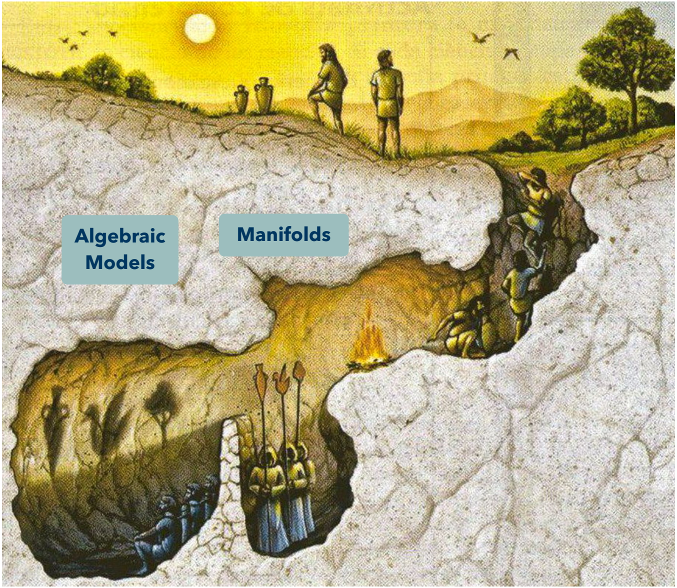
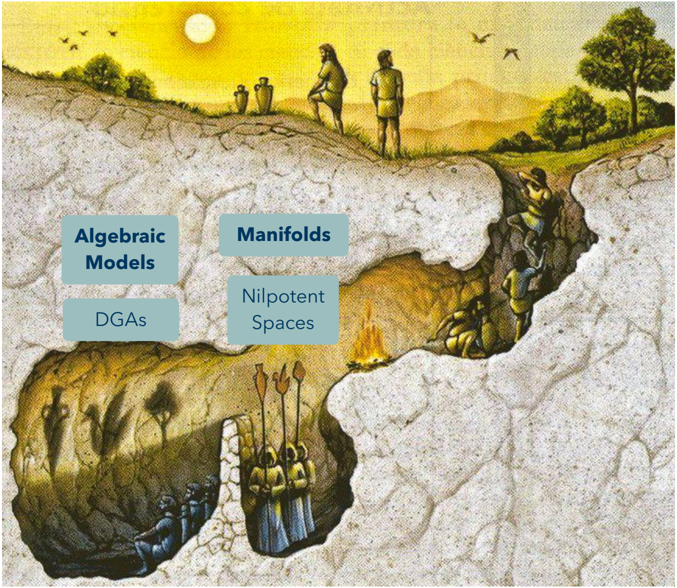
Volumes of Nullhomotopies in Nilpotent Spaces

Kyle Hansen

University of California, Santa Barbara

May 7, 2025


Algebraic Models
Manifolds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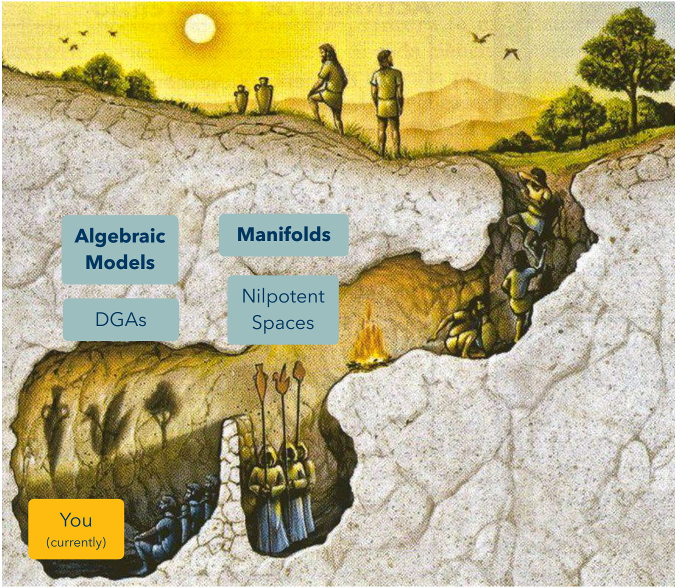
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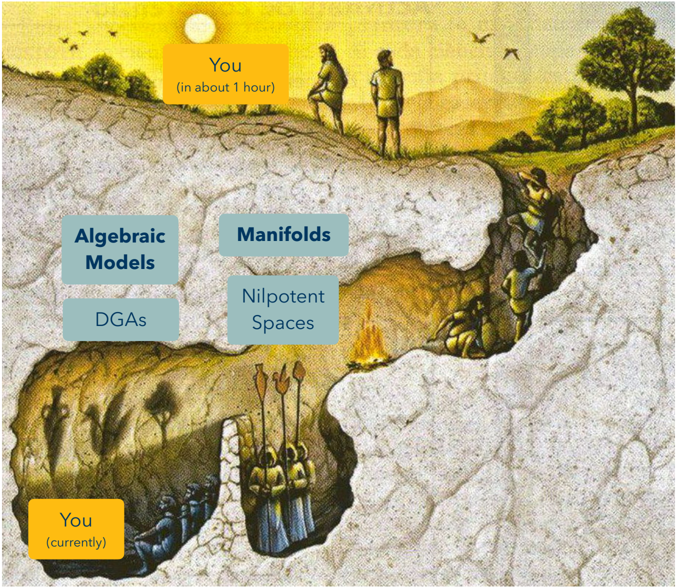
(currently)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- Chambers, Guth, Manin, Weinberger, et al. (2010s+)
  - Quantitative Homotopy Theory
  - Shadowing Principle for simply connected spaces (Manin, 2019)

