Tiling Spaces which are Fiber Bundles over Nilmanifolds

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Outline

1. Tiling Spaces of $\mathbb{R}^d$
   - Aperiodic Tilings
   - Tiling Spaces
   - Quasi-Crystal Synthesis
2. Approximate Lattices
   - Lattices in Nilpotent Lie Groups
   - The Structure of Uniform Approximate Lattices
3. Simple Tilings in Rational Nilpotent Lie Groups
   - Delaunay Tilings
   - The Structure of Nilpotent Tiling Spaces

Outline

Examples

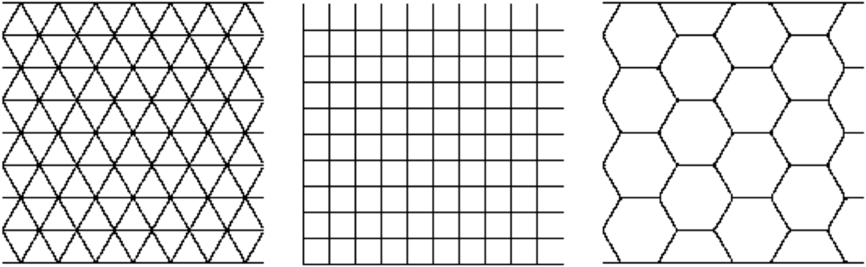

Figure: Periodic Tilings

Examples

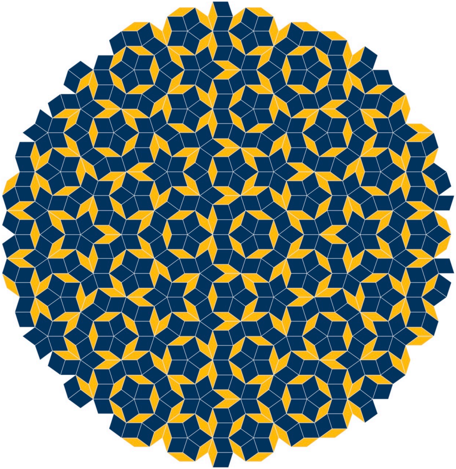

Figure: A Patch of a Penrose Tiling

Examples

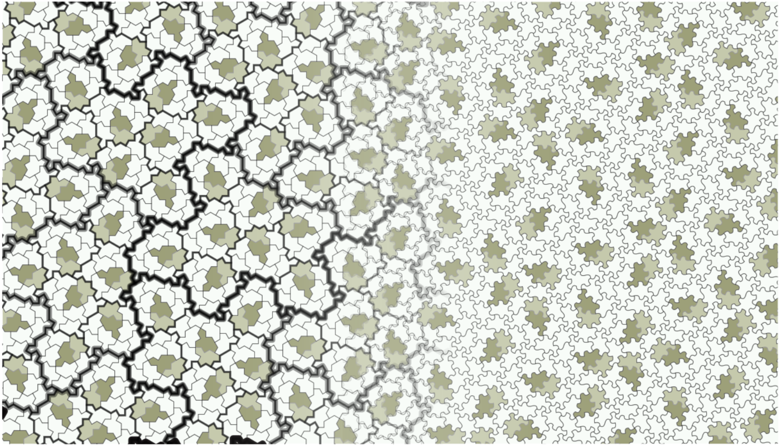

Figure: The "hat" and "spectre" aperiodic tilings.

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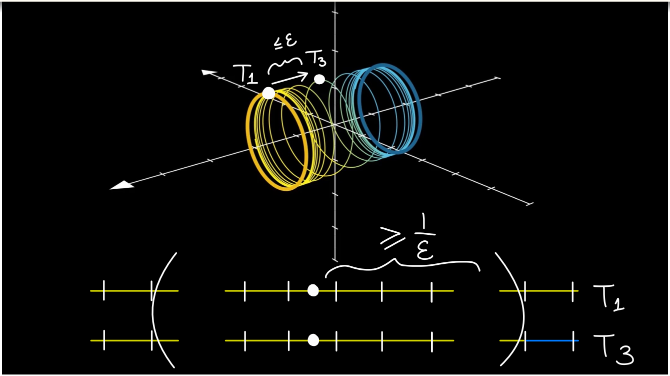
Running FLC Assumption

All tilings have **finite local complexity** and **geometrically normal**; there are finitely two-tile patches, and intersections are piecewise smooth disks.

