An Introduction to Tiling Spaces

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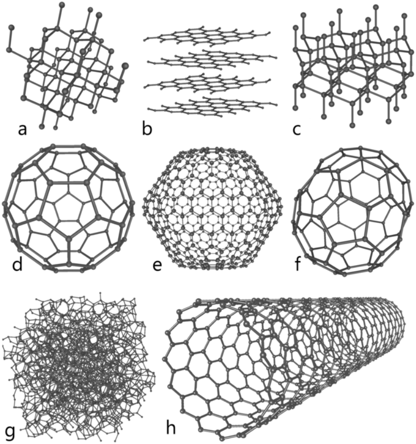
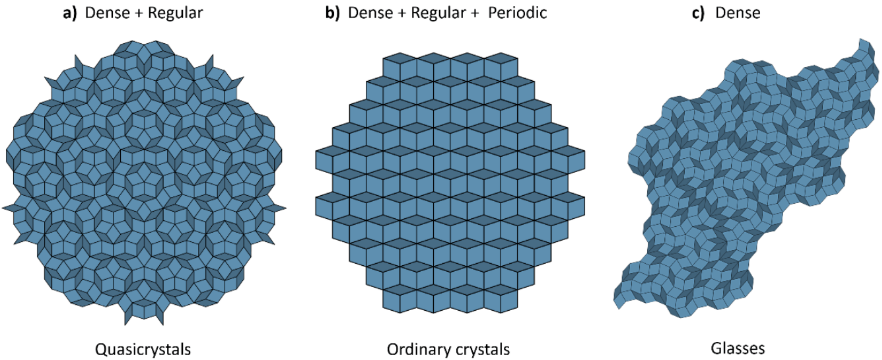
Outline



Credit: *https://courses.lumenlearning.com/introchem/chapter/allotropes-of-carbon/*

Figure: Eight Allotropes of Carbon



Credit: *https://matmatch.com/resources/blog/quasicrystals-materials-that-should-not-exist/*

Figure: Three different kinds of material

Quasicrystals and TDA

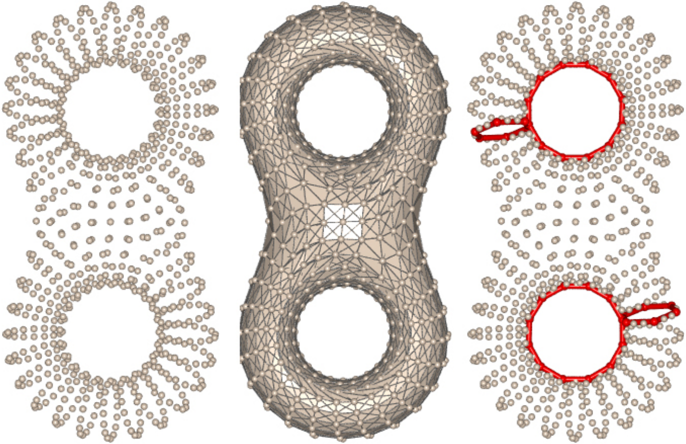


Figure: Point Clouds and Topological Data Analysis

Tilings of $\mathbb{R}^d$

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Notation

If $U \subset \mathbb{R}^d$, the *patch of U* is the set of tiles t which meet U, denoted $[U]$.

Simple Tilings

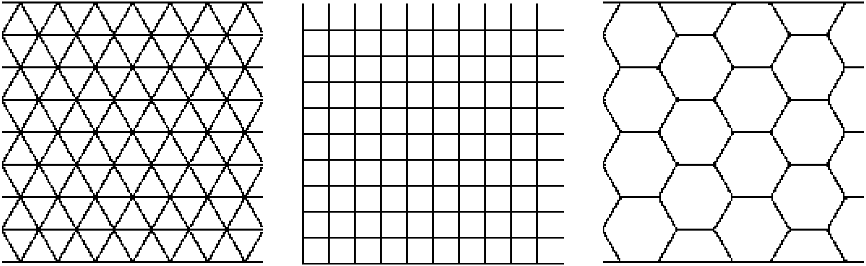

Figure: Periodic Tilings

Simple Tilings

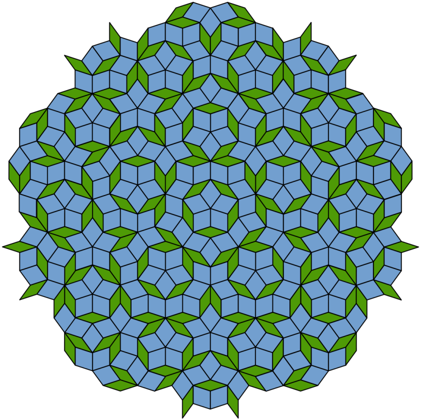

Figure: A Patch of the Penrose Tiling

Violating Hypotheses

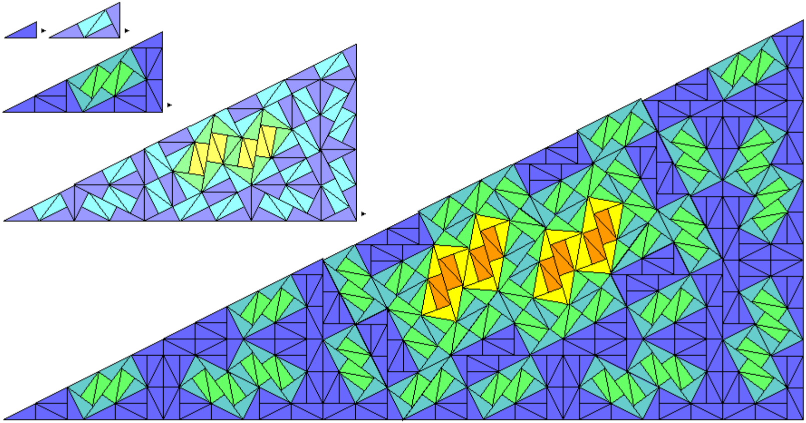

Figure: A Pinwheel Tiling. Lacks finitely many prototiles up to translation.

Violating Hypotheses

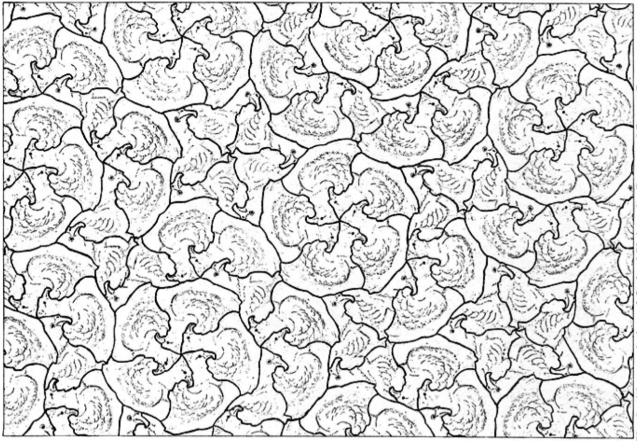

Figure: Penrose Chickens. Tiles are not polytopes.