Definition

A group G is c-**step nilpotent** if

$$G =: G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_c = \{1\}$$

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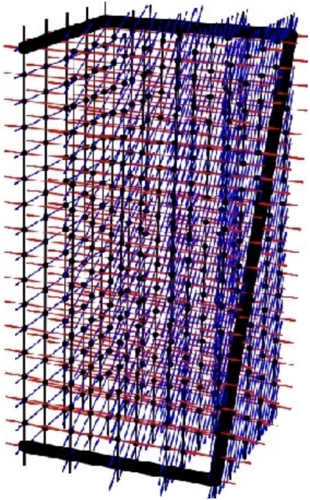
Example

$$\mathbb{H}^3_{\mathbb{Z}} = \langle X, Y, Z \mid [X, Y] = Z \rangle$$

$$\mathbb{H}^3 = \mathbb{H}^3_{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$$



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If w is a word reducing to $e \in G = \langle S \mid R \rangle$

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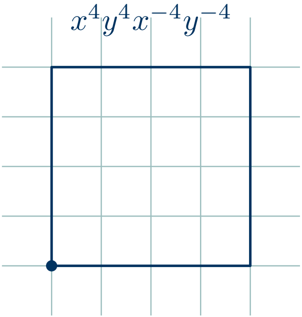

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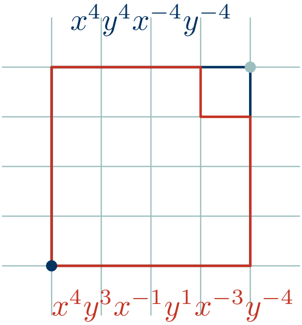

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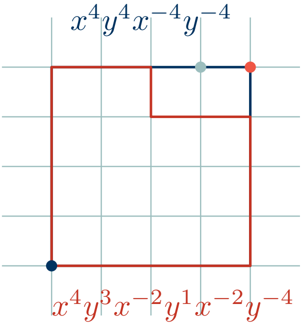

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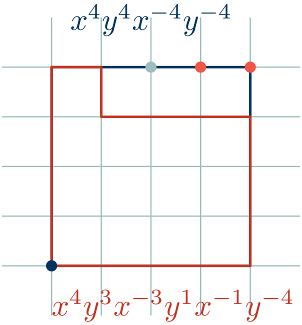

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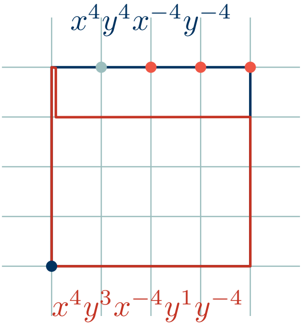

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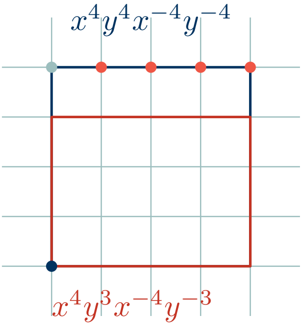

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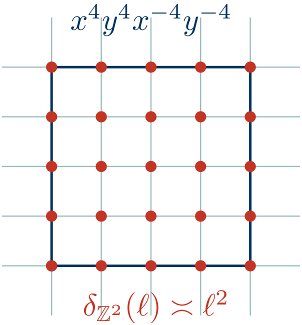

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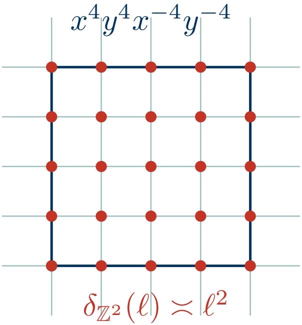

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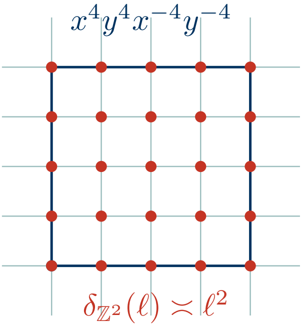

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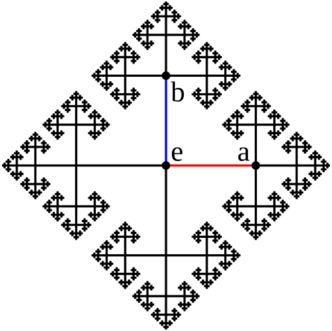
Corollary

Mal'cev $\implies \delta_{\mathbb{G}}(\ell)$ *is well-defined for* $\mathbb{G}$ *a rational nilpotent Lie group.*

Example

$$\delta_{F_r}(\ell) = \Theta(\ell)$$



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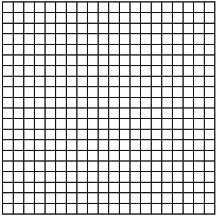

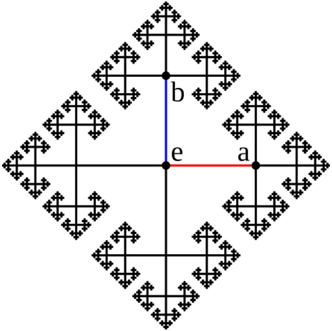

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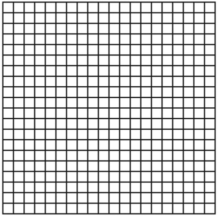

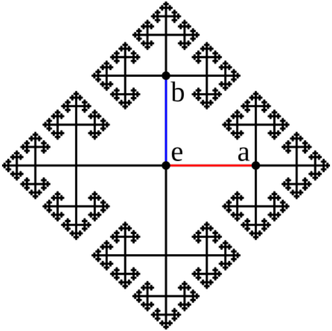

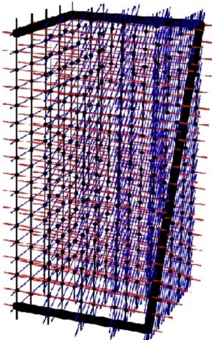

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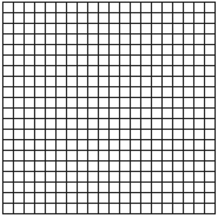

