The Hull of a Manifold with Bounded Geometry

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Outline

1 Motivation

2 Tilings of \mathbb{R}^d

Basic Notions and Examples The Hull of a Tiling Cohomology of the Hull Tilings With ILC

3 Manifolds with Bounded Geometry

A New Metric The Hull of a Manifold of BG Cohomology of the Hull The Prefoliated Structure

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Figure: Eight Allotropes of Carbon

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Figure: Three different kinds of material

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Future Directions

Tilings of \mathbb{R}^d

Definition

A *tiling* of \mathbb{R}^d is a subdivision into pieces called "tiles".

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Image: Image:

Definition

A *tiling* of \mathbb{R}^d is a subdivision into pieces called "tiles". A *simple tiling* is one in which

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Definition

A *tiling* of \mathbb{R}^d is a subdivision into pieces called "tiles".

- A simple tiling is one in which
 - 1 There is a finite collection $\{p_i\}_{i=1}^n$ of *prototiles* such that every tile is a translated copy of some p_i .

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 - 2 Each tile is a polytope
 - 3 If two tiles meet, they meet completely in one of their (d-1)-faces.

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Simple Tilings



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Figure: Periodic Tilings

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Simple Tilings



Figure: A Patch of the Penrose Tiling

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Tiling Spaces

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Violating Hypotheses



Figure: A Pinwheel Tiling. Lacks finitely many prototiles up to translation.

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Figure: Penrose Chickens. Tiles are not polytopes

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Violating Hypotheses



Figure: A chair tiling. Edges don't meet full-face to full-face

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| The Hull of a T | iling | | |

Definition (Tiling Metric)

Given two tilings, T and T', of \mathbb{R}^d , they are ε -close if up to a translation of distance ε , they agree on a ball of radius ε^{-1} around the origin.



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Definition

The *orbit* of a tiling T is its orbit under translation

$$\mathcal{O}(T) := \left\{ T - x \mid x \in \mathbb{R}^d \right\}$$

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Definition

A *tiling space* is a set of tilings which is closed under translation by \mathbb{R}^d and complete in the tiling metric.

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Definition (The Hull: Version 1)

The hull Ω_T of a tiling T is the completion of $\mathcal{O}(T)$ in the tiling metric.

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Definition (The Hull: Version 2)

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Definition (The Hull: Version 2)

Given a tiling T, for each $n \in \mathbb{N}$, the *n*th Gähler complex Γ_n is the collection of possible instructions for laying n layers of tiles around some tile at the origin consistent with T.

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The Hull of a Tiling

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Definition (The Hull: Version 2)

Given a tiling T, for each $n \in \mathbb{N}$, the *n*th Gähler complex Γ_n is the collection of possible instructions for laying n layers of tiles around some tile at the origin consistent with T. Let $f_n : \Gamma_{n+1} \to \Gamma_n$ be the forgetful map. The hull of T is the inverse limit

$$\Omega_{\mathcal{T}} := \varprojlim(\Gamma_n, f_n) = \left(\prod_{n \in \mathbb{N}} \Gamma_n\right) / \sim$$

The Hull of a Tiling

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Theorem

If T is a simple tiling, Ω_T a compact metric space.

Proof.

The Hull of a Tiling

 $\overline{\mathsf{Tilings}} \, \mathbf{of} \, \mathbb{R}^d$

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Proof.

Each Γ_n is a compact branched manifold, the inverse limit of which is a compact metric space.

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Figure: Transition Maps of a Foliated Manifold

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Figure: The Kronecker Foliation of the Torus

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Foliation of the Hull

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Foliation of the Hull

Definition

Let T be a tiling of \mathbb{R}^d , and let $T' \in \Omega_T$. An ε -transveral of T' is

$$\mathscr{T}_{\mathcal{T}',arepsilon} := \left\{ \mathit{T}'' \in \Omega_{\mathcal{T}} \mid \mathit{B}(0, arepsilon^{-1}) \cap \mathit{T}'' = \mathit{B}(0, arepsilon^{-1}) \cap \mathit{T}'
ight\}$$

 $= \begin{cases} \text{tilings which agree with } \mathcal{T}' \text{ on a ball of} \\ \text{radius } 1/\varepsilon \text{ when basepoints are aligned} \end{cases}$

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Foliation of the Hull

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 $= \left\{ \begin{array}{l} \text{tilings which agree with } \mathcal{T}' \text{ on a ball of} \\ \text{radius } 1/\varepsilon \text{ when basepoints are aligned} \end{array} \right\}$

Theorem

The hull of a simple tiling is a foliated space.

