The Hull of a Manifold with Bounded Geometry

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Outline

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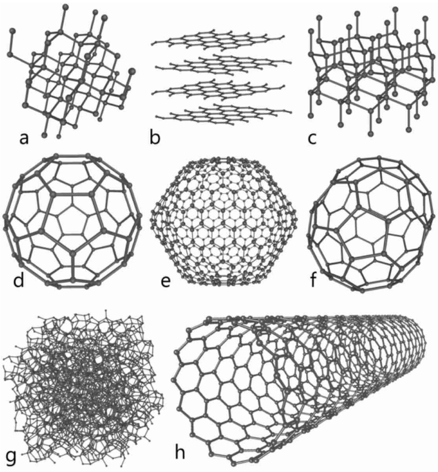

Figure: Eight Allotropes of Carbon

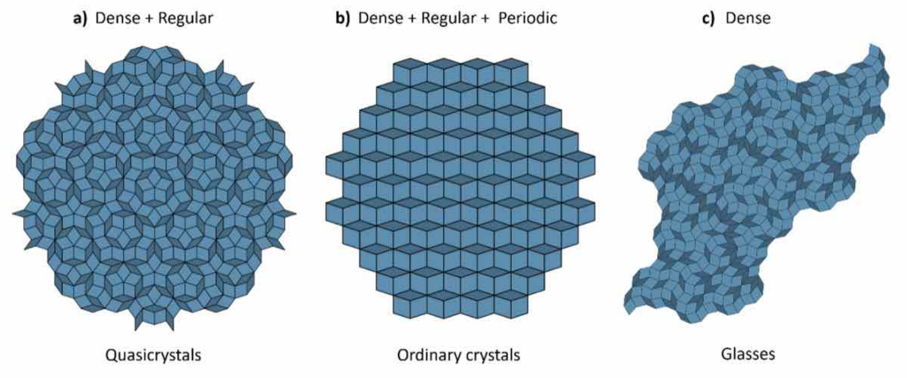

Figure: Three different kinds of material

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Tilings of $\mathbb{R}^d$

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3. If two tiles meet, they meet completely in one of their $(d-1)$-faces.

Simple Tilings

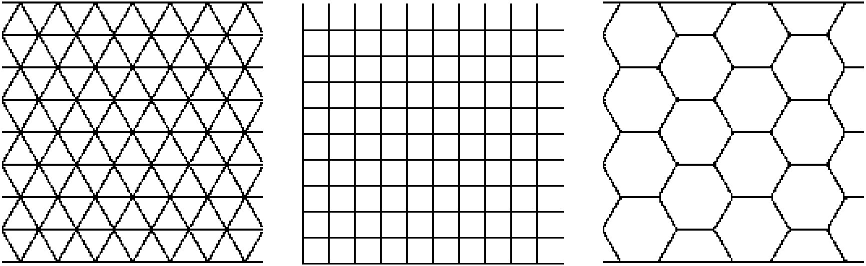


Figure: Periodic Tilings

Simple Tilings

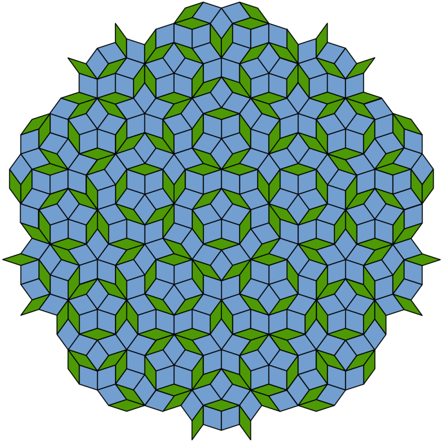


Figure: A Patch of the Penrose Tiling

Violating Hypotheses

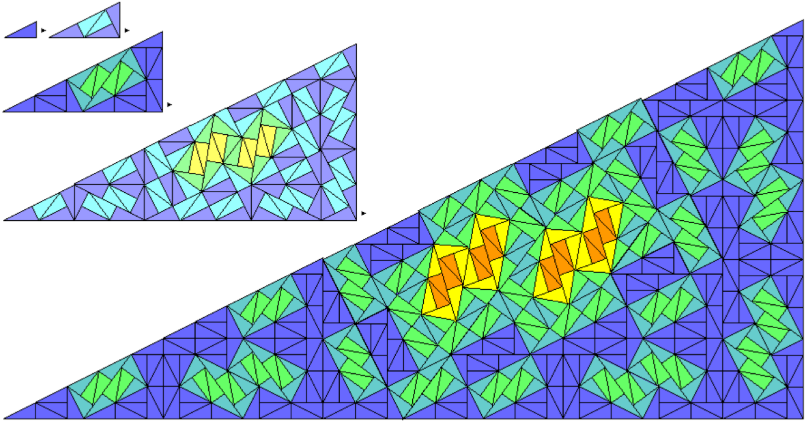


Figure: A Pinwheel Tiling. Lacks finitely many prototiles up to translation.

Violating Hypotheses

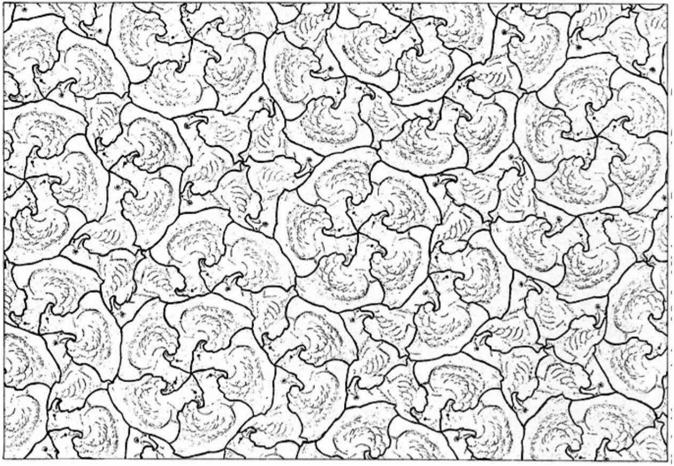


Figure: Penrose Chickens. Tiles are not polytopes

Violating Hypotheses

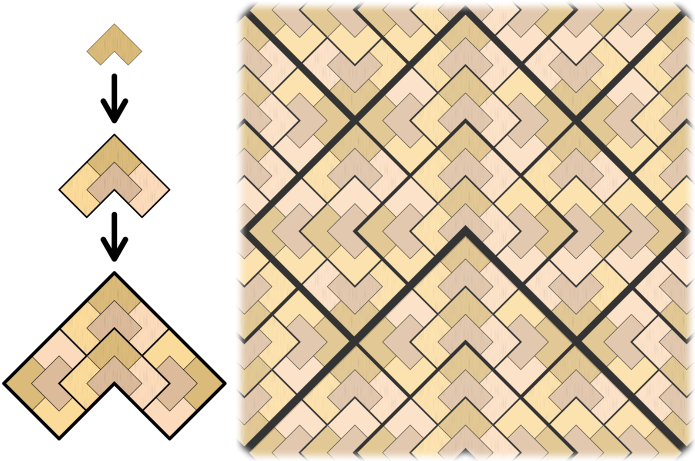
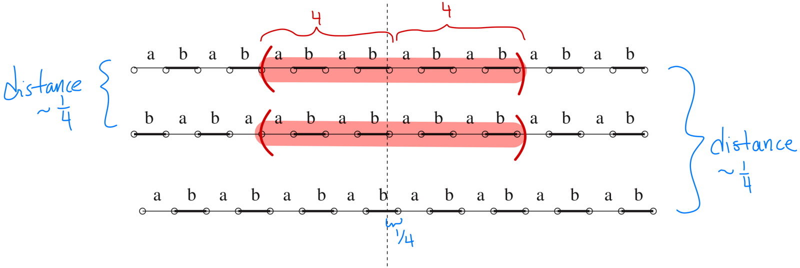


Figure: A chair tiling. Edges don't meet full-face to full-face

Definition (Tiling Metric)

Given two tilings, T and T', of $\mathbb{R}^d$, they are *ε-close* if up to a translation of distance ε, they agree on a ball of radius ε^{-1} around the origin.



Definition

The *orbit* of a tiling T is its orbit under translation

$$\mathcal{O}(T) := \left\{ T - x \mid x \in \mathbb{R}^d \right\}$$

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A *tiling space* is a set of tilings which is closed under translation by $\mathbb{R}^d$ and complete in the tiling metric.

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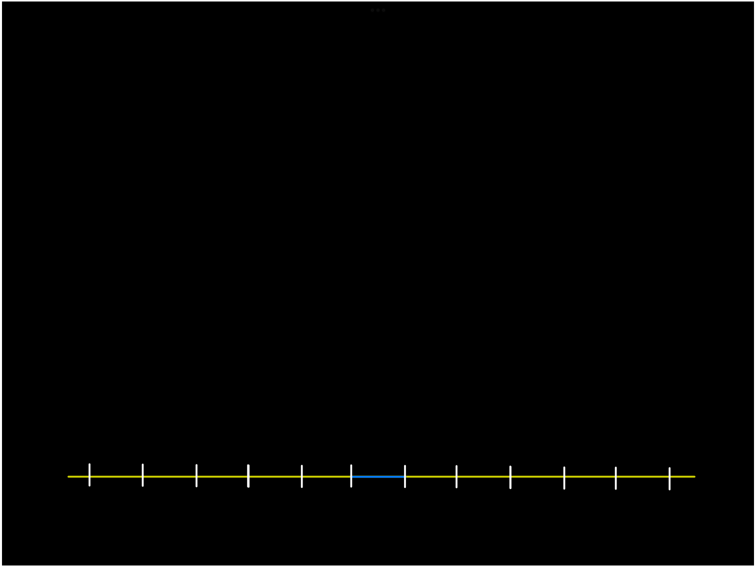
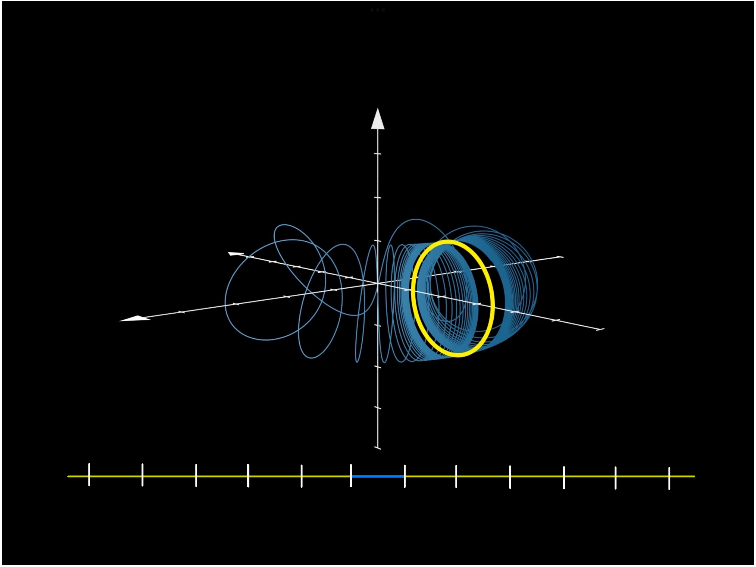
Definition (The Hull: Version 1)

The *hull* Ω_T of a tiling T is the completion of $\mathcal{O}(T)$ in the tiling metric.