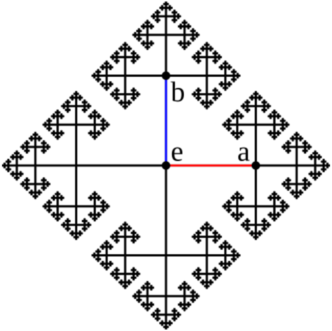

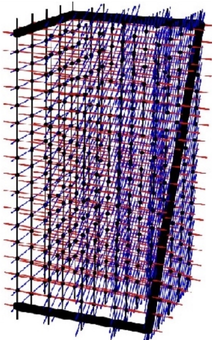

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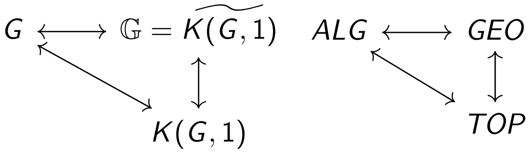

Definition

$$\text{length}(H) = \max\left\{\text{Lip}(H|_{\{x\}\times I}) \;\middle|\; x \in S^n\right\}$$

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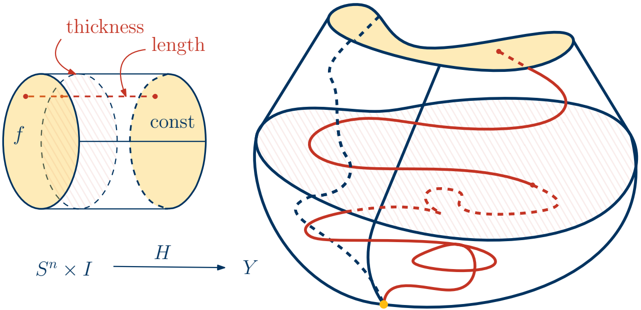

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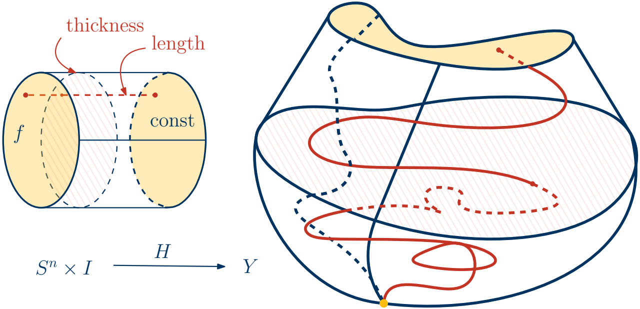

Observation/Definition

$$\text{vol}(H) \lesssim \text{thickness}(H)^n \cdot \text{length}(H).$$

Definition (Higher Order Dehn Function)

Given $f \in C_{Lip}(S^n, Y)$ with $[f] = e \in \pi_n(Y)$,

$$FV(f) = \inf \{\mathrm{vol}(F) \mid F : \mathrm{S}^n \times I \to Y \text{ is a nullhomotopy of } f\}$$

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Question

Can we bound $\delta^n_Y(L)$ using the topology of Y?

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Idea

Study $\mathsf{DGA}_{\mathbb{Q}} \rightsquigarrow$ algorithmic processes $\rightsquigarrow$ complexity bounds.

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- $\mathcal{M}_{\text{Heis}}^* = \left\langle dx^{(1)}, dy^{(1)}, \zeta^{(1)} \;\middle|\; d\zeta = dx \wedge dy \right\rangle$
  - $\text{Heis} = K(\mathbb{H}_{\mathbb{Z}}^3, 1)$

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is a **principal** $K(G,n)$ **fibration** if $\pi_1(B)$ acts trivially on fibers.

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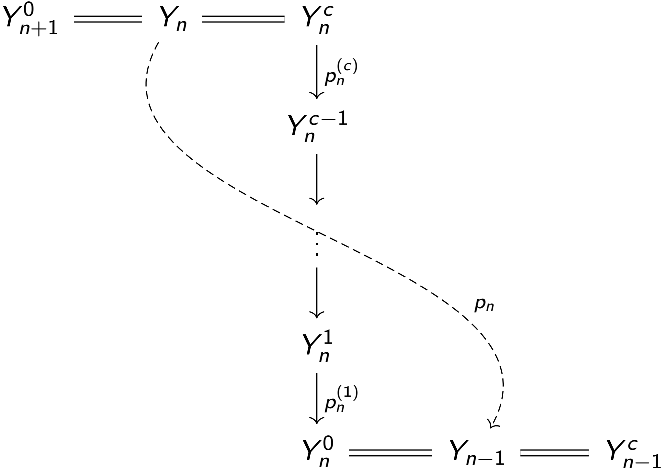
Definition

Nilpotent space: inverse limit of principal $K(G,n)$ fibrations over $\{*\}$.

Postnikov Tower of a Nilpotent Space



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$\Omega^*(B) \to \Omega^*(B) \otimes \wedge V_n$ is an **elementary extension** if $\mathrm{im}\,(D|_{V_n}) \subseteq \Omega^*(B)$:

$$\begin{array}{ccc} (\wedge V_n, 0) & \longleftarrow & (\Omega^*(B) \otimes \wedge V_n, D) \\ & & \uparrow \\ & & (\Omega^*(B), d) \end{array}$$

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Minimal Sullivan DGA: dual to Postnikov tower of nilpotent space

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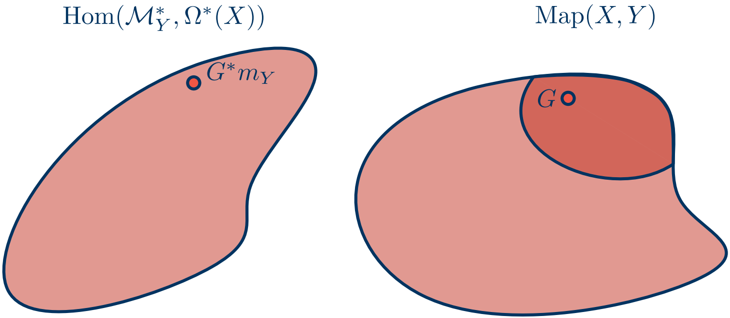
For distance $\leftrightsquigarrow$ size of homotopies: Pullbacks of genuine maps in $\mathrm{Map}(X, Y)$ have reasonably high density in $\mathrm{Hom}(\mathcal{M}_Y^*, \Omega^*(X))$.