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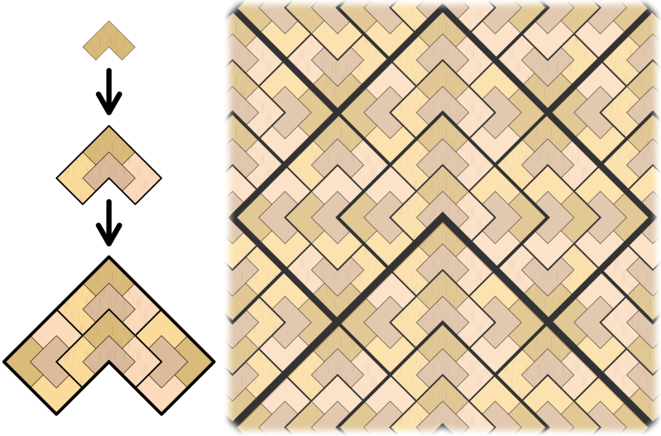

Figure: A chair tiling. Edges don't meet full-face to full-face

Equivalence of Tilings

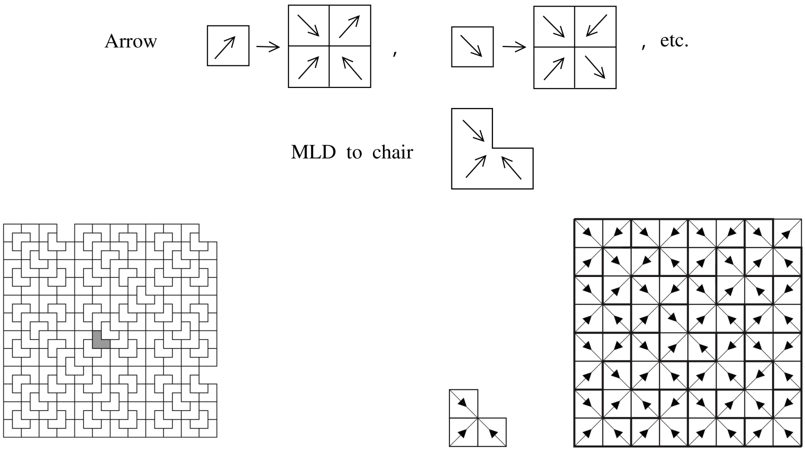

Figure: The Arrow Tiling is MLD to the chair tiling

Definition (Tiling Metric)

Given two tilings, T and T', of $\mathbb{R}^d$, they are ε-*close* ($\varepsilon > 0$) if up to a translation of distance ε, they agree on a ball of radius ε^{-1} around the origin.

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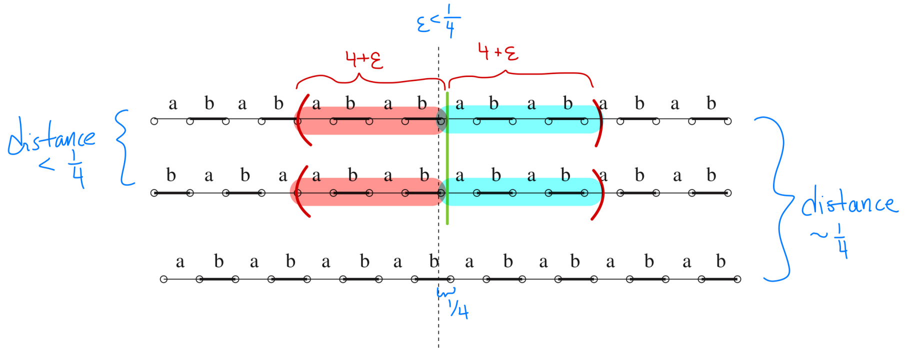
Note

Compare this definition to that of the *Gromov-Hausdorff distance* between two based metric spaces M and M'.

The Hull as an Orbit



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The *orbit* of a tiling T is the set

$$\mathcal{O}(T) := \left\{ T - x \mid x \in \mathbb{R}^d \right\}$$

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The *hull* Ω_T of a tiling T is the closure of $\mathcal{O}(T)$ in the tiling metric.

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The *hull* Ω_T of a tiling T is the closure of $\mathcal{O}(T)$ in the tiling metric.

Note

The hull Ω_T is closed under translation by $\mathbb{R}^d$, and complete in the tiling metric, and is therefore called a *tiling space*.

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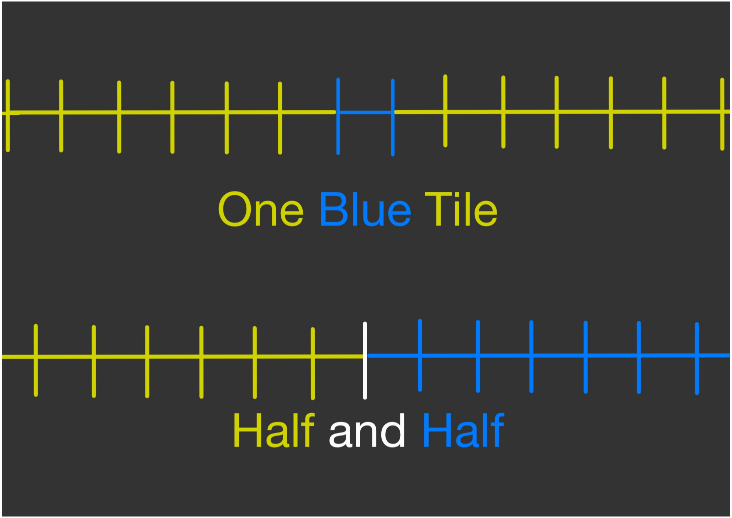

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If T is a simple tiling, Ω_T is a compact metric space.

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Take a sequence $\{T_i\} \subset \Omega_T$. For each $r > 0$, there are only finitely many patches around balls of radius r up to translation. So there is a subsequence which converges on $B_r(0)$.

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Note

There are a few other tiling spaces of interest, namely allowing the Euclidean rotations of T. In general these give new tiling spaces.

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For each $n \in \mathbb{N}$, let Γ_n be the possible instructions for laying n layers of tiles around some tile at the origin (called the n-th Gähler approximant). Let $f_n : \Gamma_{n+1} \to \Gamma_n$ be the forgetful map. The hull of T is the inverse limit

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Note

This gives some more intuition about the hull: if $T' \in \Omega_T$, then every patch of T' is found somewhere in a translate of T, and gives the same space as in orbit-closure definition, but with some more clear structure.

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We'll see some other advantages to this perspective later on when we study the cohomology of Ω_T.