The Hull of a Tiling





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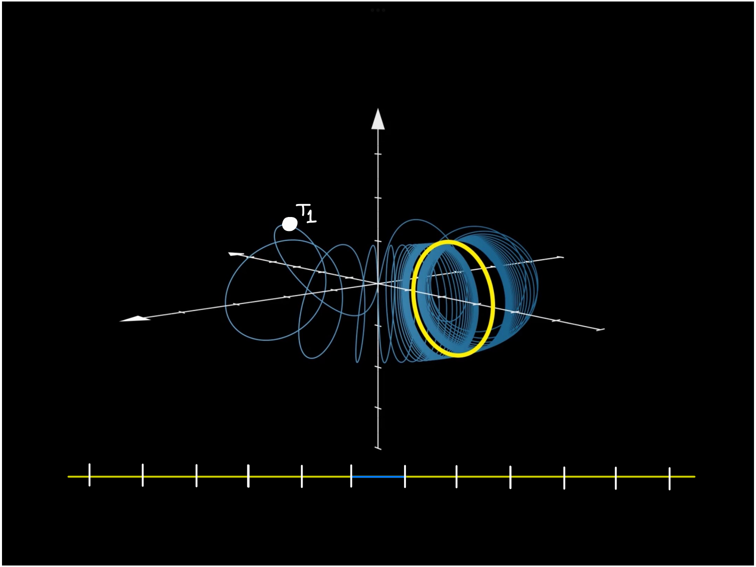

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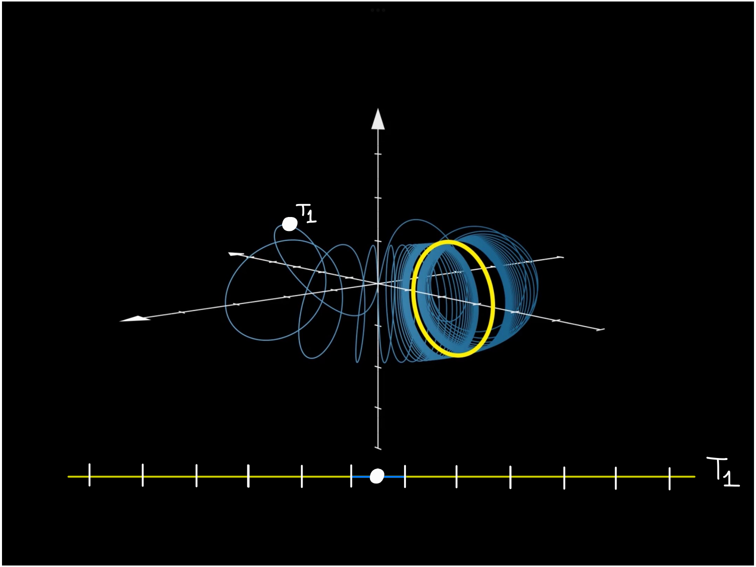

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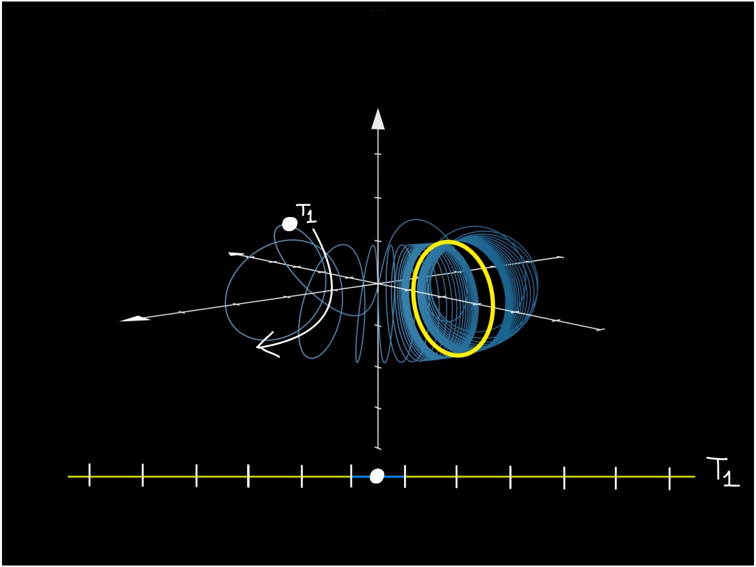

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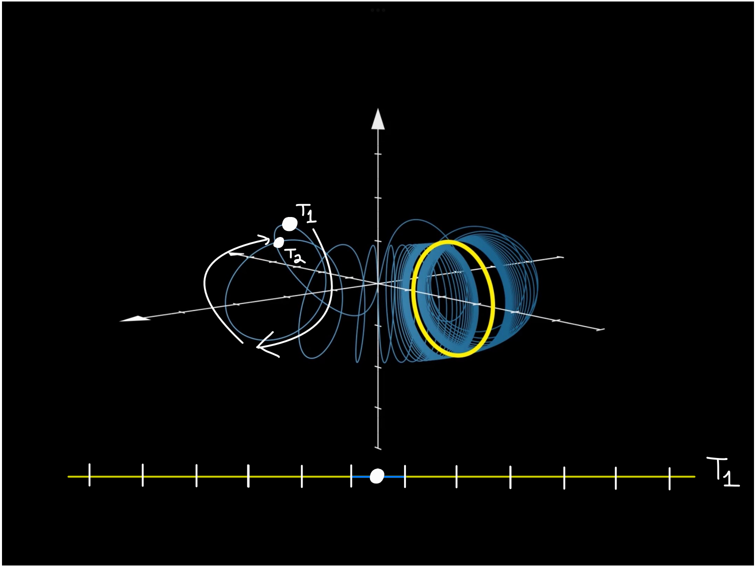

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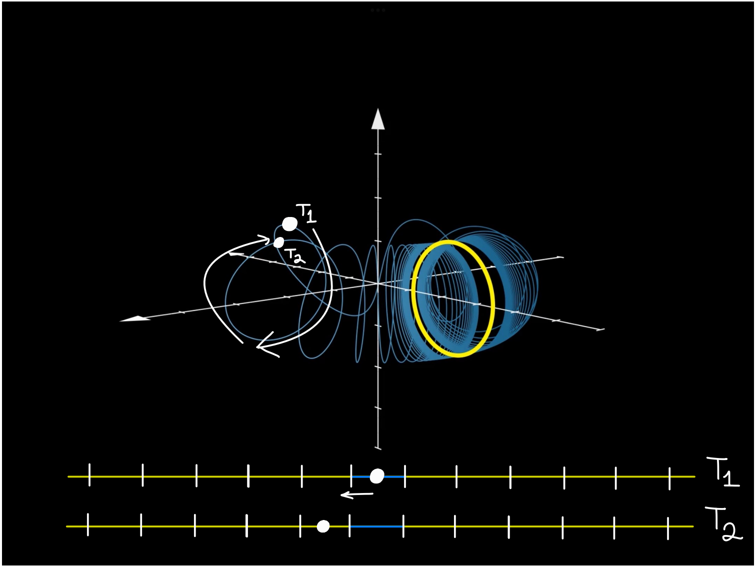
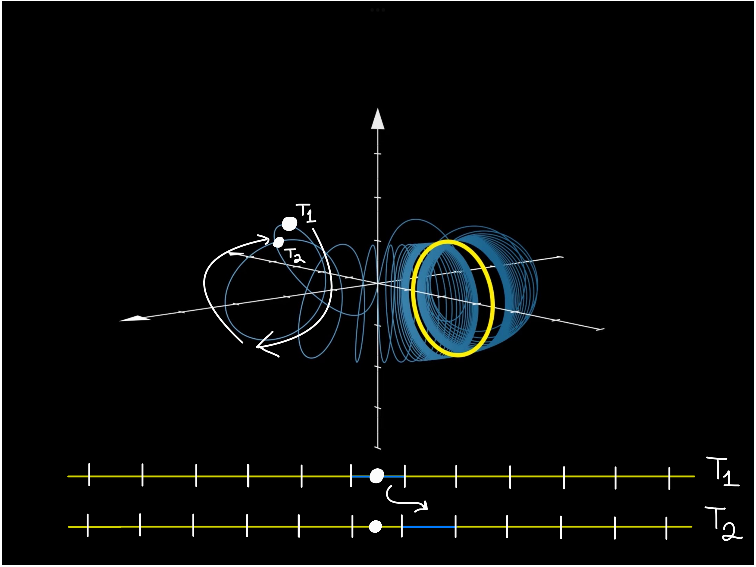
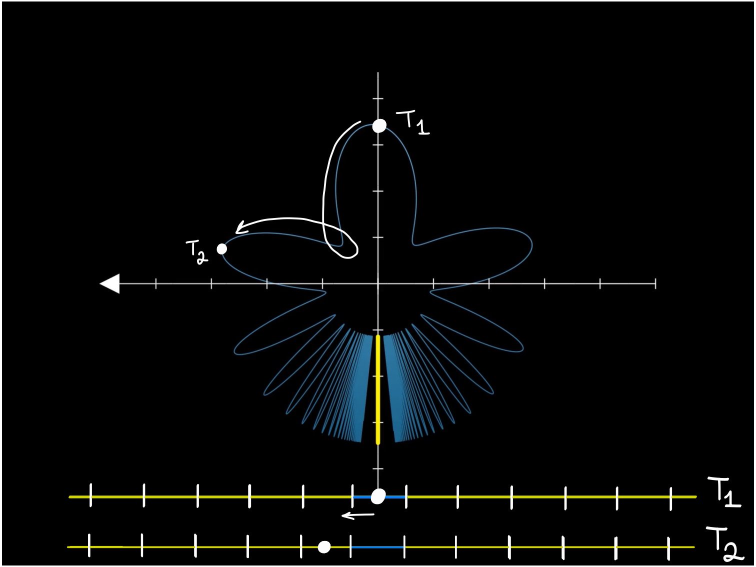



T1
T2
T1
T2


T1
T2
T1
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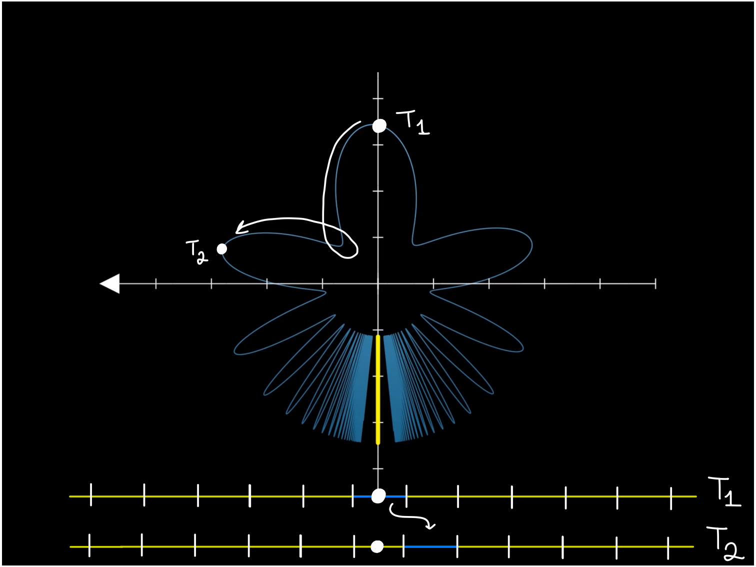