Foliation of the Hull

Definition

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Theorem

The hull of a simple tiling is a foliated space.

Proof.

Because our tilings have FLC, the action of \mathbb{R}^d is locally free. So for any $T'' \in \mathscr{T}_{T',\varepsilon}$, if $v \in \mathbb{R}^d$ with |v| small enough, $T'' - v \notin \mathscr{T}_{T',\varepsilon}$. That is, $\mathscr{T}_{T',\varepsilon}$ is transverse to the action of \mathbb{R}^d .

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Cohomology of the Hull

Tilings of \mathbb{R}^d

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Three Cohomologies of $\Omega_{\mathcal{T}}$

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Manifolds with Bounded Geometry

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Three Cohomologies of $\Omega_{\mathcal{T}}$

Čech Cohomology

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Image: A matrix

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Manifolds with Bounded Geometry

Three Cohomologies of $\Omega_{\mathcal{T}}$

- Čech Cohomology
- 2 Pattern-Equivariant Cohomology

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Three Cohomologies of $\Omega_{\mathcal{T}}$

- Čech Cohomology
- 2 Pattern-Equivariant Cohomology
- 8 Foliated Cohomology

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Tilings of \mathbb{R}^d

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Cohomology of the Hull

Čech Cohomology $\check{H}^*(\Omega_T)$



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Tilings of \mathbb{R}^d

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Cohomology of the Hull

Čech Cohomology $\check{H}^*(\Omega_T)$



Definition $\check{H}^*(X, \mathbb{R}) = \varinjlim_{\mathcal{U}} H^*(N(\mathcal{U}), \mathbb{R})$

Image: A matrix and a matrix

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

PE Cohomology



Manifolds with Bounded Geometry

Cohomology of the Hull

PE Cohomology



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Cohomology of the Hull



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| Cohomology of th | e Hull | | |

Definition (Strongly Pattern Equivariant)

A smooth function $f : T \to \mathbb{R}$ is *PE with radius* R > 0 if whenever [B(x, R)] = [B(y, R)], then f(x) = f(y).

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| Cohomology of th | e Hull | | |

Definition (Strongly Pattern Equivariant)

A smooth function $f : T \to \mathbb{R}$ is *PE with radius* R > 0 if whenever [B(x, R)] = [B(y, R)], then f(x) = f(y). A function is *strongly PE* if it is PE for some R > 0.
| Motivation | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions |
|------------------|---------------------------|---------------------------------|-------------------|
| Cohomology of th | e Hull | | |

Definition (Strongly Pattern Equivariant)

A smooth function $f : T \to \mathbb{R}$ is *PE with radius* R > 0 if whenever [B(x, R)] = [B(y, R)], then f(x) = f(y). A function is *strongly PE* if it is PE for some R > 0.

Definition (Weakly Pattern Equivariant)

A function $T \to \mathbb{R}$ which is a uniform limit of strongly-PE functions is a *weakly-PE* function.

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|------------------------|---------------------------|---------------------------------|-------------------|--|
| Cohomology of the Hull | | | | |

A strongly (weakly) PE k-form is a differential form on T

$$\omega = \sum_{|\mathcal{I}|=k} f_{\mathcal{I}} dx^{\mathcal{I}}$$

where each $f_{\mathcal{I}}$ is strongly (weakly) PE.

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| Cohomology of th | e Hull | | |

A strongly (weakly) PE k-form is a differential form on T

$$\omega = \sum_{|\mathcal{I}|=k} f_{\mathcal{I}} dx^{\mathcal{I}}$$

where each $f_{\mathcal{I}}$ is strongly (weakly) PE. The collection of such forms is denoted $C_{s-PE}^{k}(\mathcal{T})$ (resp. $C_{w-PE}^{k}(\mathcal{T})$).