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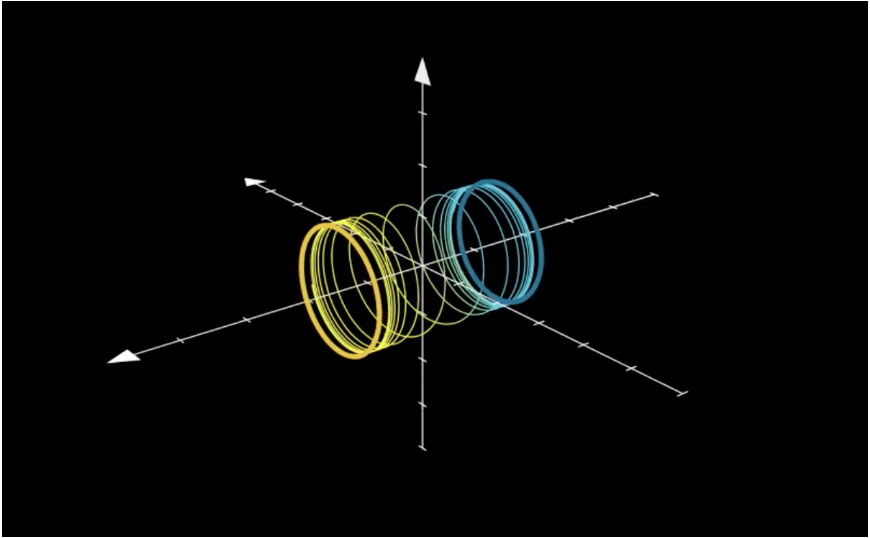
Question: How does the action of $\mathbb{R}^d$ on T interact with the hull Ω_T?

The Hull as an Inverse Limit



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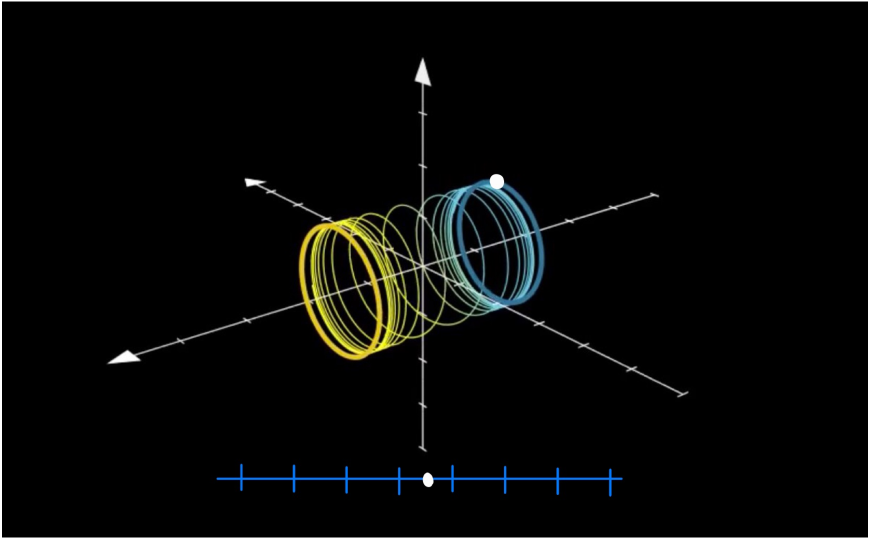

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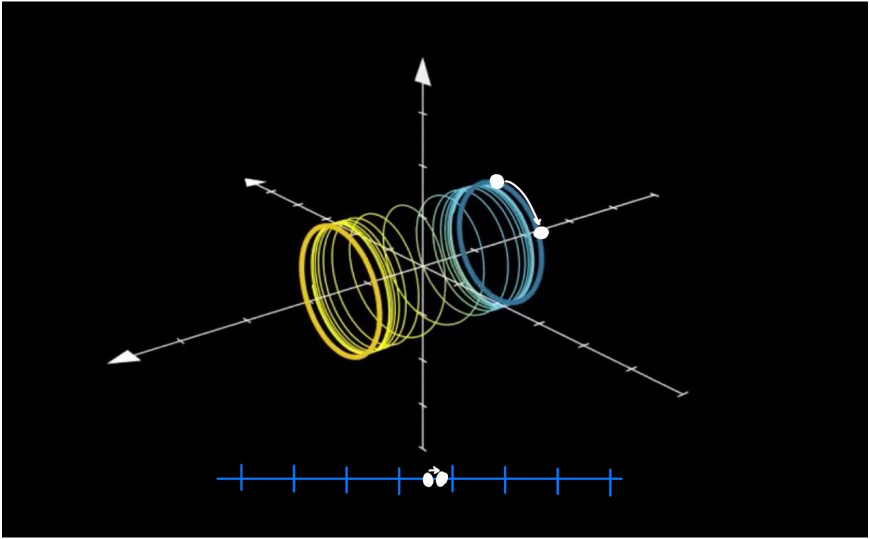

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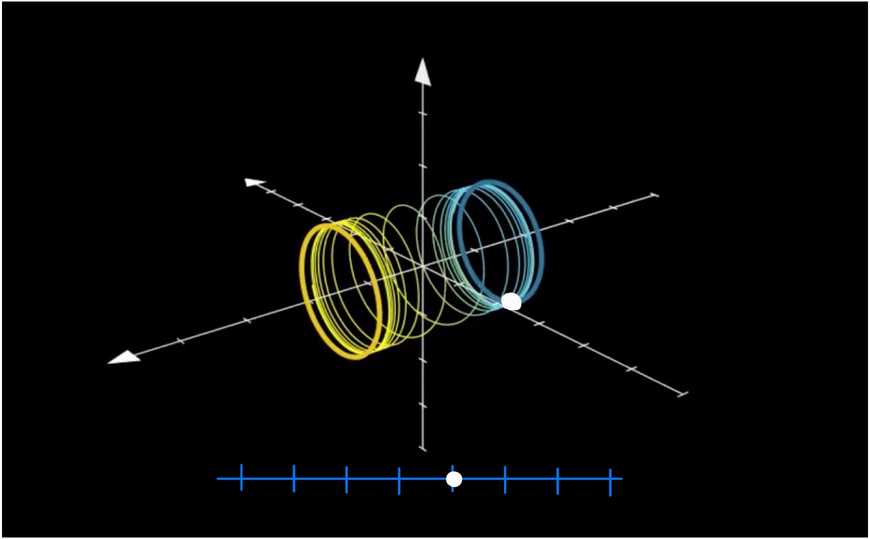