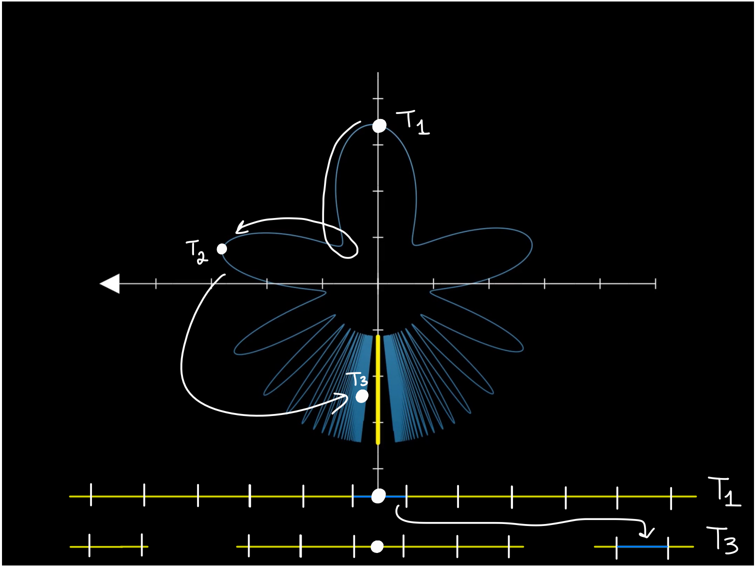
T_1
T_2
T_3
T_1
T_3

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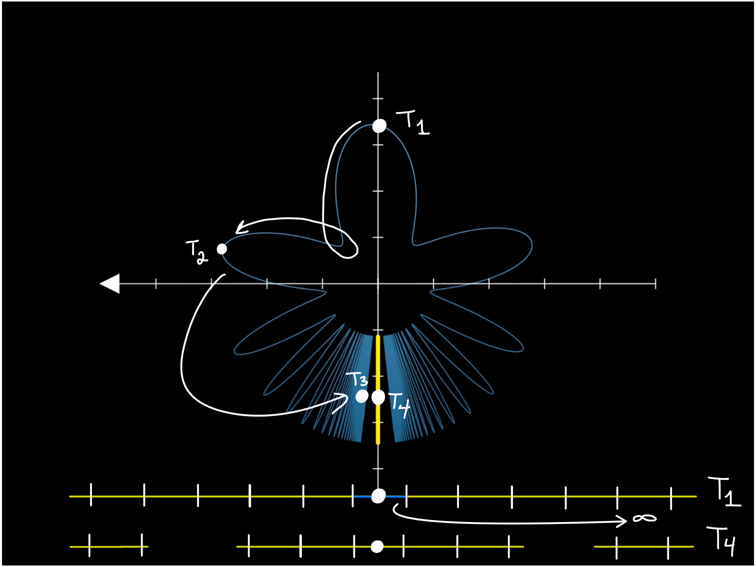
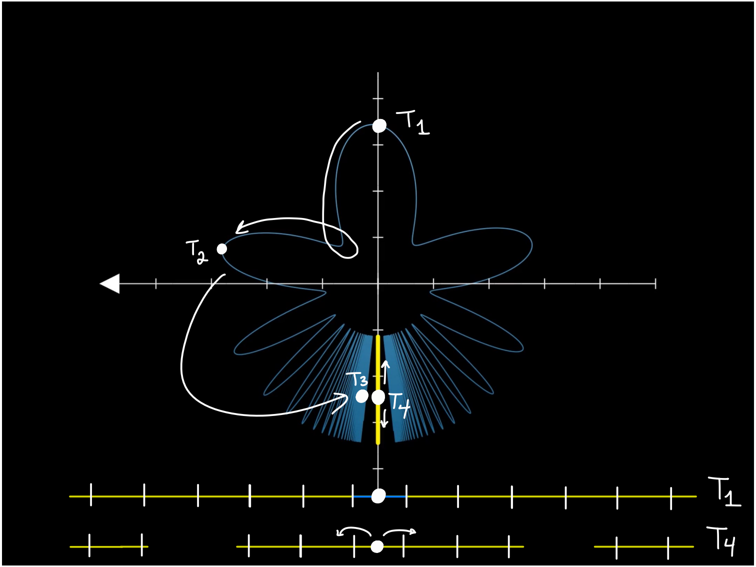
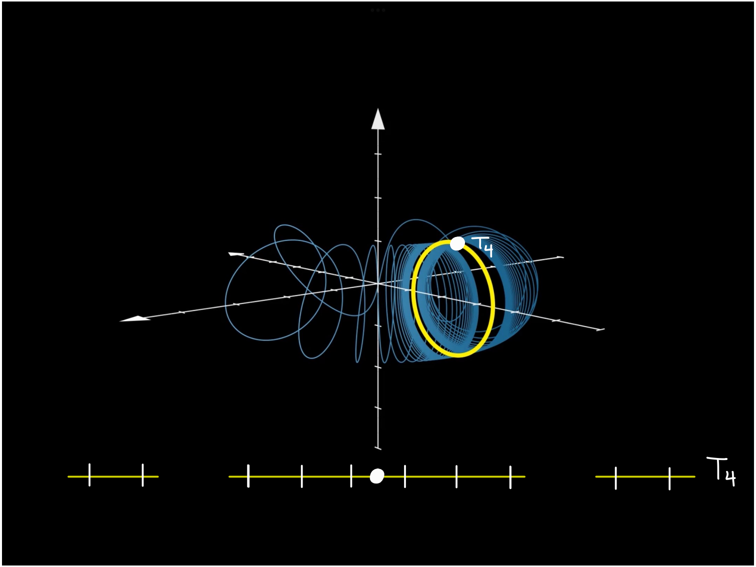



T1
T2
T3
T4
T1
T4

The Hull of a Tiling



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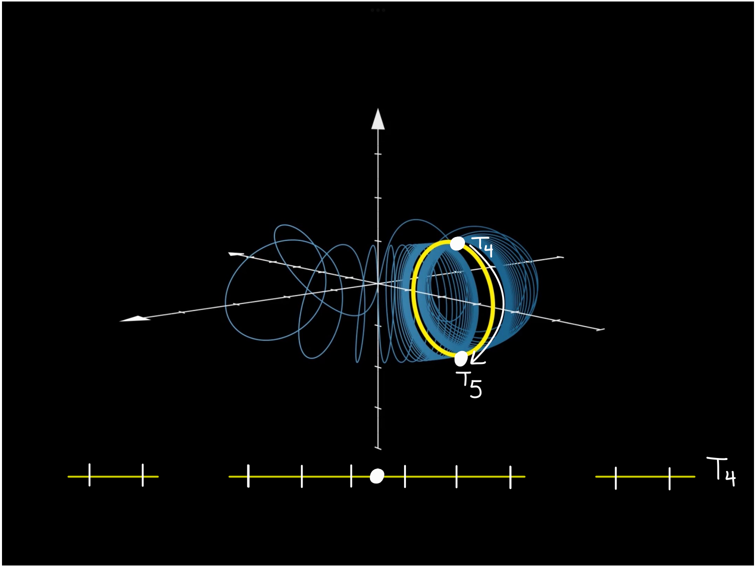

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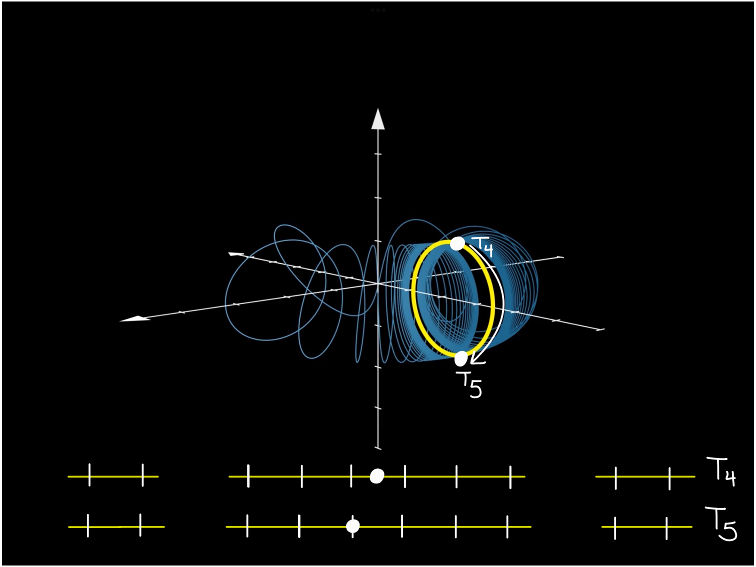

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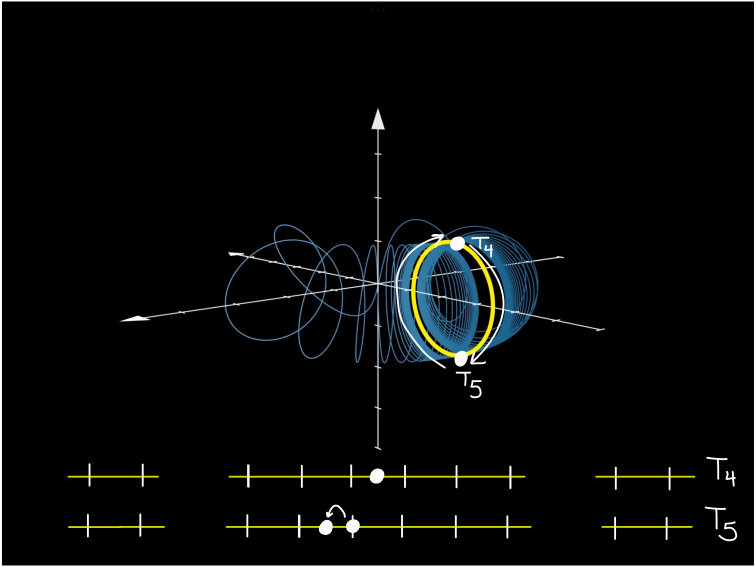