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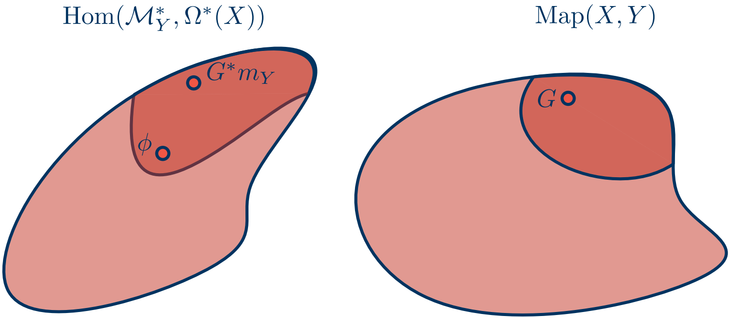

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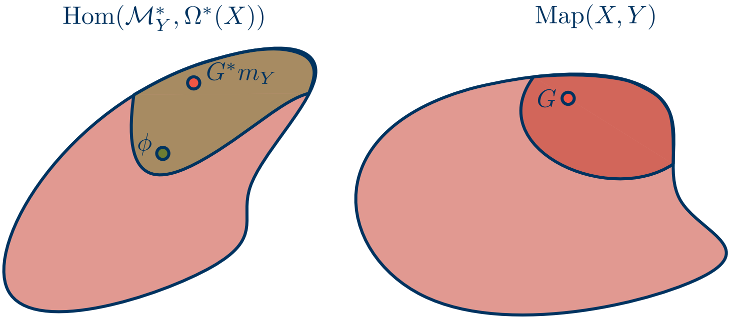

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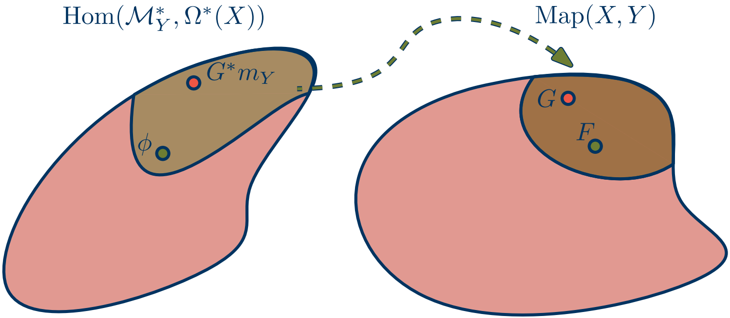

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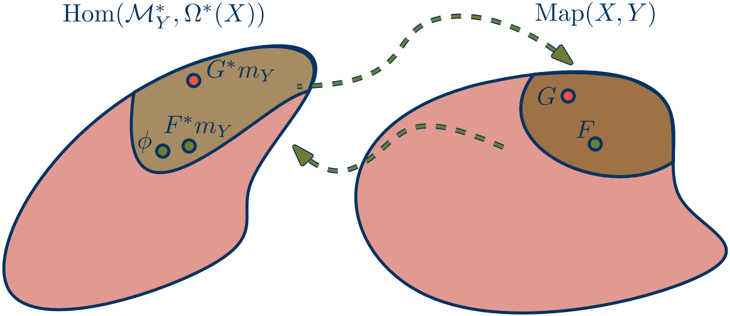

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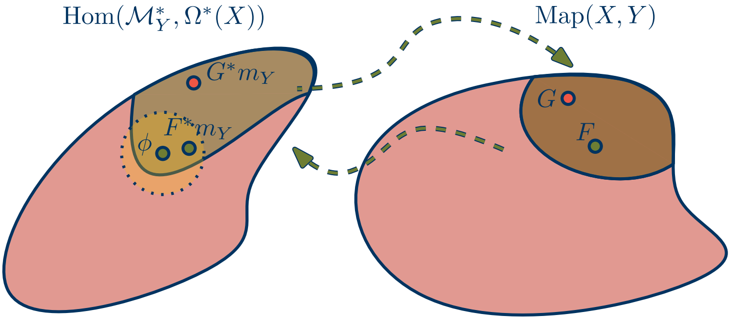

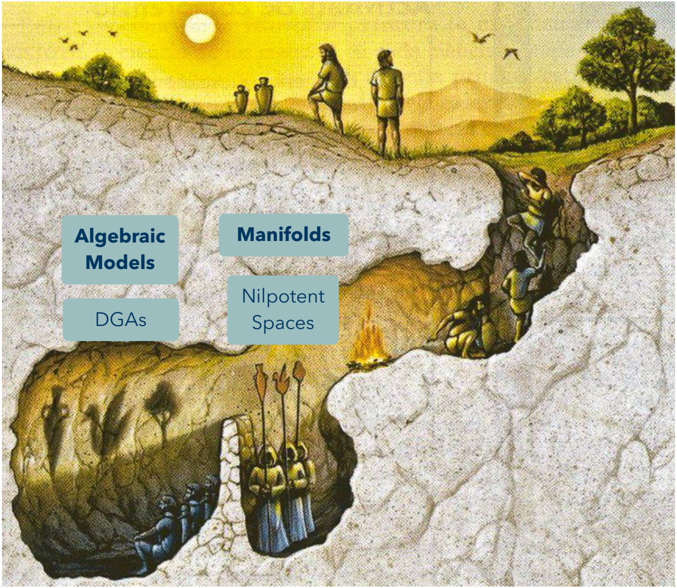
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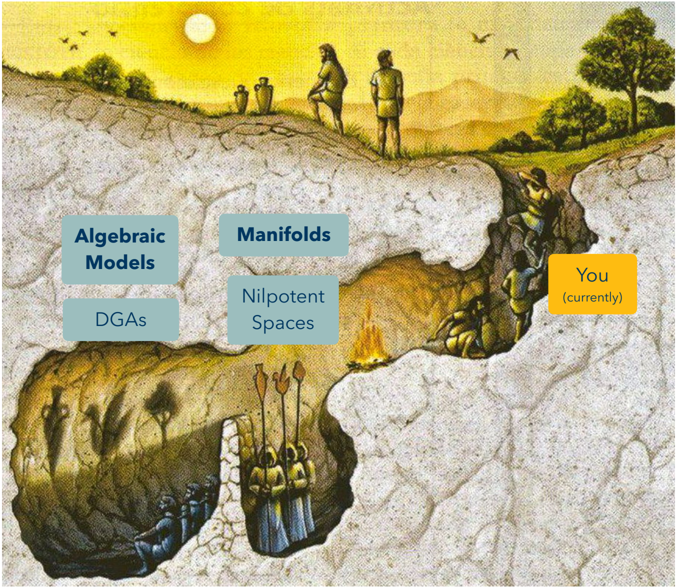
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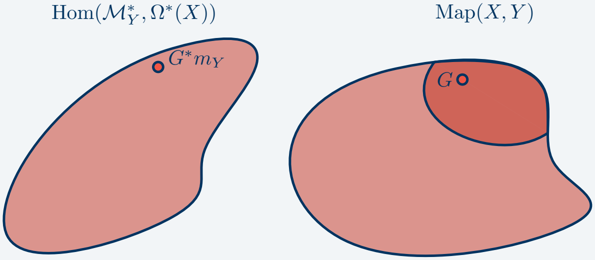
Theorem (H. 2025)