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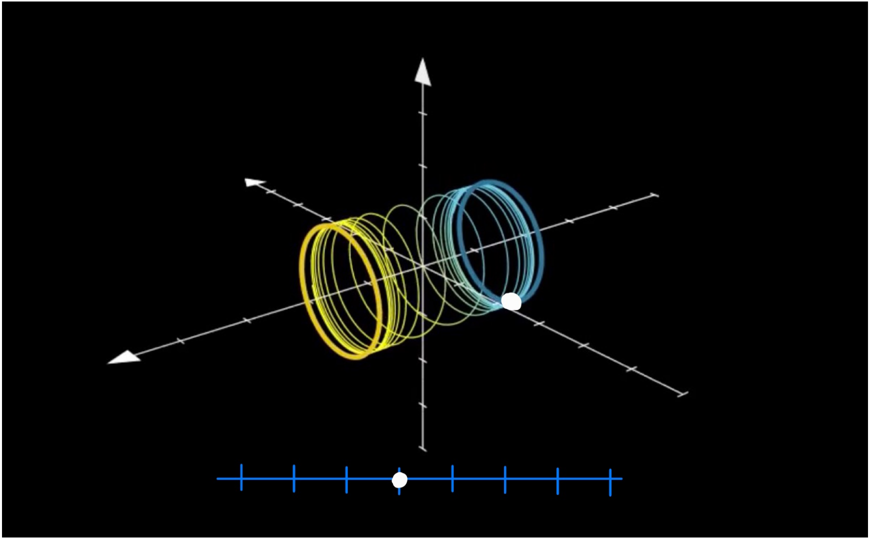

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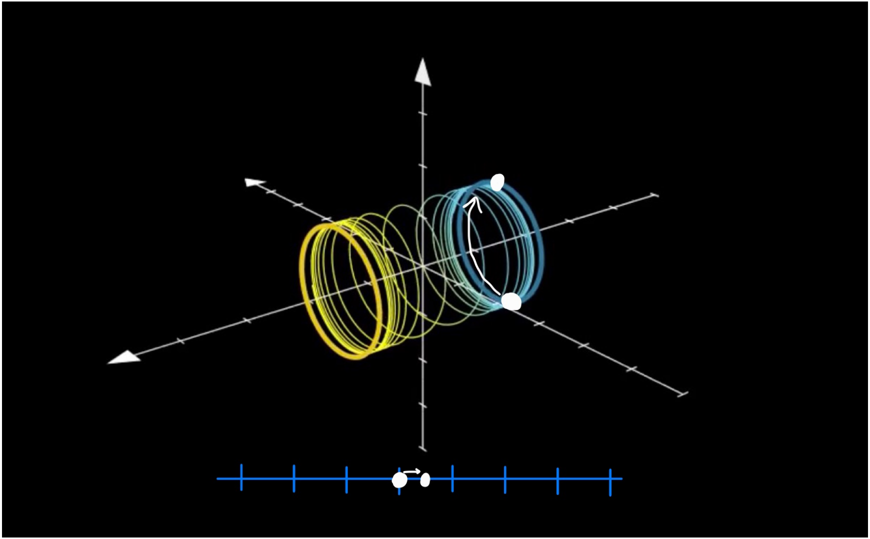

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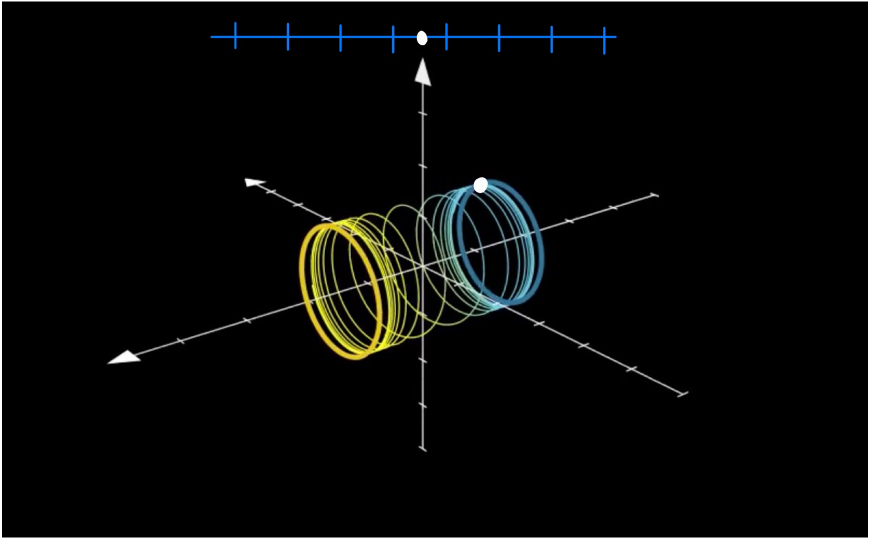

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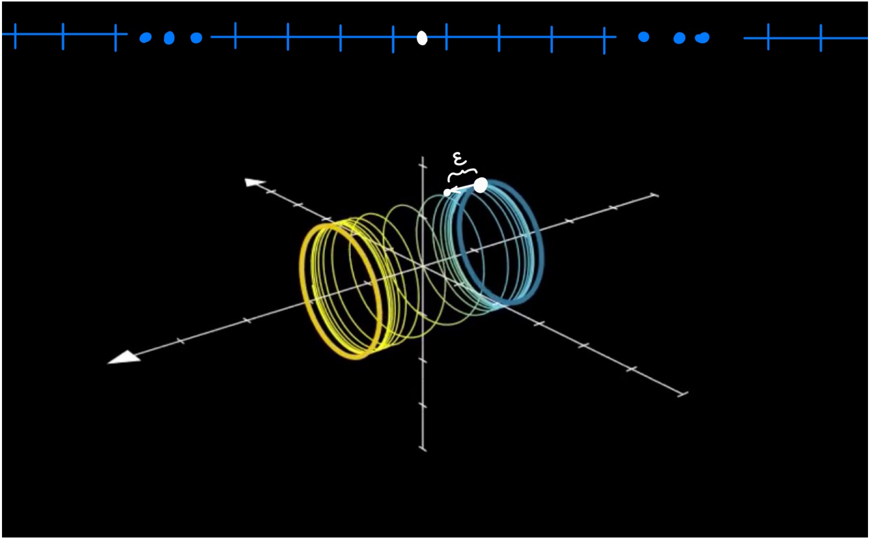

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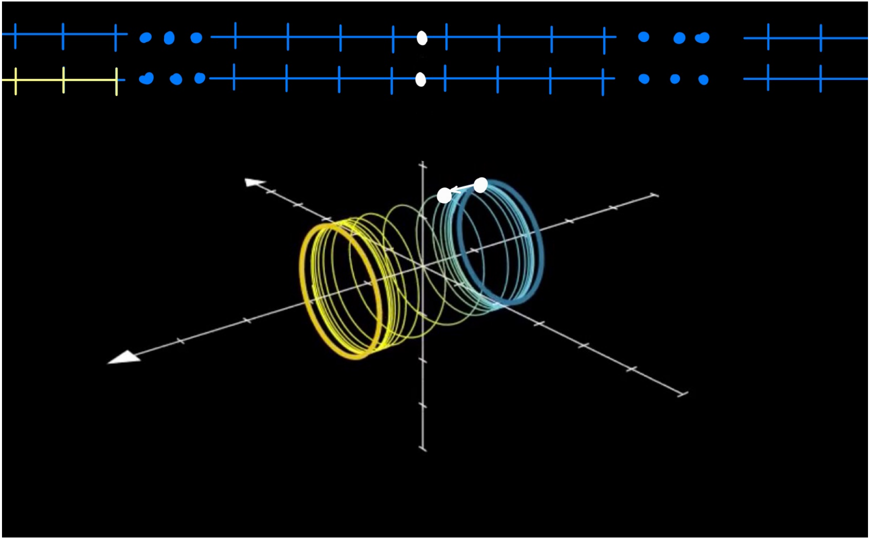