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| Cohomology of th | e Hull | | |

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Theorem (Definition)

 $C^{\bullet}_{s-PE}(T)$ and $C^{\bullet}_{w-PE}(T)$ are cochain complexes under the exterior derivative, with cohomologies $H^{*}_{s-PE}(T)$ and $H^{*}_{w-PE}(T)$ respectively.

 $\mathsf{Tilings of } \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

Foliated Cohomology

Kyle Hansen (UCSB)

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Future Directions

Foliated Cohomology

Definition

Let X be a foliated space. Let

$$C^k_{tlc}(X) = \left\{ \omega : X \to \bigwedge^k T^*X \right\}$$

 ω is leafwise-smooth, and locally constant in the transverse direction

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Future Directions

Foliated Cohomology

Definition

Let X be a foliated space. Let

 $C_{t/c}^{k}(X) = \left\{ \omega : X \to \bigwedge^{k} T^{*}X \middle| \begin{array}{c} \omega \text{ is leafwise-smooth, and locally} \\ \text{constant in the transverse direction} \end{array} \right\}$

$$C^k_{\tau}(X) = \left\{ \omega : X \to \bigwedge^k T^*X \mid \omega \text{ is transversely continuous}
ight\}$$

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Future Directions

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Theorem

 $C^{\bullet}_{tlc}(X)$ and $C^{\bullet}_{\tau}(X)$ are cochain complexes with cohomologies $H^*_{tlc}(X)$ and $H^*_{\tau}(X)$ respectively.

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Future Directions

Foliated Cohomology

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ight\}$$

Theorem

 $C^{\bullet}_{tlc}(X)$ and $C^{\bullet}_{\tau}(X)$ are cochain complexes with cohomologies $H^*_{tlc}(X)$ and $H^*_{\tau}(X)$ respectively. $\overline{H^*}_{\tau}(X)$ is the maximal Hausdorff quotient of $H^*_{\tau}(X)$.

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Cohomology of the Hull

 $\mathsf{Tilings of } \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Comparing Cohomologies

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Comparing Cohomologies

Lemma

There is a correspondence

 $\{\text{strongly PE functions } T \to \mathbb{R}\} \longleftrightarrow \varinjlim \{\text{smooth functions } \Gamma_n \to \mathbb{R}\}$

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Comparing Cohomologies

Lemma

There is a correspondence

{strongly PE functions $T \to \mathbb{R}$ } $\longleftrightarrow \lim_{n \to \infty} {smooth functions \Gamma_n \to \mathbb{R}}$

Lemma

If M is a nice enough branched manifold, $\check{H}^*(M, \mathbb{R}) = H^*_{deRham}(M)$.

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Future Directions

Comparing Cohomologies

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Lemma

If M is a nice enough branched manifold, $\check{H}^*(M, \mathbb{R}) = H^*_{deRham}(M)$.

Lemma

$$\check{H}^*(\Omega_{\mathcal{T}},\mathbb{R})=\check{H}^*(\varprojlim \Gamma_n,\mathbb{R})=\varinjlim \check{H}^*(\Gamma_n,\mathbb{R})$$

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 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

Theorem ([KP06], [Sad08])

Let T be a simple tiling. Then

$$\mathbf{1} \ H^*_{s-PE}(T) = \check{H}^*(\Omega_T, \mathbb{R})$$

2
$$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$$
 and $H^*_{w-PE}(T) = H^*_{\tau}(\Omega_T)$.

 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

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Proof of 1.

 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

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$$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$$
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Proof of 1.

$$H^*_{s-PE}(T) = \frac{B^*_{s-PE}(T)}{Z^*_{s-PE}(T)}$$

 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

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Proof of 1.

$$H_{s-PE}^{*}(T) = \frac{B_{s-PE}^{*}(T)}{Z_{s-PE}^{*}(T)}$$
$$= \frac{\lim_{\to} B^{*}(\Gamma_{n})}{\lim_{\to} Z^{*}(\Gamma_{n})}$$

 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

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Proof of 1.

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$$= \frac{\varinjlim B^{*}(\Gamma_{n})}{\varinjlim Z^{*}(\Gamma_{n})}$$
$$= \varinjlim H_{deRham}^{*}(\Gamma_{n})$$

Tilings of \mathbb{R}^d

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

Theorem ([KP06], [Sad08])

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$$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$$
 and $H^*_{w-PE}(T) = H^*_{\tau}(\Omega_T)$.