Tiling Spaces

Definition

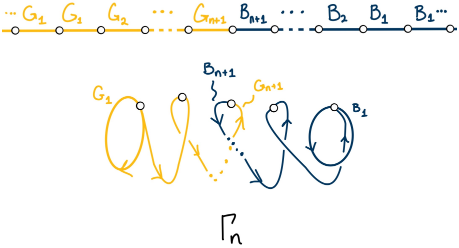
The **hull** Ω_T of a tiling T is the collection of all tilings which look like T at arbitrarily large scales around the origin. There is a **tiling metric** making Ω_T a compact metric space with $\mathbb{R}^d \curvearrowright \Omega_T$.

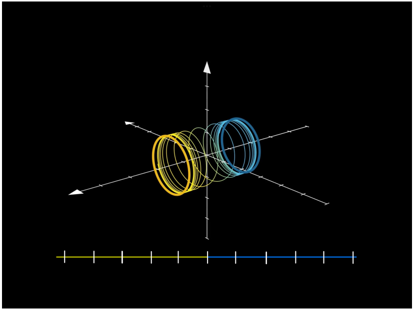


Topology of the Hull

Theorem

- *Ω_T is the inverse limit of compact branched manifolds [Sadun, 2003]*





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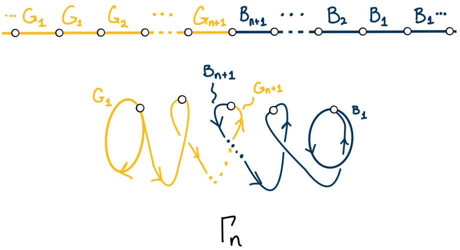

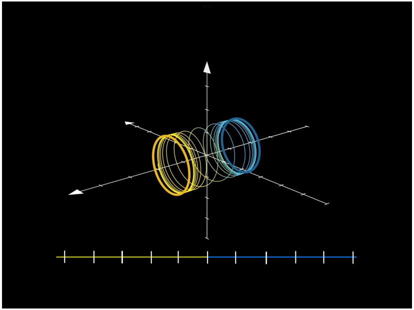

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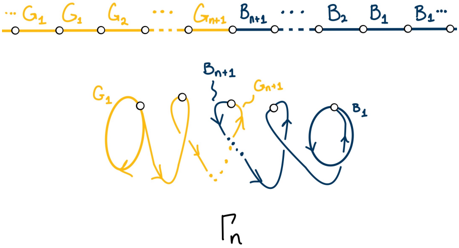

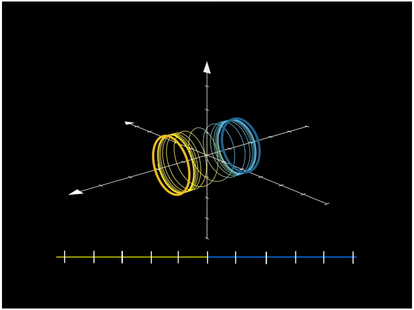

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Theorem ([Frank, 2000])

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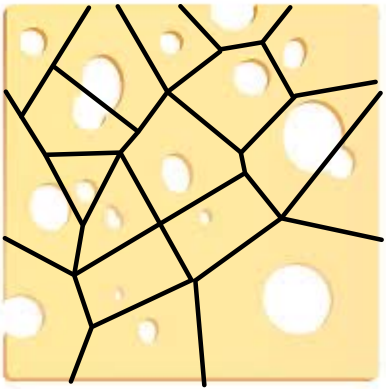