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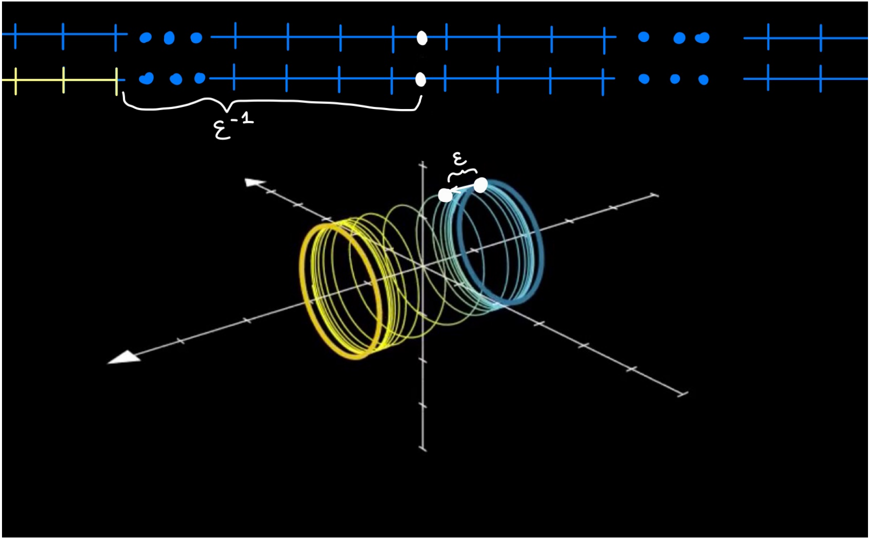

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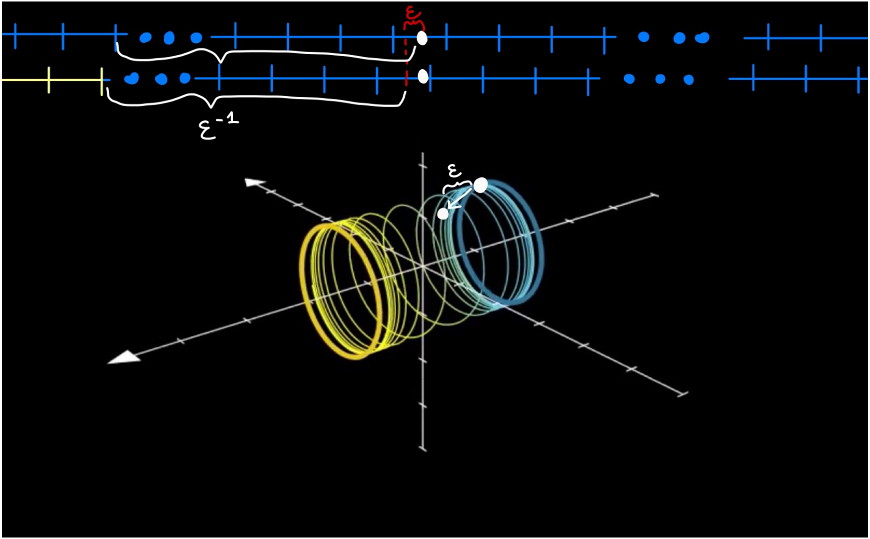

Definition (Foliated Space)

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A *foliated space X of dimension p* is a separable, metrizable space X together with a maximal collection of charts $\{\phi_\alpha : U_\alpha \to L_\alpha \times N_\alpha\}$ with $L_\alpha \subset \mathbb{R}^p$ open, where

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Note

A *level surface* is a piece $L_\alpha \times \{n\} \subset L_\alpha \times N_\alpha$. These level surfaces coalesce to create connected components called "leaves". The final condition above tells us that transition functions are smooth on leaves

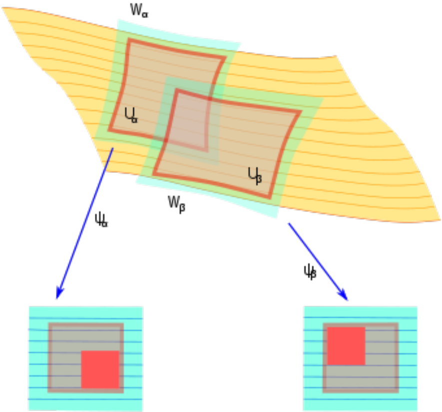

Figure: Transition Maps of a Foliated Manifold

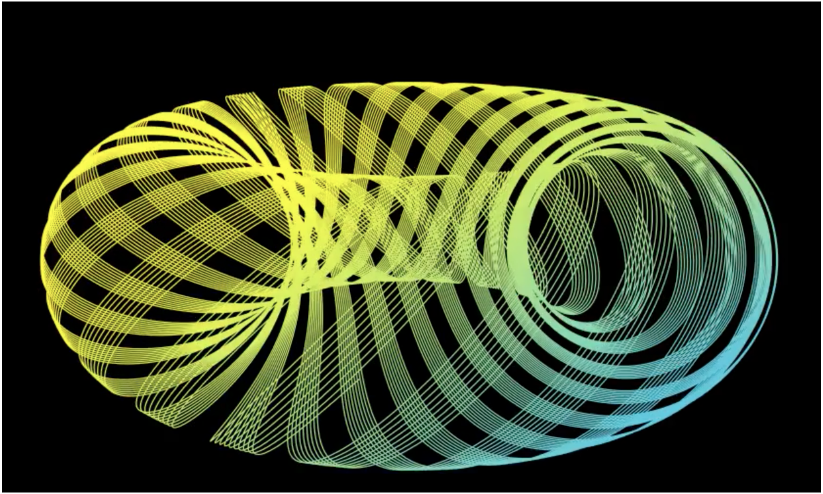

Figure: The Kronecker Foliation of the Torus

Example (Submersions and Fiber Bundles)

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If E^{p+q} is a fiber bundle

$$\begin{array}{ccc} F^p & \longrightarrow & E^{p+q} \\ & & \downarrow \\ & & B^q \end{array}$$

then E is foliated by F if F is connected.