Proof of 1.

$$\begin{aligned} H^*_{s-PE}(T) &= \frac{B^*_{s-PE}(T)}{Z^*_{s-PE}(T)} \\ &= \frac{\varinjlim B^*(\Gamma_n)}{\varinjlim Z^*(\Gamma_n)} \\ &= \varinjlim H^*_{deRham}(\Gamma_n) \\ &= \varinjlim \check{H}^*(\Gamma_n, \mathbb{R}) \end{aligned}$$

 $\overline{\mathsf{Tilings}} \, \mathsf{of} \, \mathbb{R}^d$

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Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

Theorem ([KP06], [Sad08])

Let T be a simple tiling. Then

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$$H^*_{s-PE}(T) = H^*_{tlc}(\Omega_T)$$
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Proof of 1.

$$I_{s-PE}^{*}(T) = \frac{B_{s-PE}^{*}(T)}{Z_{s-PE}^{*}(T)}$$
$$= \frac{\varinjlim B^{*}(\Gamma_{n})}{\varinjlim Z^{*}(\Gamma_{n})}$$
$$= \varinjlim H_{deRham}^{*}(\Gamma_{n})$$
$$= \varinjlim \check{H}^{*}(\Gamma_{n}, \mathbb{R})$$
$$= \check{H}^{*}(\varprojlim \Gamma_{n}, \mathbb{R})$$

Tilings of \mathbb{R}^d

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Manifolds with Bounded Geometry

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Future Directions

Cohomology of the Hull

Theorem ([KP06], [Sad08])

Let T be a simple tiling. Then

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$${}^{*}_{s-PE}(T) = \frac{B^{*}_{s-PE}(T)}{Z^{*}_{s-PE}(T)}$$
$$= \frac{\varinjlim B^{*}(\Gamma_{n})}{\varinjlim Z^{*}(\Gamma_{n})}$$
$$= \varinjlim H^{*}_{deRham}(\Gamma_{n})$$
$$= \varinjlim \check{H}^{*}(\Gamma_{n}, \mathbb{R})$$
$$= \check{H}^{*}((\varliminf \Gamma_{n}, \mathbb{R}))$$
$$= \check{H}^{*}(\Omega_{T}, \mathbb{R})$$

Future Directions

Summary of Relationships

Kyle Hansen (UCSB)

May 24, 2022 27 / 50

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| Motivation | Tilings of \mathbb{R}^d |
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| Tilings With II C | |

A tiling which contains only finitely many types of patches with diameter less than some given R > 0 has finite local complexity (FLC). Otherwise, it has infinite local complexity (ILC).

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| Tilings With ILC | |



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Tilings With ILC

Manifolds with Bounded Geometry



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Motivation Tiling

Tilings With ILC

 $\overline{\mathsf{Tilings}} \, \mathbf{of} \, \mathbb{R}^d$

Manifolds with Bounded Geometry

Future Directions

Shmuel Weinberger, et al.







Outline

Motivation

2 Tilings of \mathbb{R}^d

Basic Notions and Examples The Hull of a Tiling Cohomology of the Hull Tilings With ILC

3 Manifolds with Bounded Geometry

A New Metric The Hull of a Manifold of BG Cohomology of the Hull The Prefoliated Structure

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4 Future Directions

Smoothing Things Over

Definition

A manifold M^n is said to have *bounded geometry* if there are constants c, C > 0 such that inj(M) > c > 0 and |K(M)| < C.

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Smoothing Things Over

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Creating AnalogiesTiling TManifold M of BGTiling metricGromov-Hausdroff metric Ω_T Ω_M $H^*_{s-PE}(T)$ $H^*_{s-GE}(M)$ Foliation on Ω_T Pre-foliated structure on Ω_M

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Image: A matrix and a matrix

| Motivation 00 | Tilings of \mathbb{R}^d 000000000000000000000000000000000000 |
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| Basic Notions | |

Future Directions





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| Basic Notions | |

Future Directions





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| A New Metric | | | |

Definition (From Wikipedia)

"The Hausdorff distance [between two metric subspaces X, Y of an ambient space M] is the longest distance you can be forced to travel by an adversary who chooses a point in one of the two sets, from where you then must travel to the other set."



| Motivation | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry ○○○●○○○○○○○○ | Future Directions |
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| A New Metric | | | |

The Gromov-Hausdorff distance between two metric spaces is the infimum

$$d_{GH}(X,Y) := \inf_{f,g} d_H(f(X),g(Y))$$

over isometric embeddings $f, g: X, Y \hookrightarrow M$ into some ambient space M.