Let X be an N-dimensional finite simplicial space,



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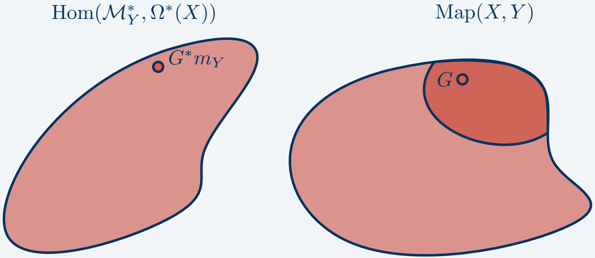

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Let X be an N-dimensional finite simplicial space, and let Y be a compact nilpotent space of finite type. Let $G : X \to Y$ be a map and let $\phi : \mathcal{M}_Y^* \to \Omega^*(X)$ be such that

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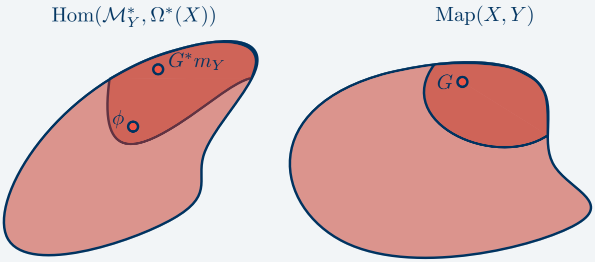

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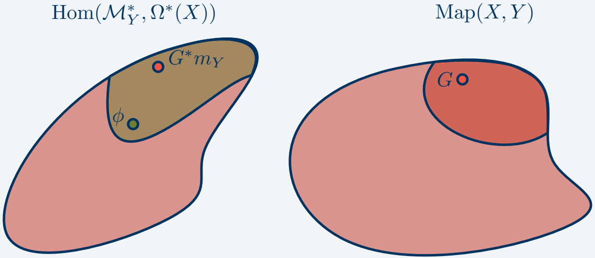

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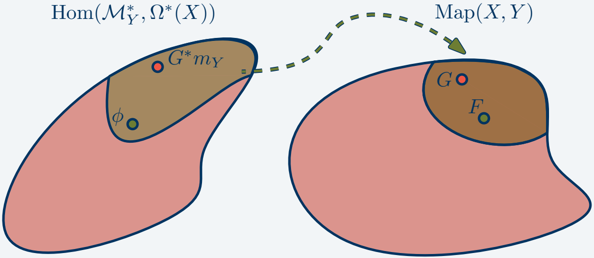

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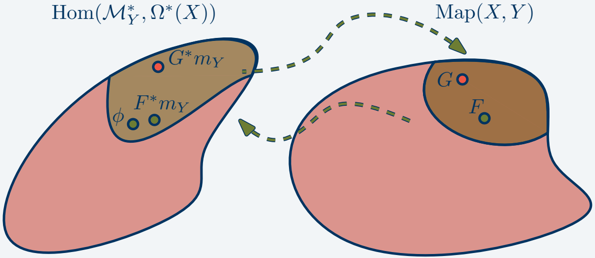

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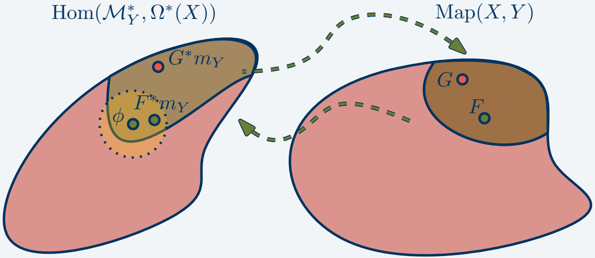

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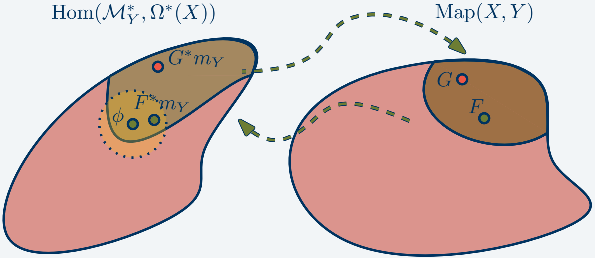

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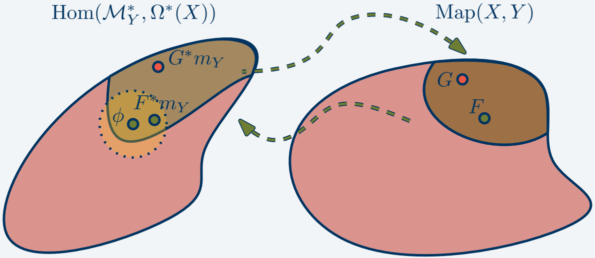