Foliation of the Hull

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Definition

Let P be a tiling of $\mathbb{R}^d$, and let $P' \in \Omega_T$. An *ε-transveral of P'* is

$$\mathcal{T}_{P',\varepsilon} := \left\{ P'' \in \Omega_T \mid B(0, \varepsilon^{-1}) \cap P'' = B(0, \varepsilon^{-1}) \cap P' \right\}$$

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Example

If T is the half-and-half tiling, and T' is an all-blue tiling, then the ε-transversal of T' is the collection of tilings which all have only blue tiles up to radius ε^{-1} around the basepoint, *and* whose basepoints align with those of T'.

Note

Because our tilings have finite local complexity, the action of $\mathbb{R}^d$ is locally free. So for any $T'' \in \mathcal{T}_{T',\varepsilon}$, the action of $\mathbb{R}^d$ takes us outside the transversal. That is, $\mathcal{T}_{T',\varepsilon}$ is transverse to the action of $\mathbb{R}^d$.

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Theorem

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The topology of Ω_T is generated by open sets of the form $B(0,\varepsilon) \times \mathcal{T}_{T',\varepsilon}$. This happens in such a way that transition functions are "nice", giving Ω_T a foliated structure. □

Three Cohomologies of Ω_T

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3. Foliated Cohomology

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If $\mathcal{U}$ is an open cover of X, and $N(\mathcal{U})$ is the *nerve of* $\mathcal{U}$, then $\check{H}^*(\mathcal{U}) = H^*(N(\mathcal{U}))$, and $\check{H}^*(X) := \varinjlim \check{H}^*(\mathcal{U})$.

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Proof.

The first equality is by definition of Ω_T. The second follows because each Γ_n is a branched manifold, and so the covers from Čech cohomology are "nice enough" for the limits to commute as they do. □

PE Cohomology

Definition (Strongly Pattern Equivariant)

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A function $T \to \mathbb{R}$ which is a uniform limit of strongly-PE functions is a *weakly-PE* function.

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A function $T \to \mathbb{R}$ which is a uniform limit of strongly-PE functions is a *weakly-PE* function.

Note

If we instead consider functions $T \to \mathbb{Z}$ we get an analogous theory for $\mathbb{Z}$-coefficients, though w-PE and s-PE are identical.

Definition

A *strongly (weakly) PE k-form* is a differential form on T

$$\omega = \sum_{|\mathcal{I}|=k} f_{\mathcal{I}} dx^{\mathcal{I}}$$

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Theorem

Let d be the exterior derivative. Then

$$C^{\bullet}_{s-PE}(T) : \qquad \cdots \to C^k_{s-PE}(T) \xrightarrow{d} C^{k+1}_{s-PE}(T) \to \cdots$$

is a chain complex (resp. $C^{\bullet}_{w-PE}(T)$), with cohomology $H^*_{s-PE}(T)$ (resp. $H^*_{w-PE}(T)$).

Foliated Cohomology

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Let M be a foliated space. Let

$$C^{\infty}_{tlc}(M) = \left\{ f : M \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continuous, leafwise-smooth, and} \\ \text{locally constant in the transverse direction} \end{array} \right\}$$

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Let $C^{\infty}_{\tau}(M) = \mathrm{closure}(C^{\infty}_{tlc}(M))$. Let $C^{k}_{tlc/\tau}(M)$ be the tlc/τ k-forms.

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$$C^\infty_{tlc}(M) = \left\{ f : M \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continuous, leafwise-smooth, and} \\ \text{locally constant in the transverse direction} \end{array} \right\}$$

Let $C^\infty_\tau(M) = \mathrm{closure}(C^\infty_{tlc}(M))$. Let $C^k_{tlc/\tau}(M)$ be the tlc/τ k-forms.

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$$C^\bullet_{tlc}(M) : \qquad \cdots \to C^k_{tlc}(M) \xrightarrow{d} C^{k+1}_{tlc}(M) \to \cdots$$

is a chain complex (resp. $C^\bullet_\tau(M)$), with cohomology $H^*_{tlc}(M)$ (resp. $H^*_\tau(M)$).

Foliated Cohomology

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is a chain complex (resp. $C^\bullet_\tau(M)$), with cohomology $H^*_{tlc}(M)$ (resp. $H^*_\tau(M)$). The maximal Hausdorff quotient of $H^*_\tau(M)$ is denoted $\overline{H}^*_\tau(M)$.

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Note

Kellendonk and Putnam's original proof does not use this fact. Instead, they apply a more general theory of foliations and dynamical systems to prove their result.

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Let $\pi_n : \mathbb{R}^d \to \Gamma_n$ *be the natural projection* $\Omega_T \to \Gamma_n$ *restricted to* $\mathcal{O}(T)$.

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Let $R_n > r_n > 0$ be such that for any $T' \in \Omega_T$, the ball $B(0, r_n)$ is contained in n layers of tiles around the origin, and $B(0, R_n)$ contains n layers of tiles around the origin.

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Let $g : \mathbb{R}^d \to \mathbb{R}$ be strongly PE with radius $R < r_n$. Then $f : \Gamma_n \to \mathbb{R}$ defined by $f(\pi_n(x)) := g(x)$ is well-defined on all of Γ_n and smooth. □

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Theorem

$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$ *and* $H^*_{w-PE}(T) = H^*_{\tau}(\Omega_T)$

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This is essentially a consequence of the lemma earlier that

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Theorem

$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$ *and* $H^*_{w-PE}(T) = H^*_{\mathcal{T}}(\Omega_T)$

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This is essentially a consequence of the lemma earlier that

$$\{\text{strongly PE functions on } T\} \longleftrightarrow \bigcup_{n\in\mathbb{N}} \{\text{smooth functions } f : \Gamma_n \to \mathbb{R}\}.$$

See [KP06] for details. □

Summary of Relationships

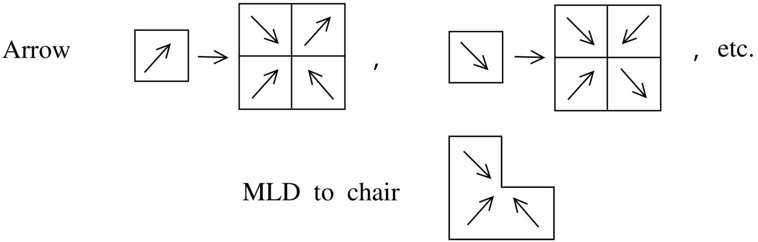
$$\begin{array}{ccccc} \check{H}^*(\Omega_T,\mathbb{R}) & = & H^*_{s-PE}(T) & = & H^*_{tlc}(\Omega_T) \\ & & & & \big\downarrow \\ & & H^*_{w-PE}(T) & = & H^*_{\tau}(\Omega_T) \\ & & & & \big\downarrow \\ & & & & \overline{H}^*_{\tau}(\Omega_T) \end{array}$$

What's the Big Deal?

Pattern-Equivariant cohomology helps us recognize the generators of cohomology.

