general, the study of complex-harmonic functions provides a fresh look into Complex Analysis, raising questions about even some of the most fundamental results in the field.

Moreover, complex-harmonic polynomials in particular provide a wonderful opportunity for research accessible to undergraduate students. As our work above suggests, with a little bit of experimentation, one can find some rather fascinating results.

Meanwhile, the Riemann Hypothesis remains one of the greatest unsolved mathematical conjectures of the past centuries, and its practical implications continue to grow. Although there is little to indicate that a breakthrough is on the horizon, there is plenty to be explored in terms of the formation, history, and implications of the Hypothesis. Of course, if one is up for the challenge of proving (or disproving) the Hypothesis, one can feel free to try his or her hand at that as well.

Appendix A Mathematica Code

The following code was created using Wolfram Mathematica 11 Student Edition, Version Number 11.3.0.0.

Level Curves

This code produces the level curves for $p_c(z) = z^n + c\bar{z}^k - 1 = 0$, with n, k and c initially set to 7, 4, and 1 respectively. It is important to note that the count provided by Mathematica cannot always be trusted, since roots of multiplicity greater than 1 may be over-counted.

```
(* Set up *)
Clear["Global*`"];
n = 7;
k = 4;
c = 1;
p[z_] = z^n + c*Conjugate[z]^k - 1;

(* Solve p_c(z) = 0 and obtain level curves *)
re = ComplexExpand[Re[p[x + I y]]];
im = ComplexExpand[Im[p[x + I y]]];
zeros = NSolve[{re == 0, im == 0}, {x, y}, Reals];
Print["The number of zeros is ", Length[zeros]];

(* Tabulate locations of roots *)
complexzeros = Table[-1000 + 1000 I, {Length[zeros]}];
For[i = 1, i <= Length[zeros], i++,
    complexzeros[[i]] = zeros[[i]][[1]][[2]] + I zeros[[i]][[2]][[2]];
]
complexzeros;

(* Plot graphs *)
c = ListPlot[ReIm[complexzeros]];
a = ContourPlot[Re[p[x + I y]], {x, -2, 2}, {y, -2, 2},
    Contours -> {0}, ContourShading -> False, ContourStyle -> Blue];
b = ContourPlot[Im[p[x + I y]], {x, -2, 2}, {y, -2, 2},
    Contours -> {0}, ContourShading -> False, ContourStyle -> Red];
d = Graphics[Circle[{0, 0}]];

Show[a, b, c, d]
```

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