Creating FLC Tilings

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Uniform approximate lattices *(certain FLC Delaunay sets) come from* ***cut-and-project schemes****.*

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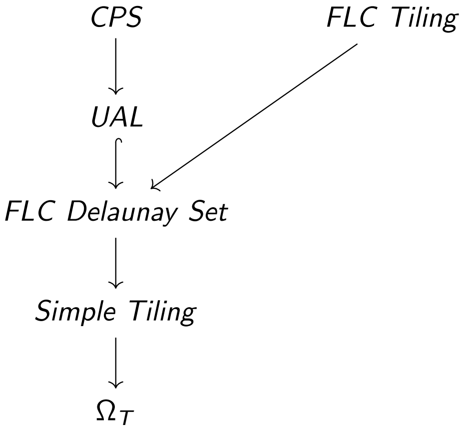

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Definition (Heuristic)

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Example

The **real Heisenberg group** $\mathbb{H}$ is

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

This is diffeomorphic to $\mathbb{R}^3$ with group law

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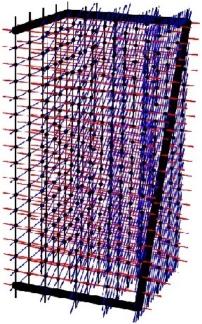

Uniform Lattices & Maltsev's Correspondence

Definition

A set of scalars S are **structure constants** for a basis $\{X_1, \ldots, X_d\}$ of the lie algebra $\mathfrak{g}$ if $[X_i, X_j] = \sum a_k X_k$ with each $a_k \in S$

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Theorem ([Maltsev, 1949])

$\mathfrak{g}$ has a basis with structure constants in $\mathbb{Q}$ $\iff$ *G has a lattice*

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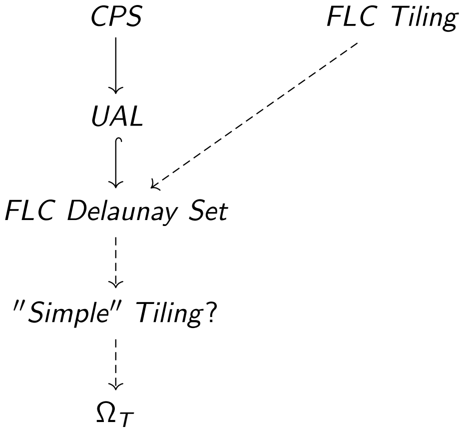
Theorem ([Machado, 2018])

Every UAL $\Lambda \subseteq G$ comes from a cut and project scheme.

The Lie of the Land in G



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Benefits in $\mathbb{R}^d$ absent in G

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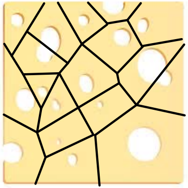

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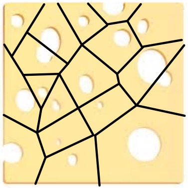

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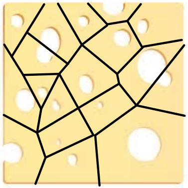

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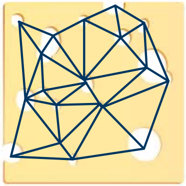

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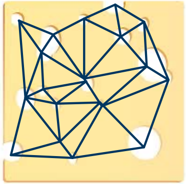

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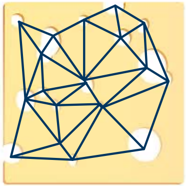

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Theorem ([H., 2024])

There is a tiling equivalence: Tiling $\rightarrow$ Delaunay Set $\rightarrow$ Simple Tiling

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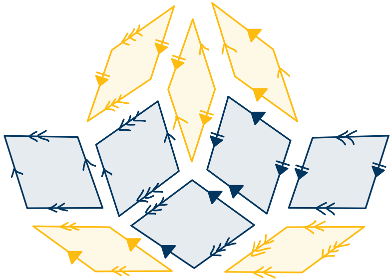