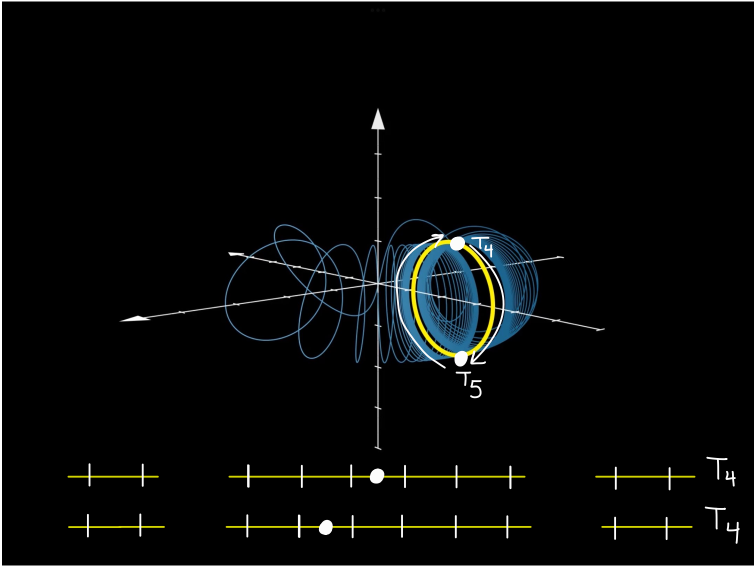
T4
T5
T4
T4

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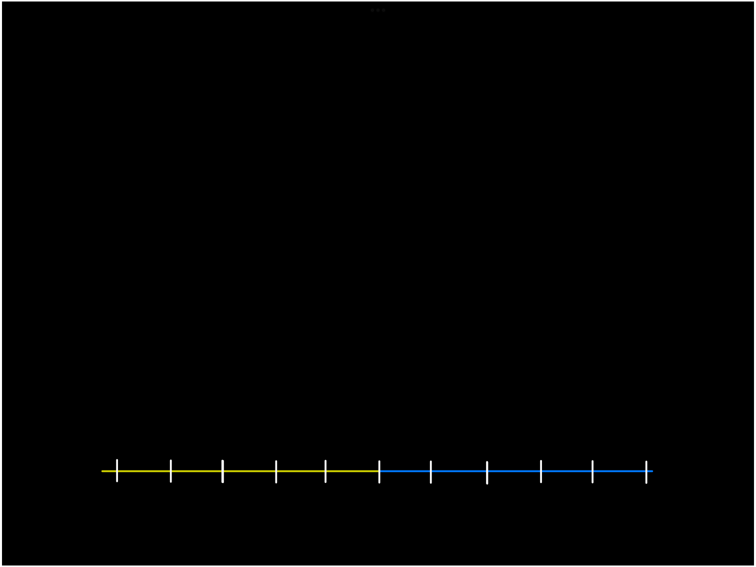

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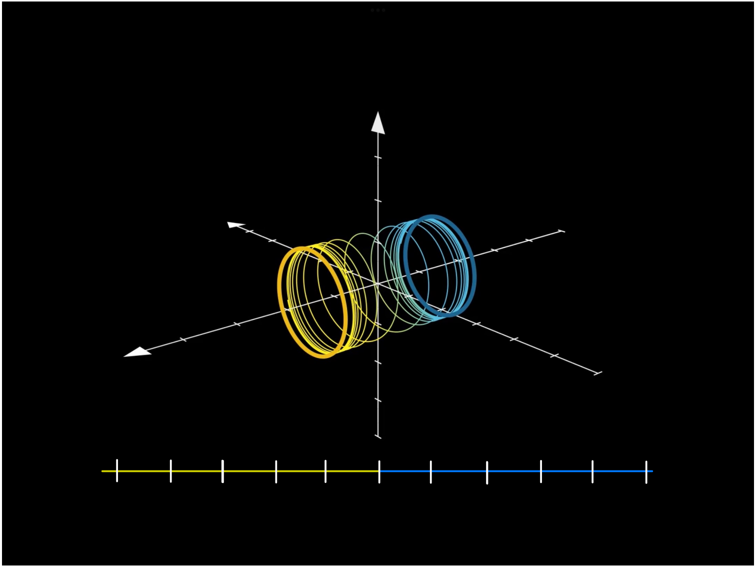

Definition (The Hull: Version 2)

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Given a tiling T, for each $n \in \mathbb{N}$, the *nth Gähler complex* Γ_n is the collection of possible instructions for laying n layers of tiles around some tile at the origin consistent with T.

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$$\Omega_T := \varprojlim(\Gamma_n, f_n) = \left(\prod_{n\in\mathbb{N}} \Gamma_n\right) / \sim$$

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If T is a simple tiling, Ω_T a compact metric space.

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Each Γ_n is a compact branched manifold, the inverse limit of which is a compact metric space. □

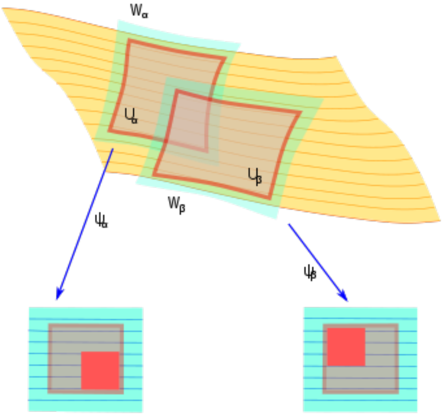

Figure: Transition Maps of a Foliated Manifold

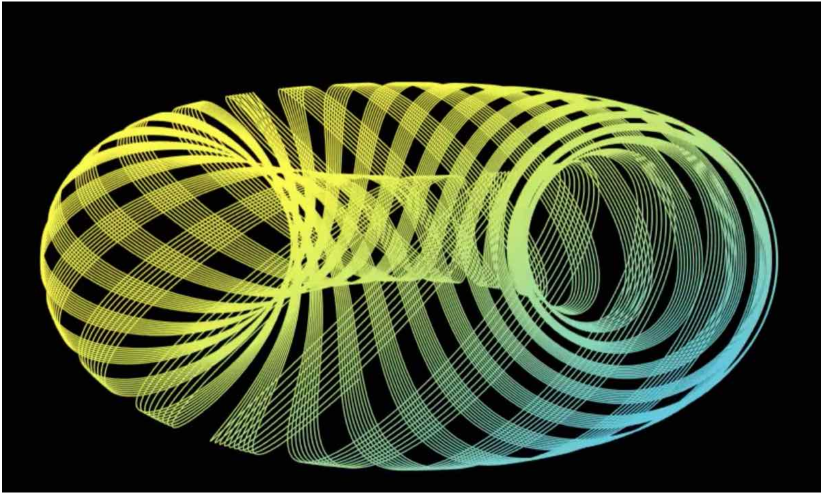

Figure: The Kronecker Foliation of the Torus

Foliation of the Hull

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Definition

Let T be a tiling of $\mathbb{R}^d$, and let $T' \in \Omega_T$. An *ε-transveral of T'* is

$$\mathscr{T}_{T',\varepsilon} := \{ T'' \in \Omega_T \mid B(0,\varepsilon^{-1}) \cap T'' = B(0,\varepsilon^{-1}) \cap T' \}$$
$$= \left\{ \begin{matrix} \text{tilings which agree with } T' \text{ on a ball of} \\ \text{radius } 1/\varepsilon \text{ when basepoints are aligned} \end{matrix} \right\}$$

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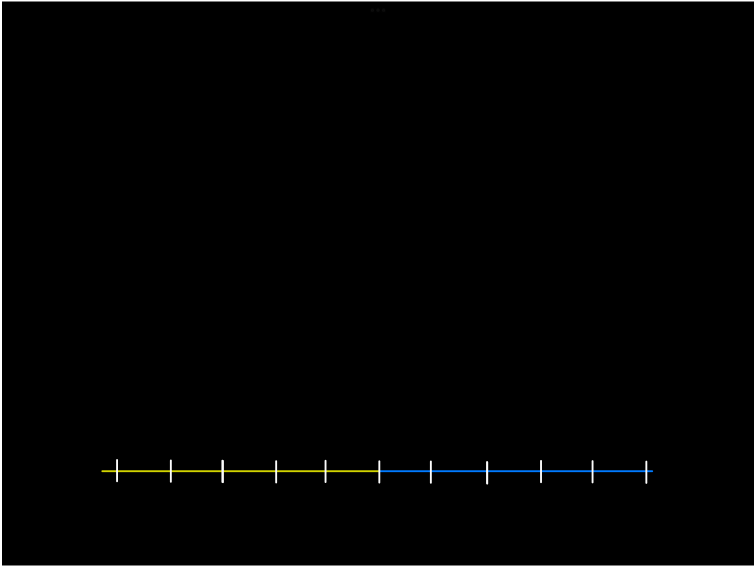
Proof.

Because our tilings have FLC, the action of $\mathbb{R}^d$ is locally free. So for any $T'' \in \mathscr{T}_{T',\varepsilon}$, if $v \in \mathbb{R}^d$ with $|v|$ small enough, $T'' - v \notin \mathscr{T}_{T',\varepsilon}$. That is, $\mathscr{T}_{T',\varepsilon}$ is transverse to the action of $\mathbb{R}^d$. □



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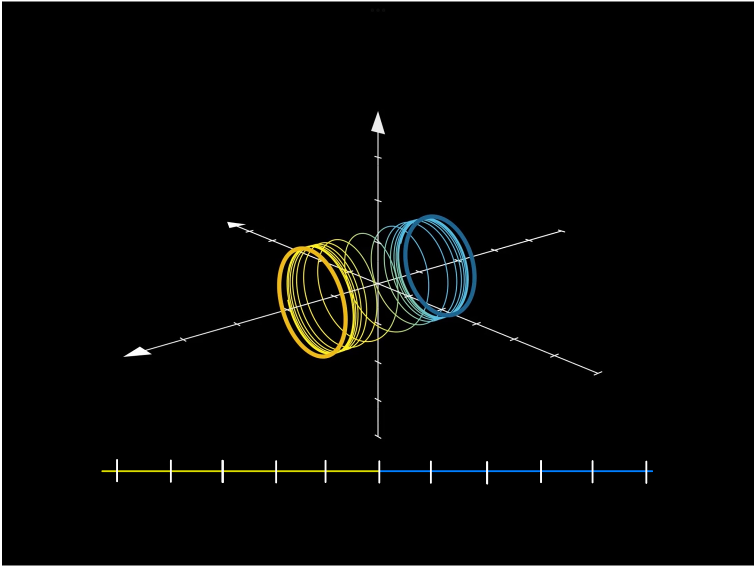

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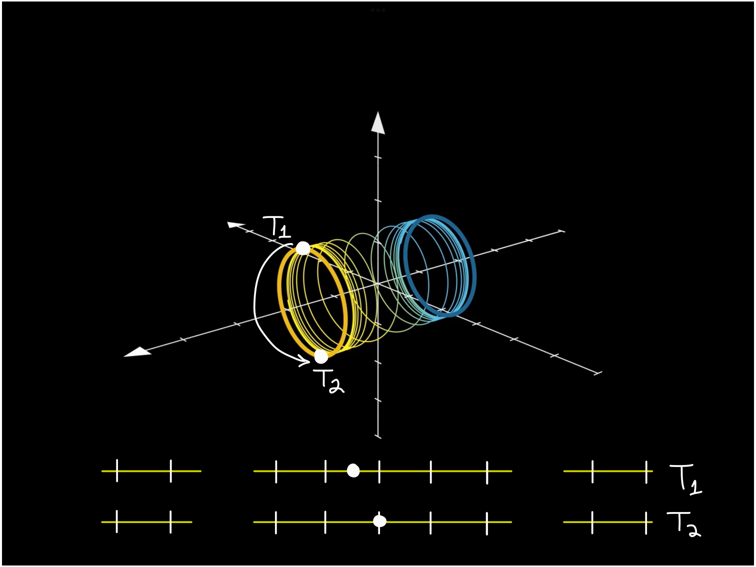