Recall

The "chair tiling" is the same as the "arrow tiling". We can describe the cohomology of the chair tiling using the arrow tiling.



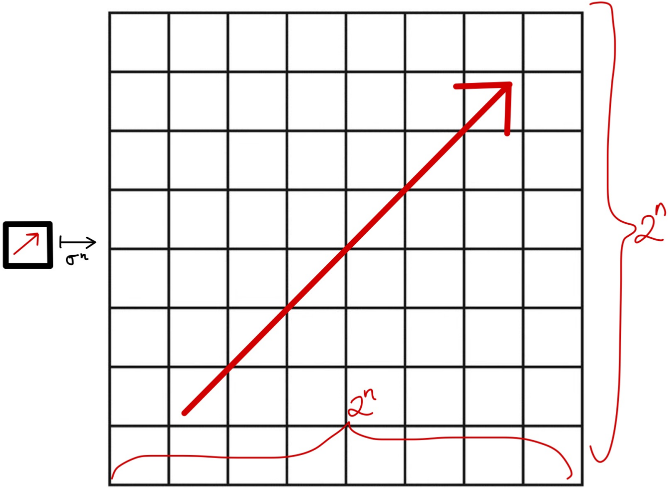
Representatives of the Arrow Tiling Cohomology

Proposition

If T is the arrow tiling, then the Čech cohomology groups of Ω_T with integer coefficients is given by

$$\check{H}^0(\Omega_T) = \mathbb{Z}$$
$$\check{H}^1(\Omega_T) = \mathbb{Z}[1/2]^2$$
$$\check{H}^2(\Omega_T) = \frac{1}{3}\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2$$

Example of PE-Cohomology



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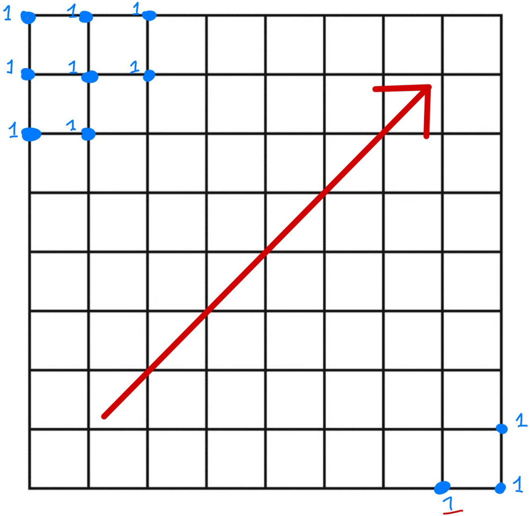
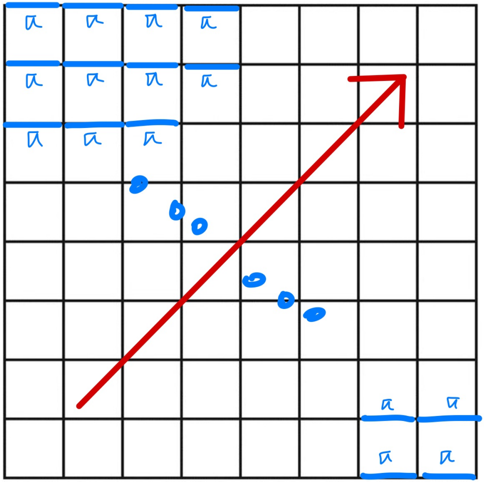


$$1 \in \check{H}^0(\Omega_T)$$

Example of PE-Cohomology



$$(a, b) \in \check{H}^1(\Omega_T)$$

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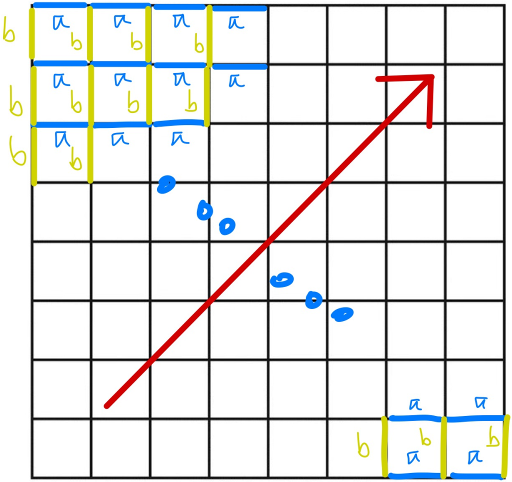

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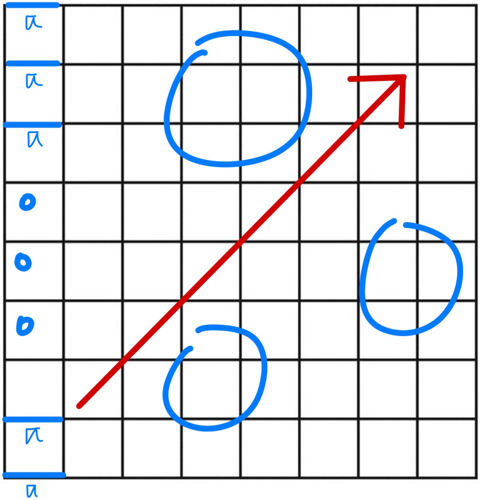
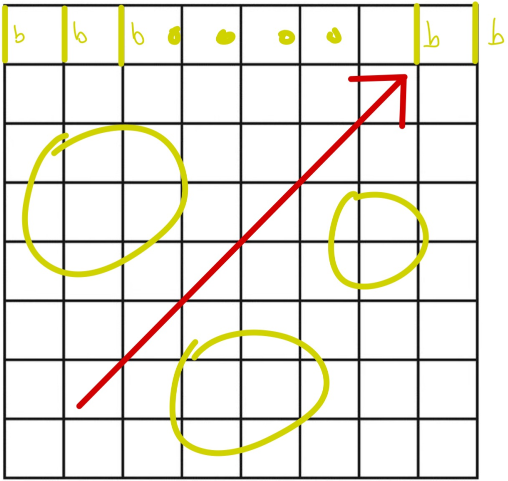