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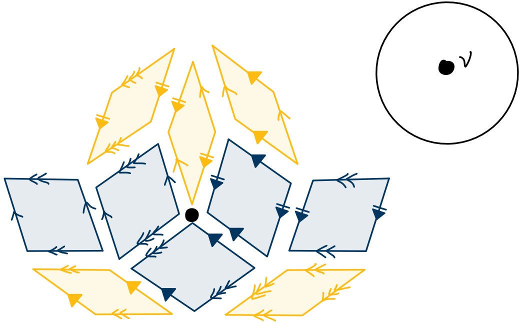

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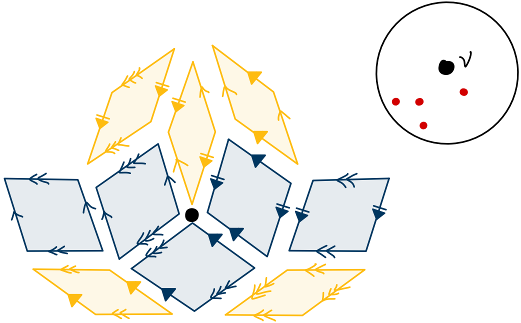

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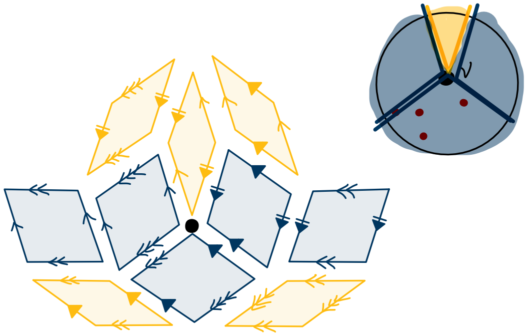

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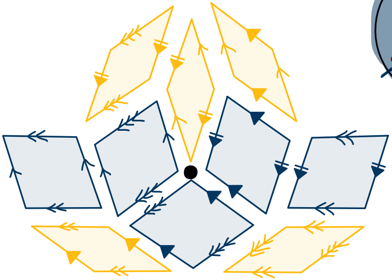

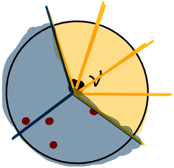

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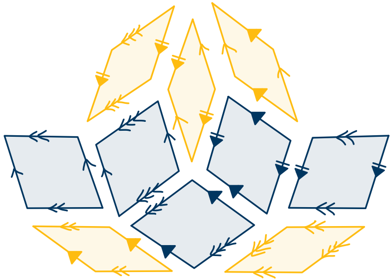

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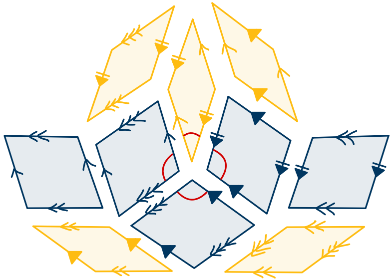

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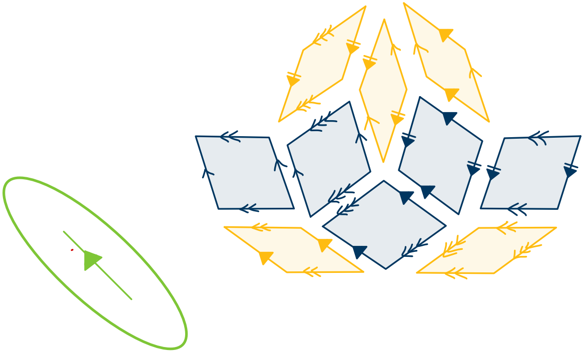

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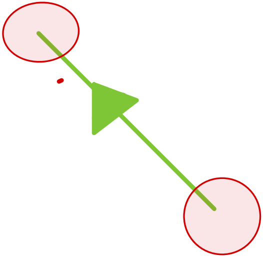