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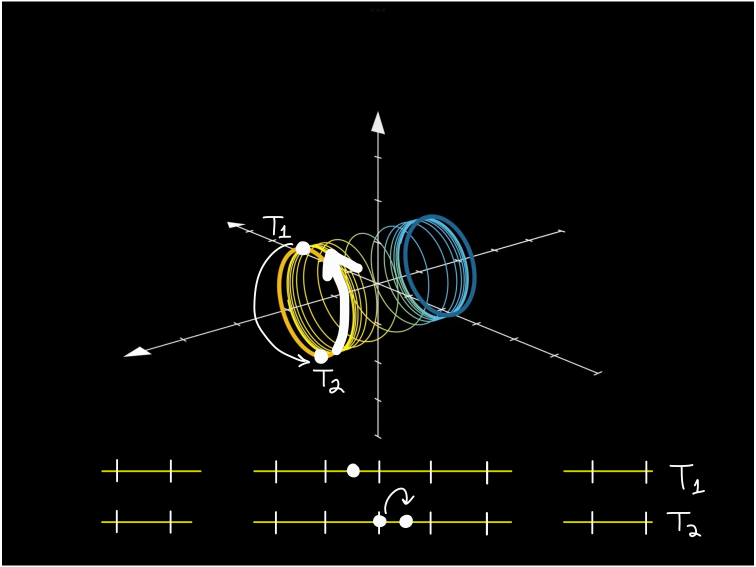

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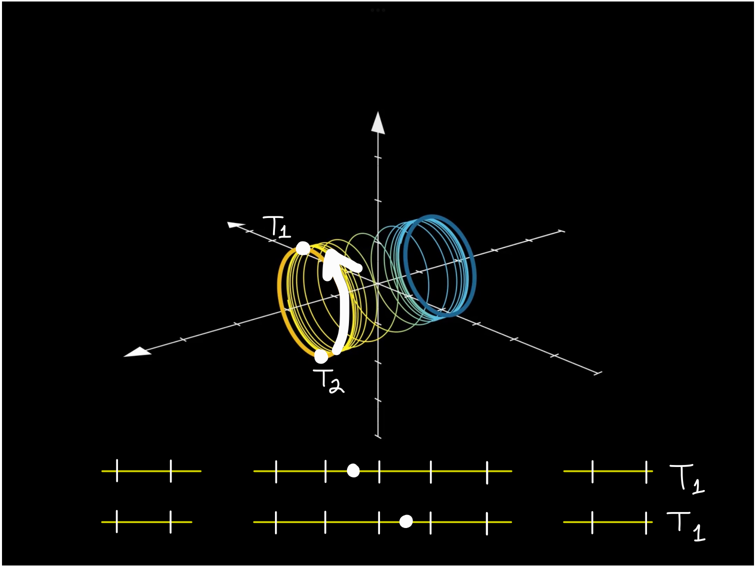

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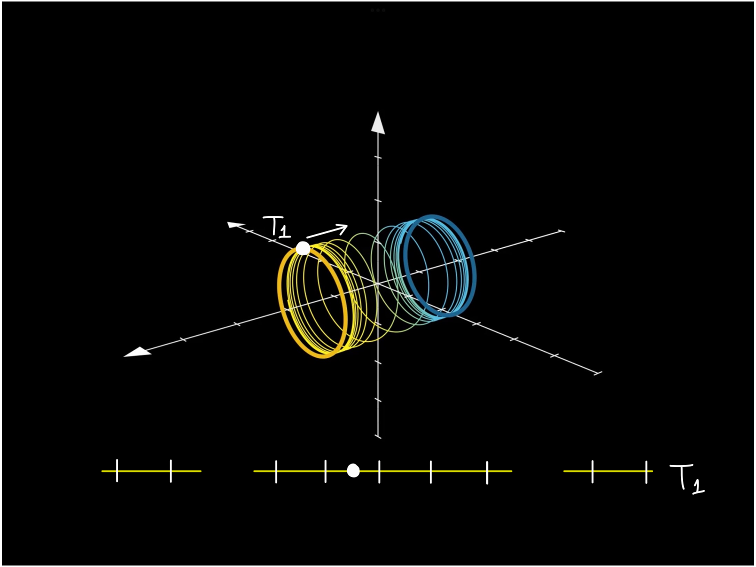

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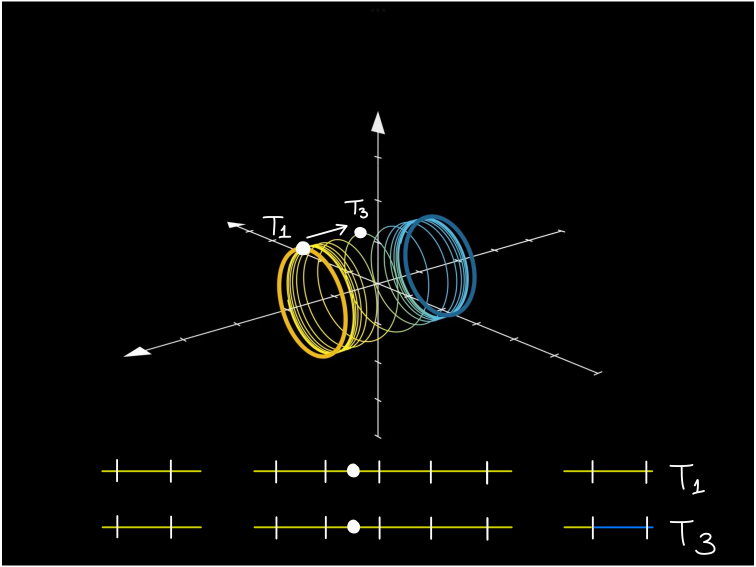

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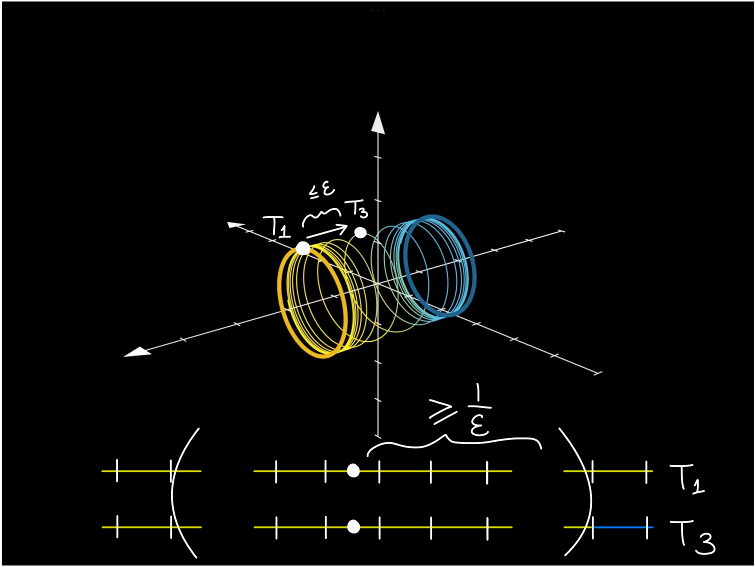

Three Cohomologies of Ω_T

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1. Čech Cohomology

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2. Pattern-Equivariant Cohomology

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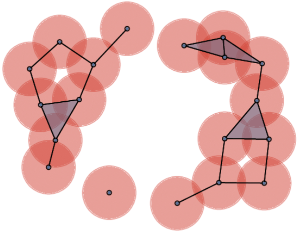
1. Čech Cohomology
2. Pattern-Equivariant Cohomology
3. Foliated Cohomology

Čech Cohomology $\check{H}^*(\Omega_T)$



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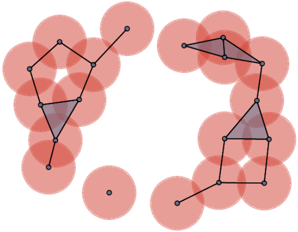
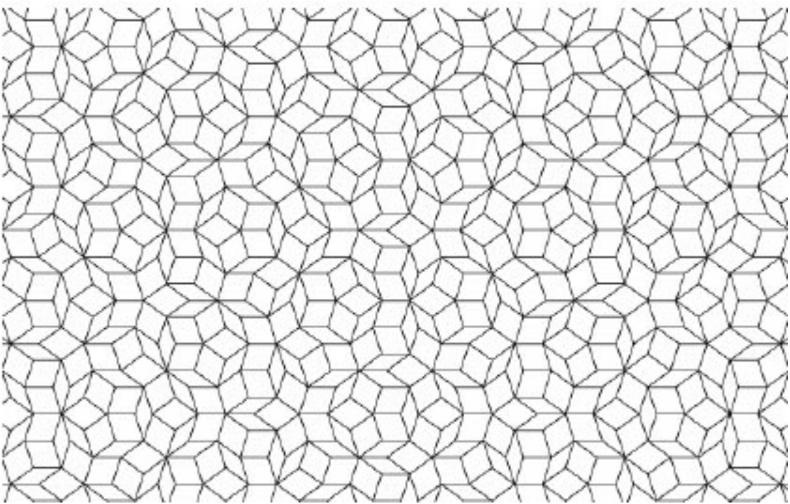


Definition

$$\check{H}^*(X, \mathbb{R}) = \varinjlim_{\mathcal{U}} H^*(N(\mathcal{U}), \mathbb{R})$$

PE Cohomology



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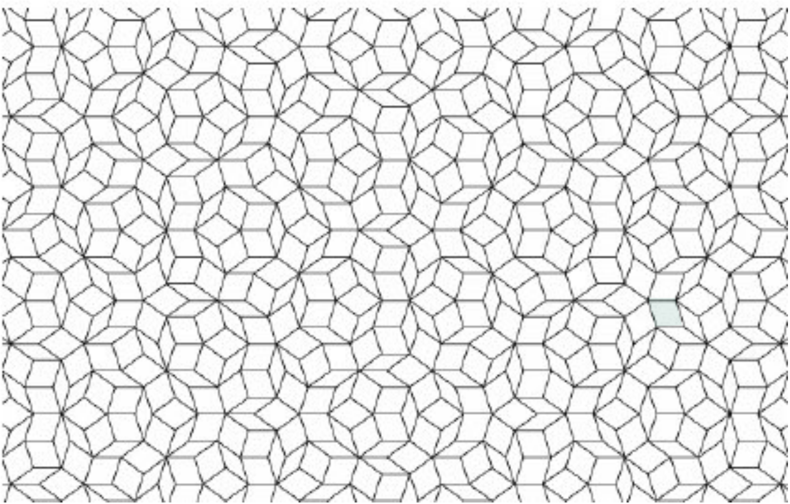