$$\left(\frac{a}{2^n}, 0\right) \in \check{H}^1(\Omega_T)$$

a cocycle in Γ_n

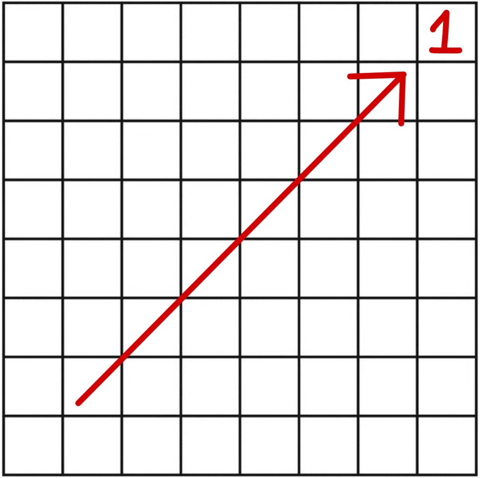
Example of PE-Cohomology



$$\left(0, \frac{b}{2^m}\right) \in \check{H}^1(\Omega_T)$$

a cocycle in Γ_m

Example of PE-Cohomology



$$\left(\frac{1}{4^n}, 0, 0\right) \in \check{H}^2(\Omega_T)$$

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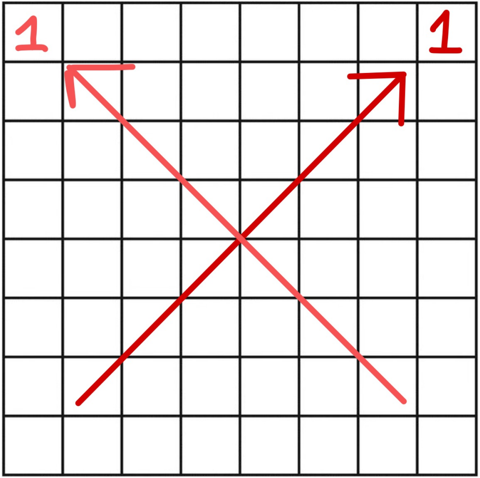

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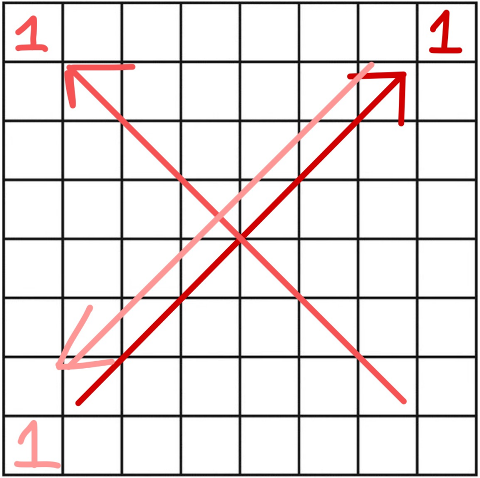

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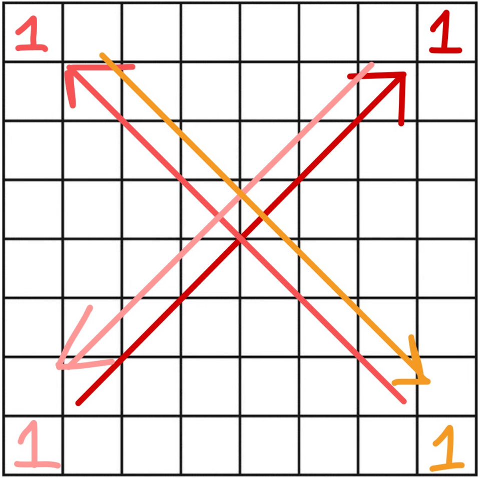

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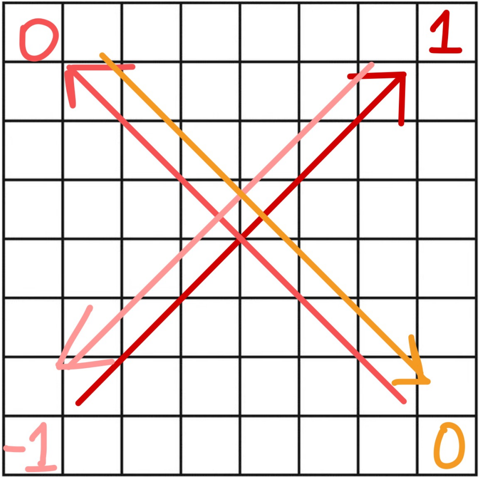

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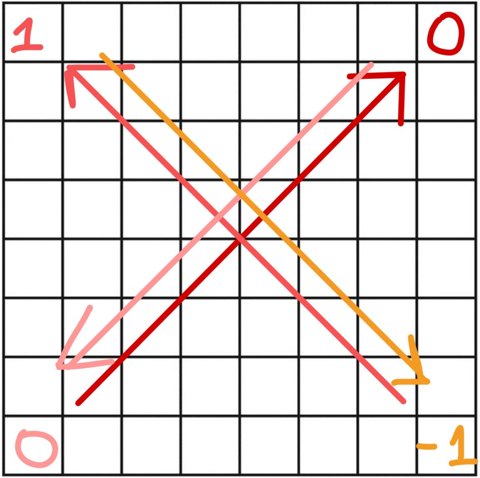

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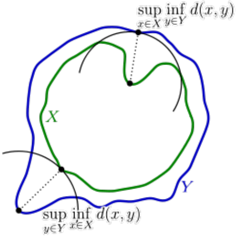
Crete-ising The Discete

Definition

From Wikipedia: "The Hausdorff distance [between two metric subspaces X, Y of an ambient space M] is the longest distance you can be forced to travel by an adversary who chooses a point in one of the two sets, from where you then must travel to the other set."



$$d_{\mathrm{H}}(X,Y) = \max\left\{ \sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(X,y) \right\},$$

Definition

The *Gromov-Hausdorff distance* between two metric spaces is the infimum

$$d_{GH}(X, Y) := \inf_{f,g} d_H(f(X), g(Y))$$

over isometric embeddings $f, g : X, Y \hookrightarrow M$ into some ambient space M. In other words, it is the smallest possible separation between X and Y on any metric on their union.

The Idea

Rather than comparing tilings of $\mathbb{R}^d$ using the tiling metric, we can compare $\mathbb{R}^d$ with a given metric, using the Gromov-Hausdorff Distance. *Pointed* or *Based* GH space (*GHB*) tries to keep basepoints close together as well.

Definition

Let M be a manifold of bounded geometry (i.e., $\mathrm{inj}(M) > c > 0$ and $|K| < C$), and let $GHB(D)$ be Pointed Gromov-Hausdorff space of balls of radius $D/2$. Define $\Psi_D : M \to BGH$ by $\Psi_D(m) = B(m, \frac{D}{2})$.

Theorem

The image $\Psi_D(M) \subseteq GBH(D)$ is precompact.

Proof.

Any uniformly totally bounded class of compact metric spaces is pre-compact in GH space. See [BBI01, 264f.] for more details. □

Definition

The *hull of a Manifold with Bounded Geometry* is a subspace of GH space

$$\Lambda(M) := \varprojlim \mathrm{Closure}(\Psi_D(M))$$

Definition

The complex of differential forms which are continuous under GH correspondence creates a cohomology $H^*_{bg}(M)$. Compare this to the foliated and weakly-PE cohomologies.

Where'd All the Tilings Go?

Tiling $\longrightarrow$ Voronoi Diagram $\longrightarrow$ Geometry $\longrightarrow$ Mfld with BG

References I

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Thank you!