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| The Hull of a Manifold of BG | | | |

Let M be a manifold with bounded geometry and let GHB(D) be Pointed Gromov-Hausdorff space of balls of diameter D. Define

 $\Psi_D: M o GHB(D)$ $m \mapsto B(m, D/2)$
| Motivation | Tilings of \mathbb{R}^d 000000000000000000000000000000000000 | Manifolds with Bounded Geometry | Future Directions 0 |
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| The Hull of a Ma | nifold of BG | | |

Let M be a manifold with bounded geometry and let GHB(D) be Pointed Gromov-Hausdorff space of balls of diameter D. Define

 $\Psi_D: M o GHB(D)$ $m \mapsto B(m, D/2)$

Theorem

The image $\Psi_D(M) \subseteq GHB(D)$ is precompact.

Proof.

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| The Hull of a Ma | nifold of BG | | |

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 $\Psi_D: M o GHB(D)$ $m \mapsto B(m, D/2)$

Theorem

The image $\Psi_D(M) \subseteq GHB(D)$ is precompact.

Proof.

Any uniformly totally bounded class of compact metric spaces is pre-compact in GH space. See [BBI01, 264f.] for details.

| Motivation | Tilings of \mathbb{R}^d 000000000000000000000000000000000000 | Manifolds with Bounded Geometry | Future Directions |
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| The Hull of a M | anifold of BG | | |



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| Motivation | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions |
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| The Hull of a Ma | nifold of BG | | |

Let *M* be a manifold with bounded geometry. The hull Ω_M of *M* is the subspace of *GHB* defined by $\Omega_M := \varprojlim \overline{\Psi_D(M)}$.

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| The Hull of a Manifold of BG | | | | |

Let *M* be a manifold with bounded geometry. The hull Ω_M of *M* is the subspace of *GHB* defined by $\Omega_M := \varprojlim \overline{\Psi_D(M)}$.

Example

| Motivation | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions | |
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| The Hull of a Manifold of BG | | | | |

Let *M* be a manifold with bounded geometry. The hull Ω_M of *M* is the subspace of *GHB* defined by $\Omega_M := \varprojlim \overline{\Psi_D(M)}$.

Example

1 *M* is homogeneous iff $\Psi_D(M) = \bullet$ for all D > 0.

| Motivation | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions | |
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| The Hull of a Manifold of BG | | | | |

Let *M* be a manifold with bounded geometry. The hull Ω_M of *M* is the subspace of *GHB* defined by $\Omega_M := \lim_{M \to \infty} \overline{\Psi_D(M)}$.

Example

- M is homogeneous iff $\Psi_D(M) = \bullet$ for all D > 0.
- **2** If *M* arises from a simple tiling *T*, then $\Omega_M \simeq \Omega_T$.

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| The Hull of a Manifold of BG | | | | |

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Example

- M is homogeneous iff $\Psi_D(M) = \bullet$ for all D > 0.
- **2** If *M* arises from a simple tiling *T*, then $\Omega_M \simeq \Omega_T$.
- S If X → M is a covering and M is compact and not too homogeneous, then Ω_X ≃ M. In particular, Ω_M ≃ M.

Future Directions

Almost-Flat \mathbb{R}^n



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Future Directions

Almost-Flat \mathbb{R}^n



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Almost-Flat \mathbb{R}^n



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Cohomology of the Hull

Tilings of $\mathbb{R}^{\prime\prime}$

Manifolds with Bounded Geometry

Future Directions

Cohomology of the Hull

Kyle Hansen (UCSB)

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Cohomology of the Hull

Recall

There is a correspondence

{*PE* forms on *T* with radius n} \leftrightarrow {smooth forms on Γ_n }

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Future Directions

Cohomology of the Hull

Recall

There is a correspondence

{*PE* forms on *T* with radius n} \leftrightarrow {smooth forms on Γ_n }

Definition

A smooth k-form ω on M is Geometry Equivariant with radius D/2 if there is a continuous k-form $\widetilde{\omega}$ on $\overline{\Psi_D(M)}$ such that $\widetilde{\omega}(\Psi_D(p)) = \omega(p)$.