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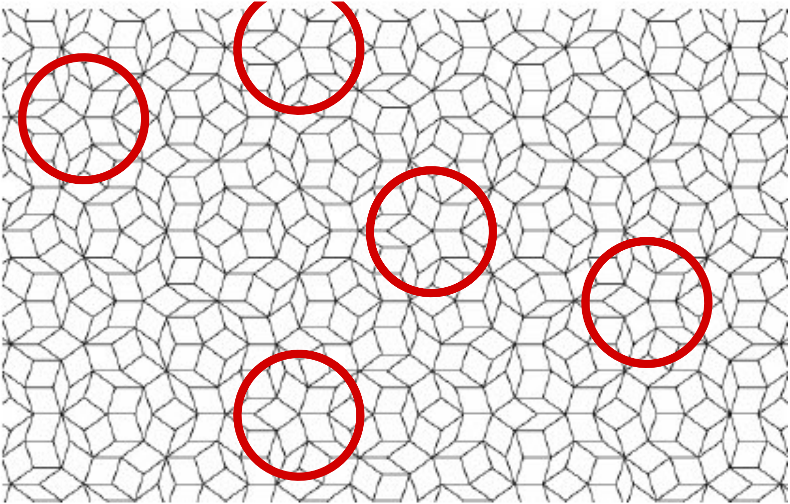

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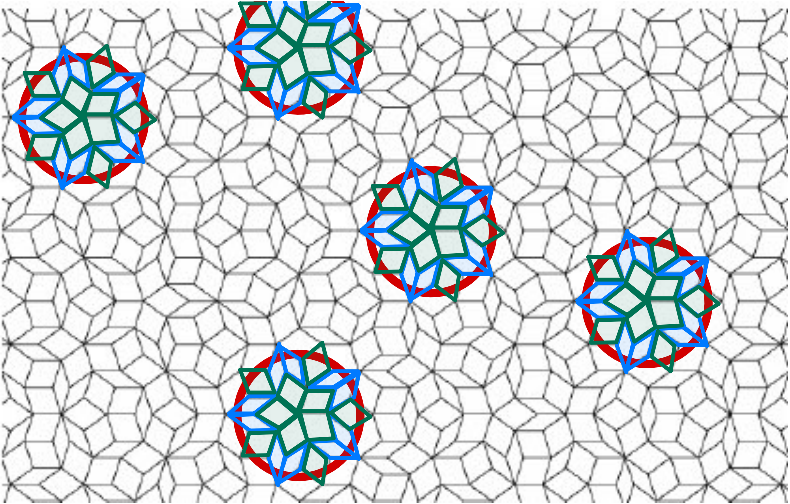

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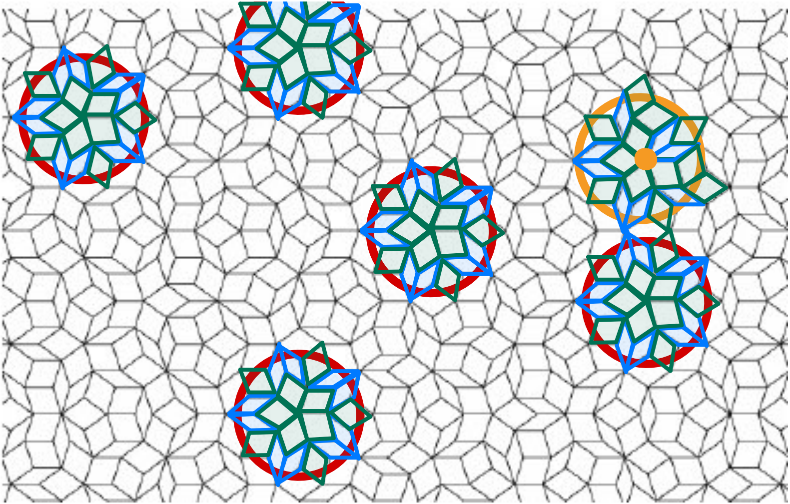

Definition (Strongly Pattern Equivariant)

A smooth function $f : T \to \mathbb{R}$ is *PE with radius* $R > 0$ if whenever $[B(x, R)] = [B(y, R)]$, then $f(x) = f(y)$.

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Definition (Weakly Pattern Equivariant)

A function $T \to \mathbb{R}$ which is a uniform limit of strongly-PE functions is a *weakly-PE* function.

Definition

A *strongly (weakly) PE k-form* is a differential form on T

$$\omega = \sum_{|\mathcal{I}|=k} f_{\mathcal{I}} dx^{\mathcal{I}}$$

where each $f_{\mathcal{I}}$ is strongly (weakly) PE.

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Theorem (Definition)

$C^\bullet_{s-PE}(T)$ and $C^\bullet_{w-PE}(T)$ are cochain complexes under the exterior derivative, with cohomologies $H^*_{s-PE}(T)$ and $H^*_{w-PE}(T)$ respectively.

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$$C^k_{tlc}(X) = \left\{ \omega : X \to \bigwedge^k T^*X \;\middle|\; \begin{array}{l} \omega \text{ is leafwise-smooth, and locally} \\ \text{constant in the transverse direction} \end{array} \right\}$$

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Let T be a simple tiling. Then

1. $H^*_{s-PE}(T) = \check{H}^*(\Omega_T, \mathbb{R})$
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□

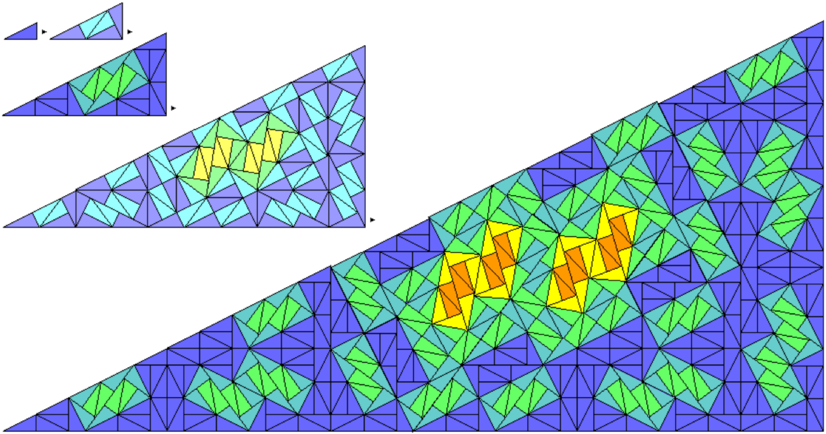
Summary of Relationships

$$
\begin{array}{ccccc}
\check{H}^*(\Omega_T, \mathbb{R}) & = & H^*_{s-PE}(T) & = & H^*_{tlc}(\Omega_T) \\
 & & & & \downarrow \\
 & & H^*_{w-PE}(T) & = & H^*_{\mathcal{T}}(\Omega_T) \\
 & & & & \twoheaddownarrow \\
 & & & & \overline{H}^*_{\mathcal{T}}(\Omega_T)
\end{array}
$$

Definition

A tiling which contains only finitely many types of patches with diameter less than some given $R > 0$ has *finite local complexity* (FLC). Otherwise, it has *infinite local complexity* (ILC).

Tilings With ILC



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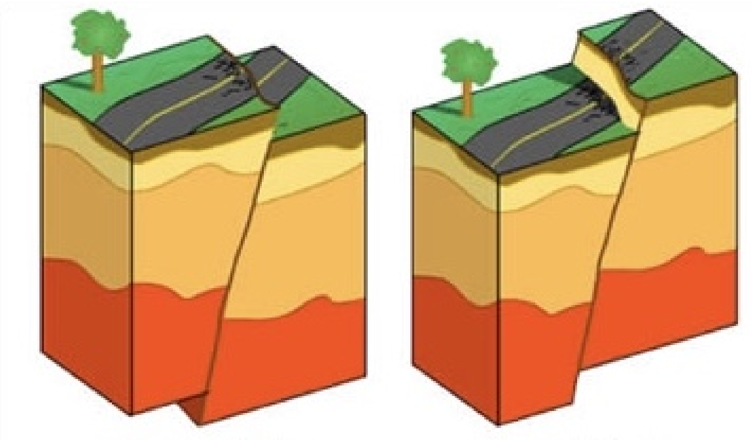

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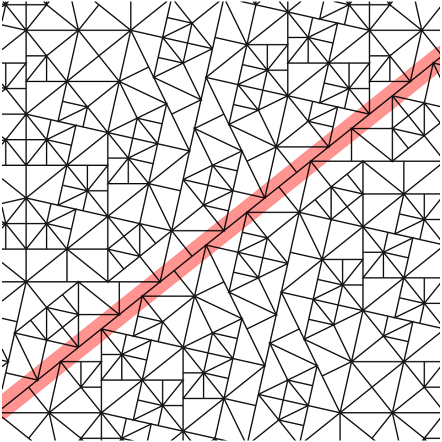

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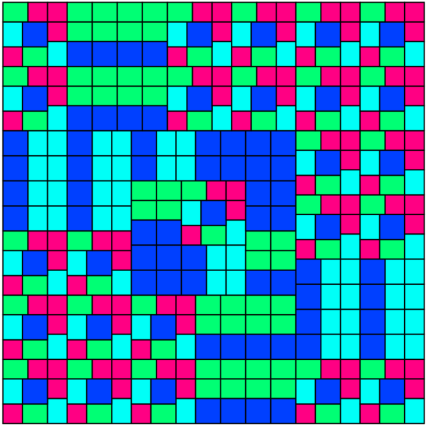
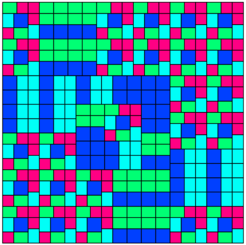
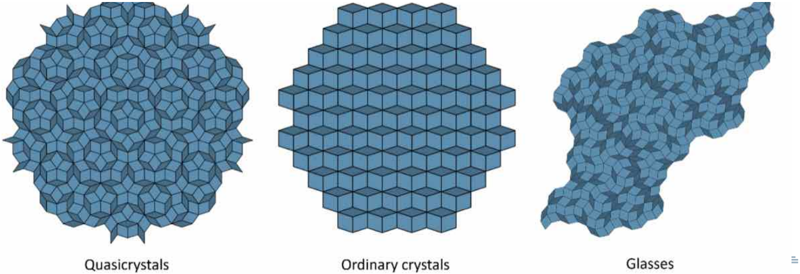


Shmuel Weinberger, et al.





Outline

Smoothing Things Over

Definition

A manifold M^n is said to have *bounded geometry* if there are constants $c, C > 0$ such that $\mathrm{inj}(M) > c > 0$ and $|K(M)| < C$.