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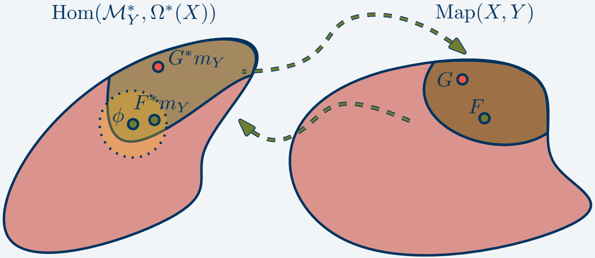

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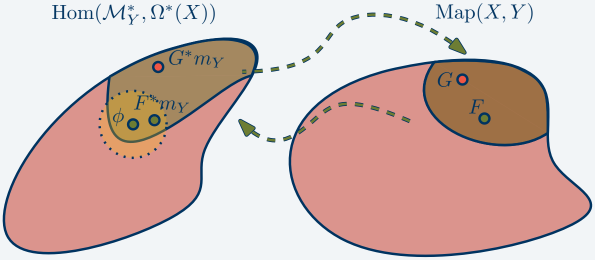

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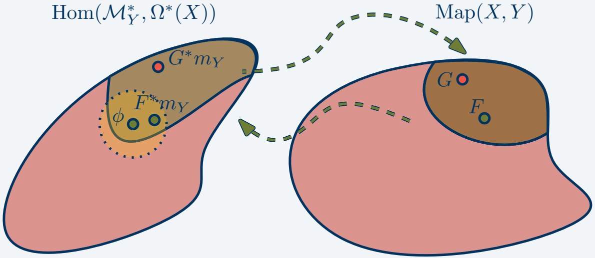

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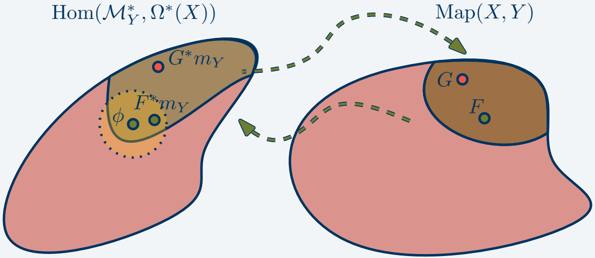

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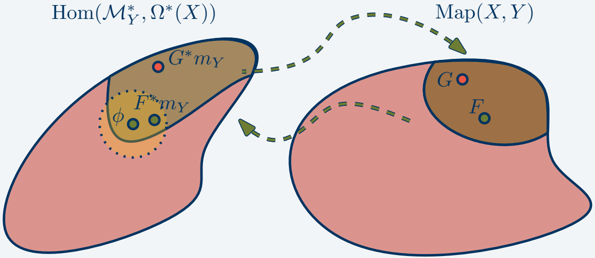

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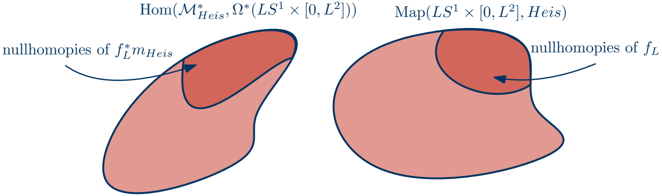
$\delta^1_{Heis}(L) = O(L^3)$ *using the Shadowing Principle*

Proof.



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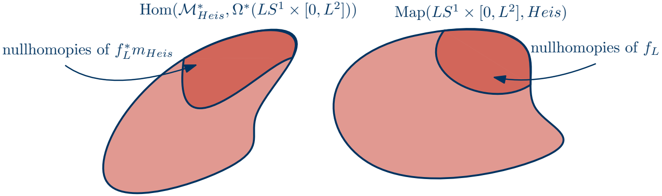

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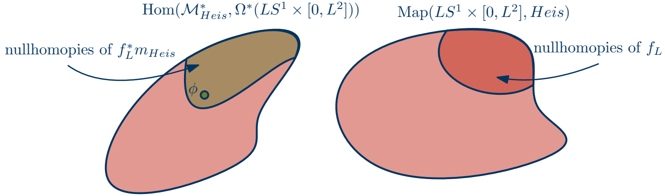

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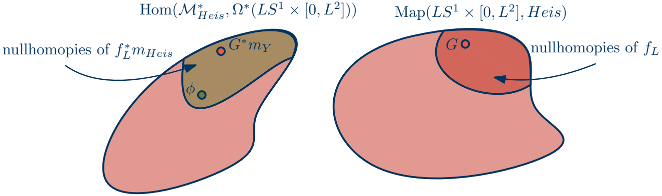

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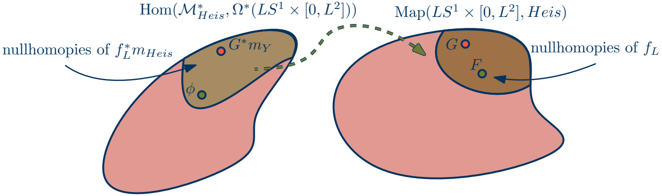

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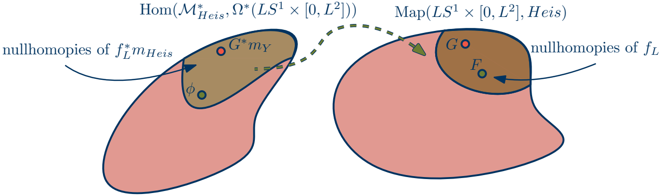
5) $\text{vol}(F \circ \text{rescale}) = O(L) \cdot O(L^2)$ *nullhomotopy of* f □



$\mathcal{M}^*_{Heis} = \langle dx, dy, \zeta \mid d\zeta = dx \wedge dy \rangle$

Lemma

If $\psi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1)$ is nullhomotopic with $\|\psi\|_{op} \le L$, then there is a nullhomotopy $\phi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1 \times [0, L])$ with $\|\phi\|_{op} \le CL$.