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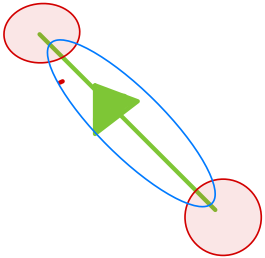

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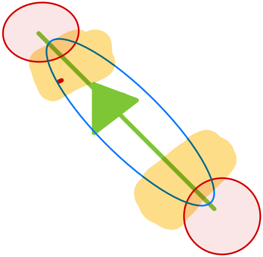

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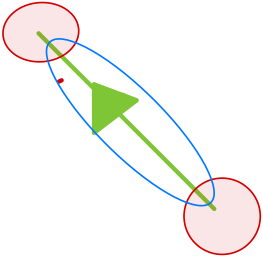

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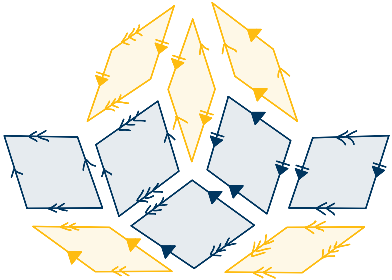

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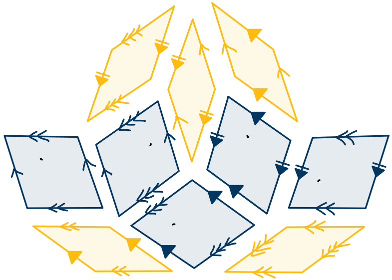

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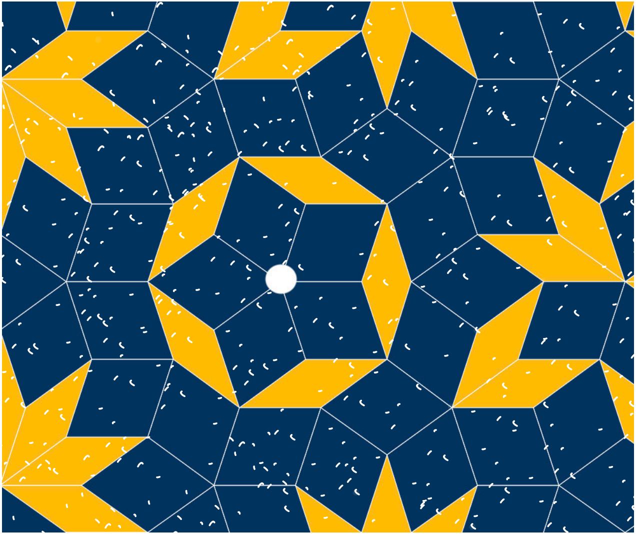

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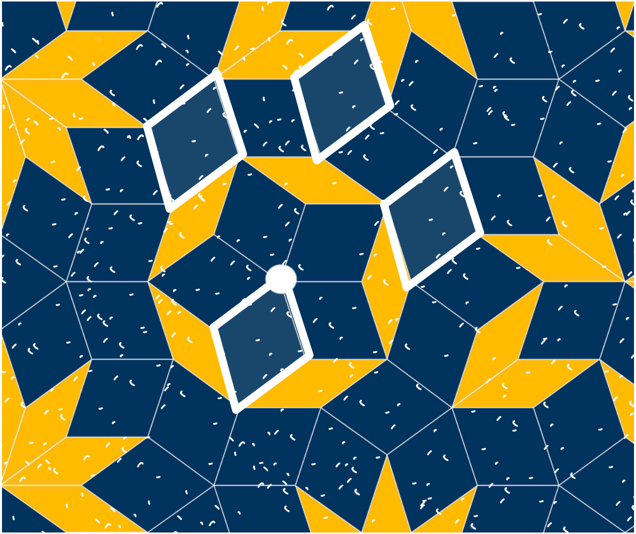