Cohomology of the Hull

Recall

There is a correspondence

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Definition

A smooth k-form ω on M is Geometry Equivariant with radius D/2 if there is a continuous k-form $\widetilde{\omega}$ on $\overline{\Psi_D(M)}$ such that $\widetilde{\omega}(\Psi_D(p)) = \omega(p)$.

Definition

Let $C_{s-GE}^{k}(M) = \{ \omega \in \Omega^{k}(M) \mid \omega \text{ and } d\omega \text{ are } GE \text{ with some radius } R \}.$ Let $H_{s-GE}^{*}(M)$ be the cohomology of the cochain complex $C_{s-GE}^{\bullet}(M)$.

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| Cohomology of the Hull | | | | |

If *M* is homogeneous, then $C_{s-GE}^{k}(M) = \{\omega \in \Omega^{k}(M) \mid \omega \text{ is constant}\}.$

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| Motivation 00 | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions |
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| Cohomology of the Hull | | | |

If *M* is homogeneous, then
$$C_{s-GE}^{k}(M) = \{\omega \in \Omega^{k}(M) \mid \omega \text{ is constant}\}.$$

Example

Let *M* be symmetric of noncompact type, tiled by a Γ -equivariant tiling, where $\Gamma \curvearrowright M$ is geometric. Then $\Omega_M = M/\Gamma$ and $H^*_{s-GE}(M) = H^*(M/\Gamma)$.

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| Cohomology of the Hull | | | |

If *M* is homogeneous, then
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Let *M* be symmetric of noncompact type, tiled by a Γ -equivariant tiling, where $\Gamma \curvearrowright M$ is geometric. Then $\Omega_M = M/\Gamma$ and $H^*_{s-GE}(M) = H^*(M/\Gamma)$.

Example

If *M* as above has a single impurity, then $H^*_{s-GE}(M) = H^*(M/\Gamma) \oplus \mathbb{R}[n]$.

Proof.

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| Motivation 00 | Tilings of \mathbb{R}^d | Manifolds with Bounded Geometry | Future Directions |
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| Cohomology of the | e Hull | | |

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Example

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Example

If *M* as above has a single impurity, then $H^*_{s-GE}(M) = H^*(M/\Gamma) \oplus \mathbb{R}[n]$.

Proof.

$$C^k_{s-GE}(M) = C^k(M/\Gamma) \oplus C^k_c(M)$$

The Prefoliated Structure

Manifolds with Bounded Geometry

The Prefoliated Structure on the Hull

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Future Directions

The Prefoliated Structure on the Hull

Definition (Leaves of the Hull) For each $\tilde{p} = (N, p) \in \Omega_M$, define $\Psi : (N, p) \to \Omega_M$ by $\Psi(q) = \lim B(q, D) = (N, q).$

$$\Psi(q) = \lim_{GHB} B(q, D) = (N, q)$$

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The prefoliated structure on Ω_M is the collection $\{\Psi : \tilde{p} \to \mathcal{L}(\tilde{p})\}$.

The Prefoliated Structure

Back to Almost-Flat \mathbb{R}^n



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Back to Universal Covers



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Outline

Motivation

2 Tilings of \mathbb{R}^d

Basic Notions and Examples The Hull of a Tiling Cohomology of the Hull Tilings With ILC

Manifolds with Bounded Geometry

A New Metric The Hull of a Manifold of BG Cohomology of the Hull The Prefoliated Structure

4 Future Directions

We plan to make a systematic study of Ω_M . In particular we aim to...

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- ① Determine obstructions to the prefoliated structure being a foliation
- ② Develop tools for calculating cohomology of manifolds arising from tilings of ILC
- 3 Study perturbations of metrics using the hull
- Develop an index theory for elliptic operators on prefoliated spaces analogous to the index theory of foliated spaces in [MS88]

5 1 SQC

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