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$$\phi(dx) := \psi(dx)\left(1 - \frac{t}{L}\right) - c(dx)dt$$

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$$\phi(dx) := \psi(dx)\left(1 - \frac{t}{L}\right) - c(dx)dt$$

$$\phi(dy) := \psi(dy)\left(1 - \frac{t}{L}\right) - c(dy)dt$$

$\mathcal{M}^*_{Heis} = \langle dx, dy, \zeta \mid d\zeta = dx \wedge dy \rangle$

Lemma

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$$\phi(dy) \coloneqq \psi(dy)\left(1 - \frac{t}{L}\right) - c(dy)dt$$

$$\phi(\zeta) \coloneqq \psi(\zeta)\left(1 - \frac{t}{L}\right) - Lc(\zeta)dt + (\psi(dx)c(dy) - c(dx)\psi(dy))\frac{Lt - t^2}{2L^2}$$

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Lemma

If $\psi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1)$ *is nullhomotopic with* $\|\psi\|_{op} \leq L$*, then there is a nullhomotopy* $\phi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1 \times [0, L])$ *with* $\|\phi\|_{op} \leq CL$.

Proof.

$$\|\phi(dx)\| \lesssim \|\psi(dx)\| + 1 \lesssim L$$

$$\phi(dy) \coloneqq \psi(dy)\left(1 - \frac{t}{L}\right) - c(dy)dt$$

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Corollary

$\mathrm{Lip}(f) \leq L \rightsquigarrow \|f^* m_{Heis}\|_{op} \lesssim L \rightsquigarrow \phi$ *with* $\|\phi\|_{op} \lesssim L$

$\mathcal{M}^*_{Heis} = \langle dx, dy, \zeta \mid d\zeta = dx \wedge dy \rangle$

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If $\psi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1)$ is nullhomotopic with $\|\psi\|_{op} \leq L$, then there is a nullhomotopy $\phi : \mathcal{M}^*_{Heis} \to \Omega^*(S^1 \times [0, L])$ with $\|\phi\|_{op} \leq CL$.

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Corollary

$\mathrm{Lip}(f) \leq L \rightsquigarrow \|f^* m_{Heis}\|_{op} \lesssim L \rightsquigarrow \phi$ *with* $\|\phi\|_{op} \lesssim L$

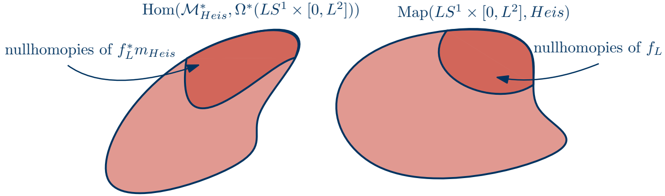
$\implies \phi_L : \mathcal{M}^*_{Heis} \to \Omega^*(LS^1 \times [0, L^2])$ *of* $f_L^* m_{Heis}$ *with* $\|\phi_L\|_{op} \leq C$.

Theorem

$\delta^1_{Heis}(L) = O(L^3)$ *using the Shadowing Principle*

Proof.

1) *Replace* $f : S^1 \xrightarrow{L-\text{Lip}} Heis$ *nullhomotopic with* $f_L : LS^1 \xrightarrow{1-\text{Lip}} Heis$
2) *Find bounded nullhomotopy* $\phi : \mathcal{M}^*_{Heis} \to \Omega^*(LS^1 \times [0, L^2])$ *of* $f_L^* m_{Heis}$
3) *Guess a nullhomotopy* $G : LS^1 \times [0, L^2] \to Heis$ *of* f_L
4) *Adjust* G *into a bounded* F *using* ϕ
5) $\text{vol}(F \circ \text{rescale}) = O(L) \cdot O(L^2)$ *nullhomotopy of* f □



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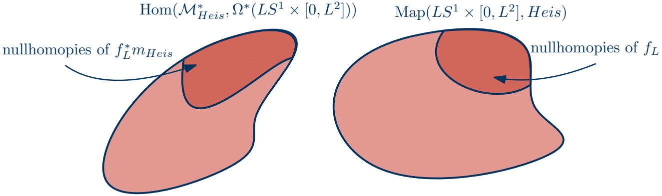

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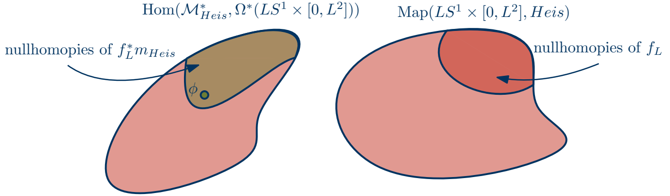

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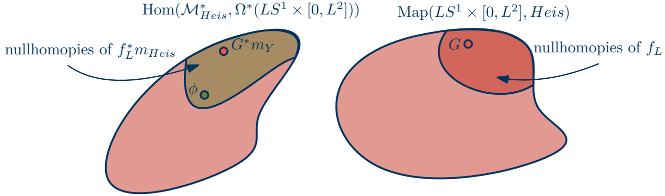

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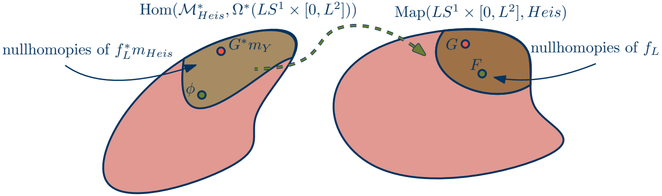

$X = LS^1 \times [0, L^2]$

- Guess $G : X \to Heis$

$$X \xrightarrow{G} Heis$$

$X = LS^1 \times [0, L^2]$

- Guess $G : X \to Heis$
- Adjust inductively on principal refinement of Postnikov tower of *Heis*

$$\begin{array}{ccc} X & \xrightarrow{G} & Heis \\ & & \downarrow \\ & & \mathbb{T}^2 \\ & & \downarrow \\ & & \{*\} \end{array}$$