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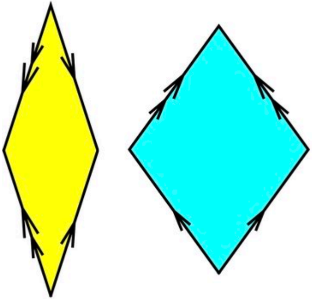
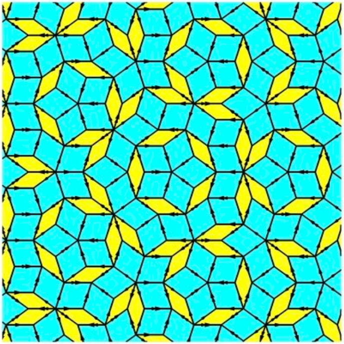
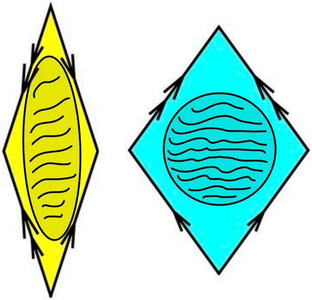
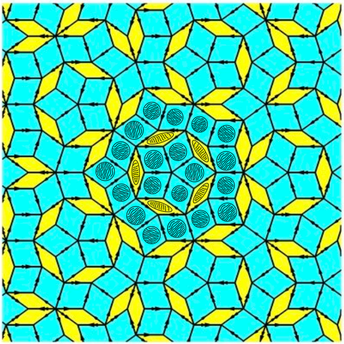
Creating Analogies

Tiling T	**Manifold M of BG**
Tiling metric	Gromov-Hausdroff metric
Ω_T	Ω_M
$H^*_{s-PE}(T)$	$H^*_{s-GE}(M)$
Foliation on Ω_T	Pre-foliated structure on Ω_M



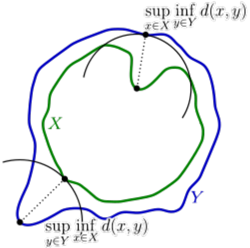






Definition (From Wikipedia)

"The Hausdorff distance [between two metric subspaces X, Y of an ambient space M] is the longest distance you can be forced to travel by an adversary who chooses a point in one of the two sets, from where you then must travel to the other set."



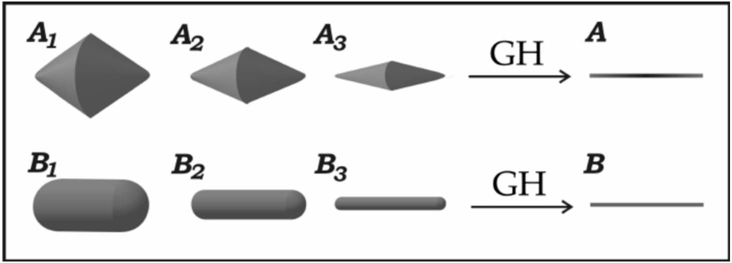
$$d_{\mathrm{H}}(X,Y) = \max\left\{ \sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(X,y) \right\},$$

Definition

The *Gromov-Hausdorff distance* between two metric spaces is the infimum

$$d_{GH}(X, Y) := \inf_{f,g} d_H(f(X), g(Y))$$

over isometric embeddings $f, g : X, Y \hookrightarrow M$ into some ambient space M.



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Let M be a manifold with bounded geometry and let $GHB(D)$ be Pointed Gromov-Hausdorff space of balls of diameter D. Define

$$\begin{aligned} \Psi_D : M &\to GHB(D) \\ m &\mapsto B(m, D/2) \end{aligned}$$

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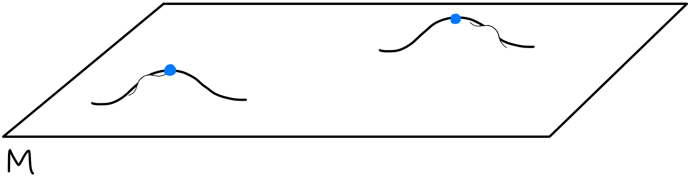
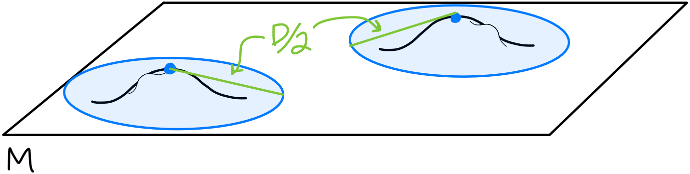
Any uniformly totally bounded class of compact metric spaces is pre-compact in GH space. See [BBI01, 264f.] for details. □

The Hull of a Manifold of BG




D/2
M

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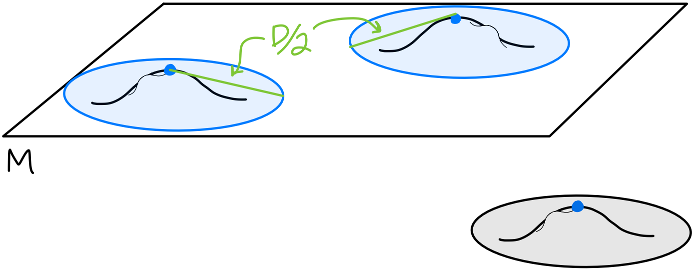

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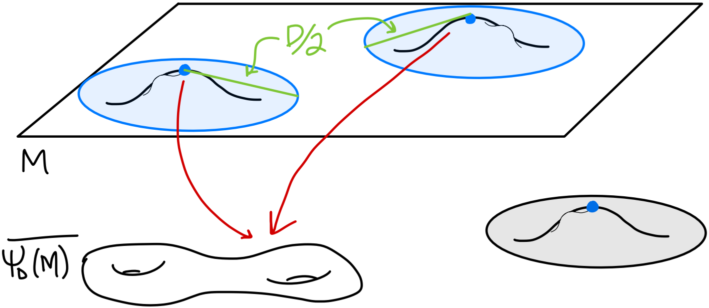

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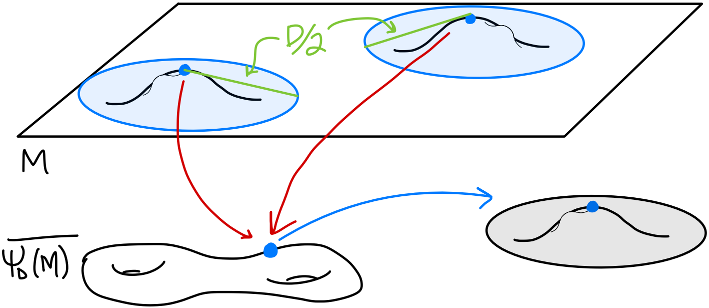

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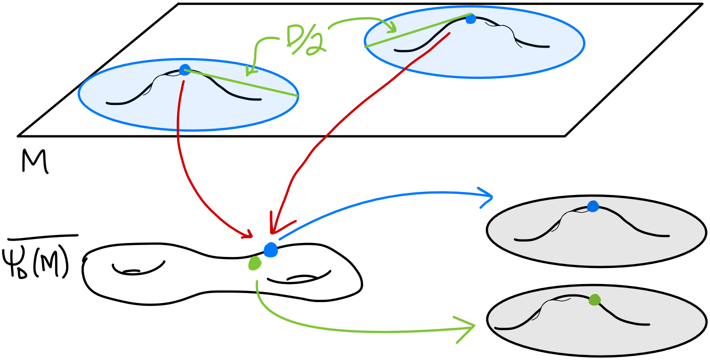

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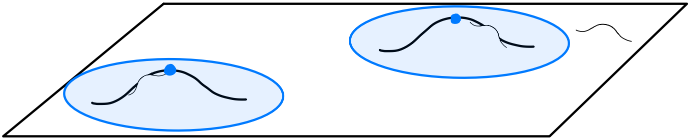

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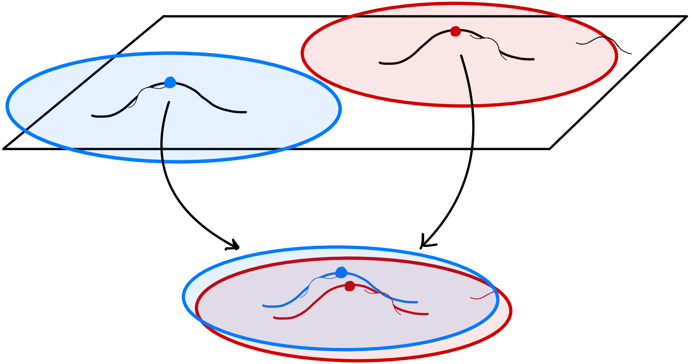

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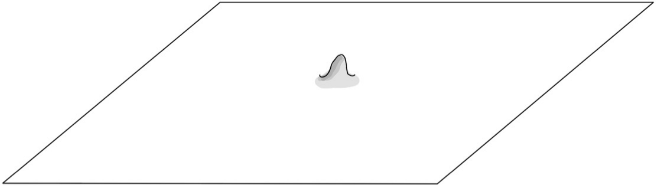
3. If $X \to M$ is a covering and M is compact and not too homogeneous, then $\Omega_X \simeq M$. In particular, $\Omega_M \simeq M$.

Almost-Flat $\mathbb{R}^n$



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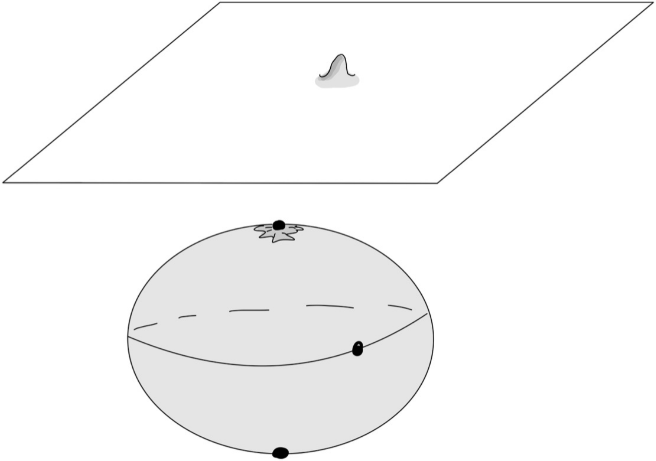