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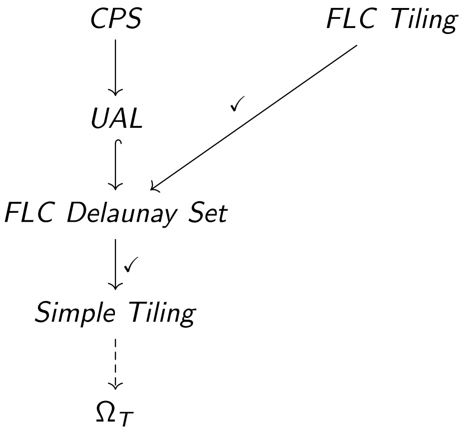

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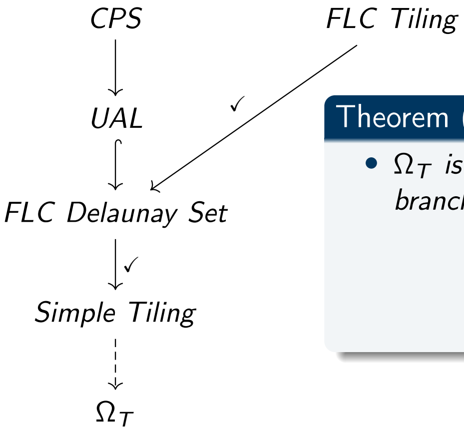

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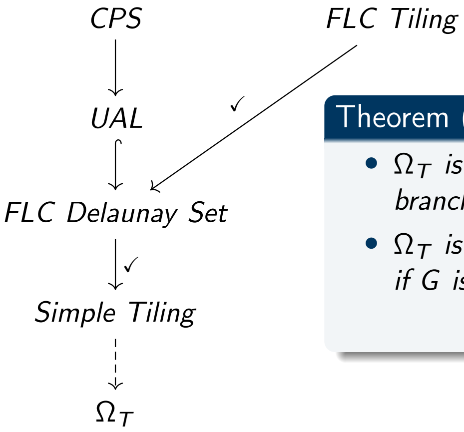

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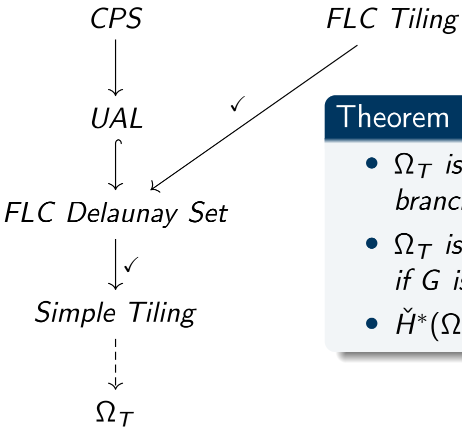

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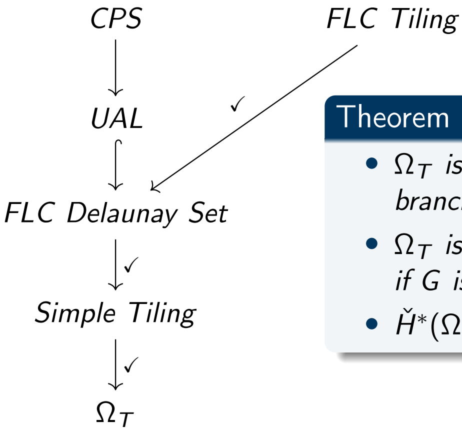

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Lemma

If T is a tiling of a rational G, there is a tiling $T_{\mathbb{Q}}$ whose adjacent vertices have rational displacement, such that $\Omega_T \approx \Omega_{T_{\mathbb{Q}}}$ are homeomorphic.

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Let T_0 be a simple tiling on a connected, simply connected, rational nilpotent Lie group G. Then Ω_{T_0} is a fiber bundle over a nilmanifold.

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- Study topology of specific tilings using $\check{H}^*(\Omega_T; \mathbb{R}) \approx H^*_{PE}(T)$.

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Thank you!

Figure: *Fishes and Scales* (M.C. Escher, 1959)

Questions?