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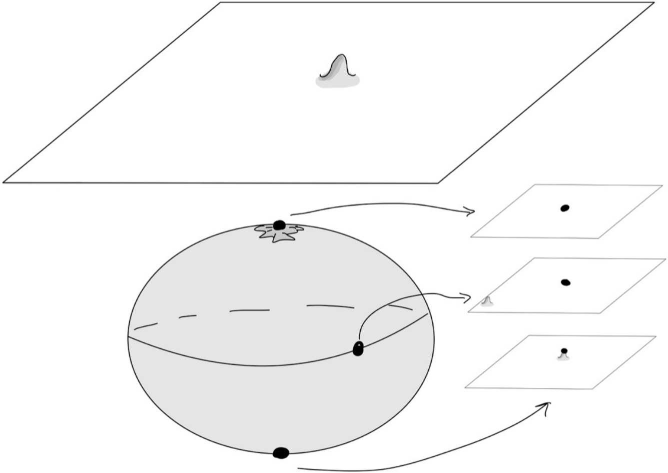
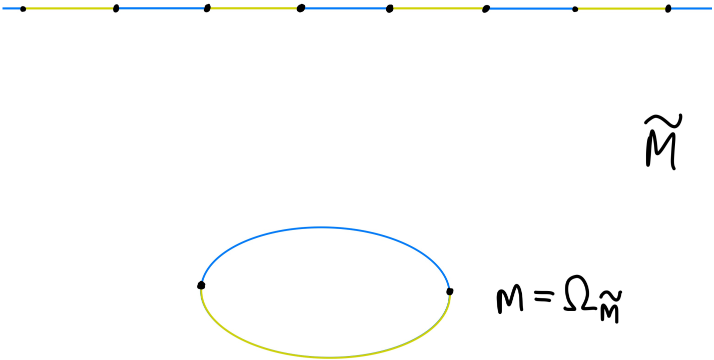
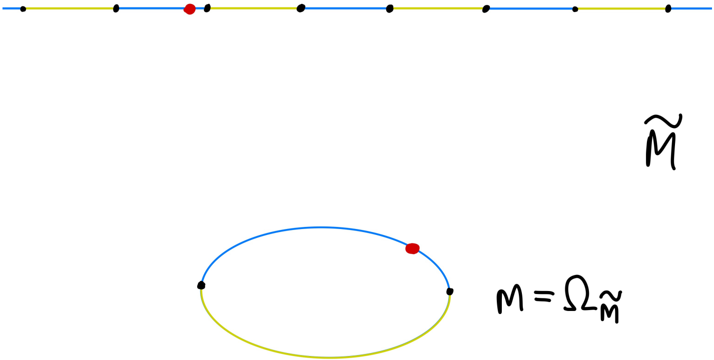


Universal Cover



Universal Cover



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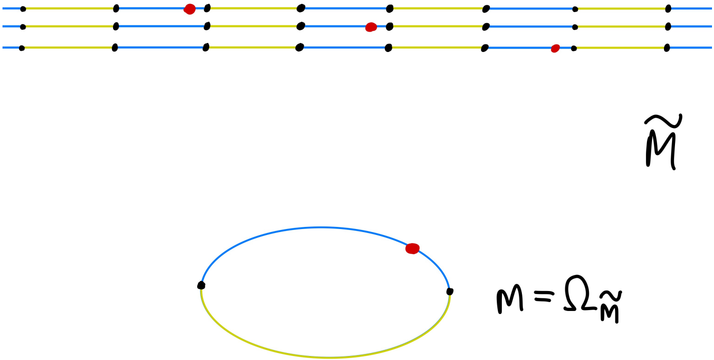

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Let $C^k_{s-GE}(M) = \{\omega \in \Omega^k(M) \mid \omega \text{ and } d\omega \text{ are } GE \text{ with some radius } R\}$.
Let $H^*_{s-GE}(M)$ be the cohomology of the cochain complex $C^\bullet_{s-GE}(M)$.

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Proof.

$C^k_{s-GE}(M) = C^k(M/\Gamma) \oplus C^k_c(M)$ □

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Definition (Leaves of the Hull)

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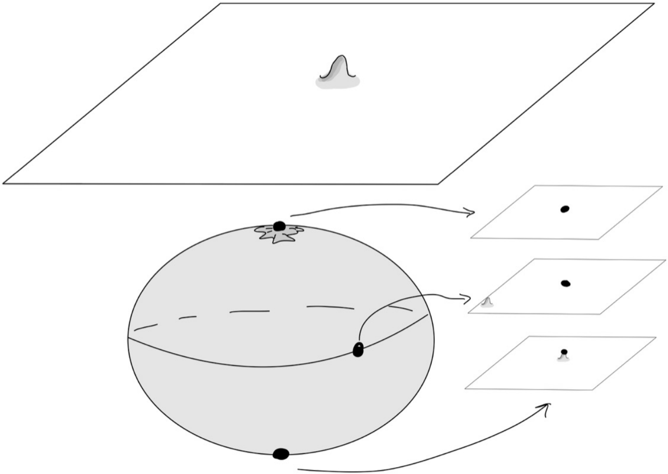
The *prefoliated structure* on Ω_M is the collection $\{\Psi : \widetilde{p} \to \mathcal{L}(\widetilde{p})\}$.

Back to Almost-Flat $\mathbb{R}^n$



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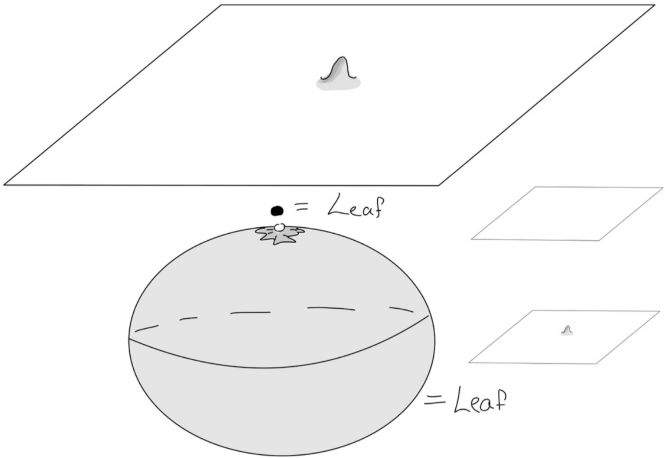
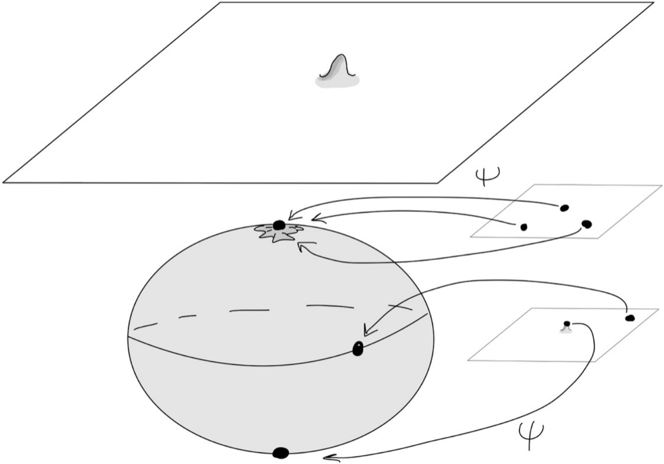


Back to Almost-Flat $\mathbb{R}^n$



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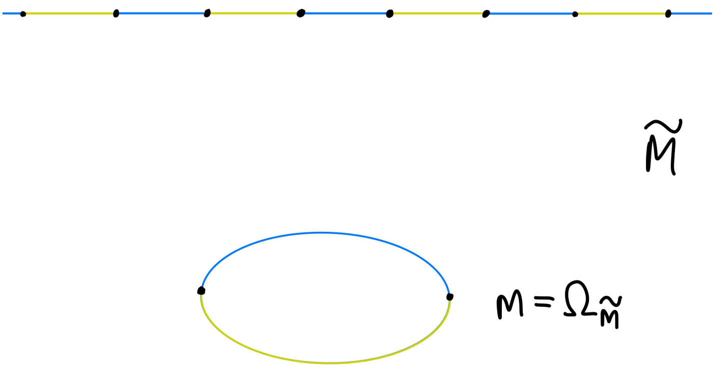

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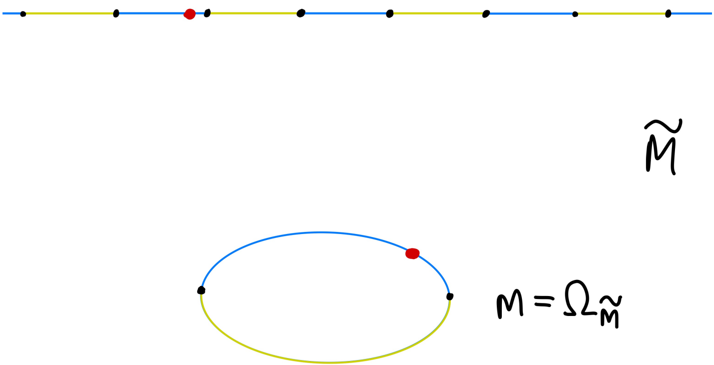

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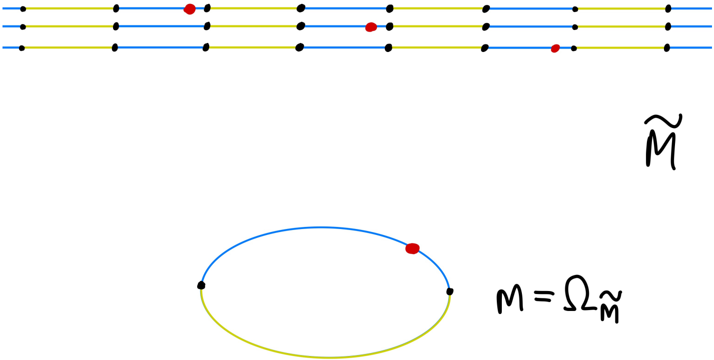

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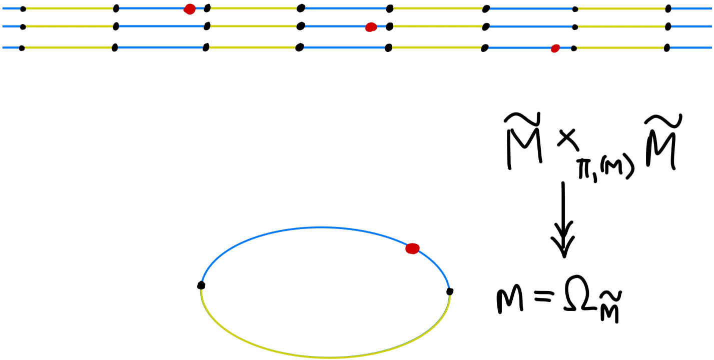

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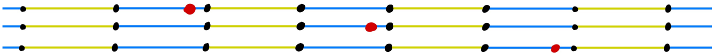
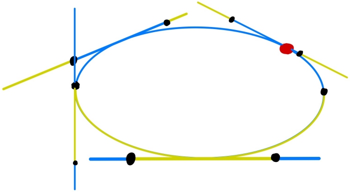
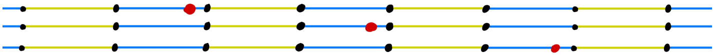
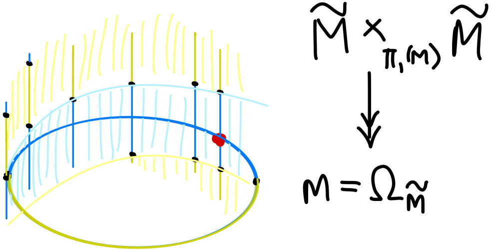




$$\tilde{M} \times_{\pi_1(M)} \tilde{M}$$
$$\downarrow$$
$$M = \Omega_{\tilde{M}}$$

Back to Universal Covers





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4. Develop an index theory for elliptic operators on prefoliated spaces analogous to the index theory of foliated spaces in [MS88]

References I

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Thank you!