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- Guess $G : X \to Heis$
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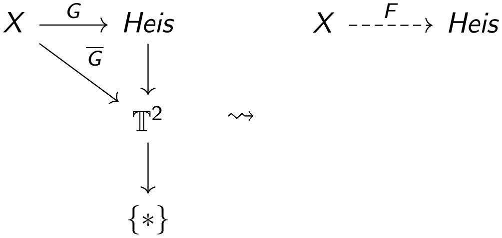
  - E.g., if $\overline{G} : X \to \mathbb{T}^2$ is bounded, lift to bounded $F : X \to Heis$



$X = LS^1 \times [0, L^2]$

- Guess $G : X \to \mathit{Heis}$
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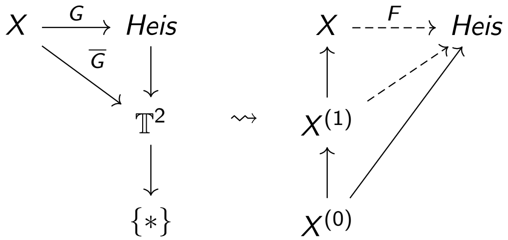
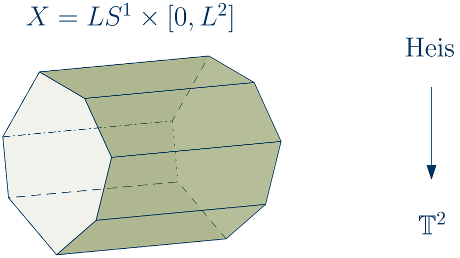
  - E.g., if $\overline{G} : X \to \mathbb{T}^2$ is bounded, lift to bounded $F : X \to \mathit{Heis}$
- During each stage, adjust inductively on skeleta of X



Data

- $X = LS^1 \times [0, L^2]$



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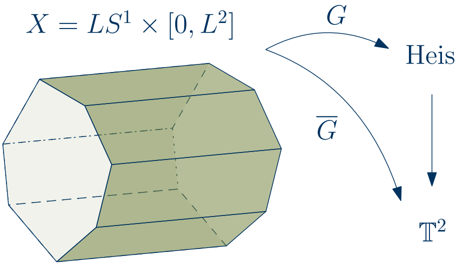
- $G : X \to \mathrm{Heis}$
- $\overline{G} : X \to \mathbb{T}^2$



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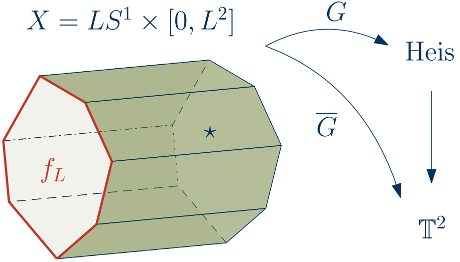
- $G|_{LS^1 \times \{0, L^2\}} = f_L \sqcup \star$



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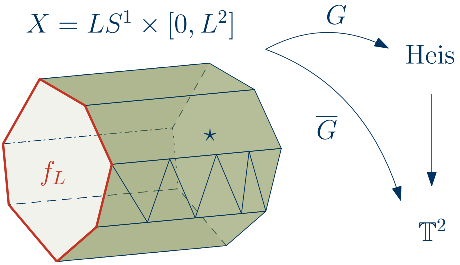

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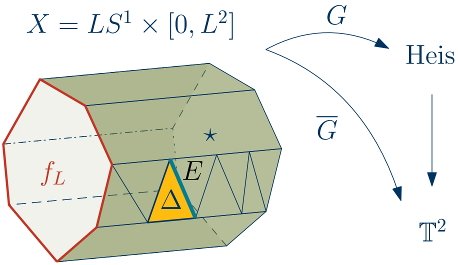

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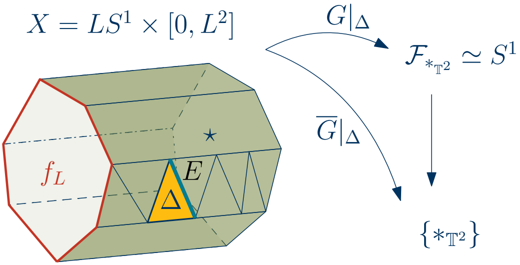
Simplifying Assumptions

- $\overline{G}(\Delta) = \{*_{\mathbb{T}^2}\}$

$\Longrightarrow$ $G(\Delta) \subseteq \mathcal{F}_* \simeq S^1$



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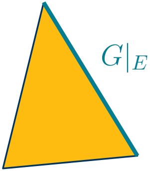
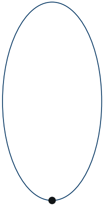


$S^1 \simeq \mathcal{F}_* \subseteq \mathrm{Heis}$



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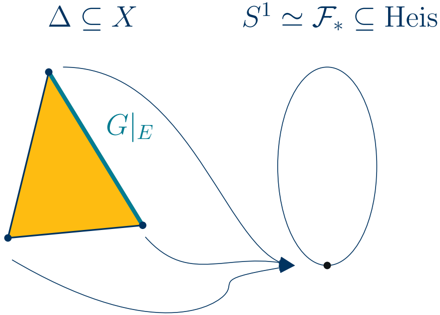

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$\Longrightarrow$ $G(\Delta) \subseteq \mathcal{F}_* \simeq S^1$

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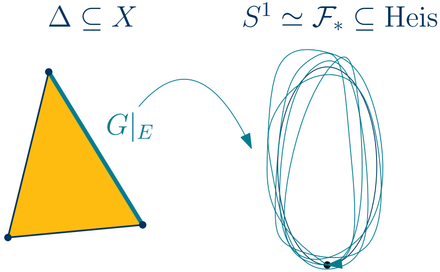
$\Longrightarrow$ $[G|_E]_z \in \pi_1(S^1) \cong \mathbb{Z}$



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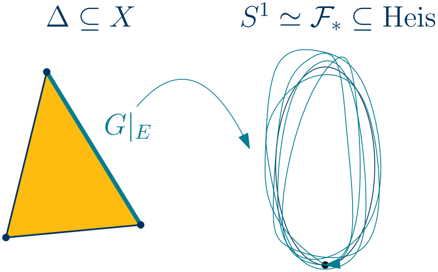
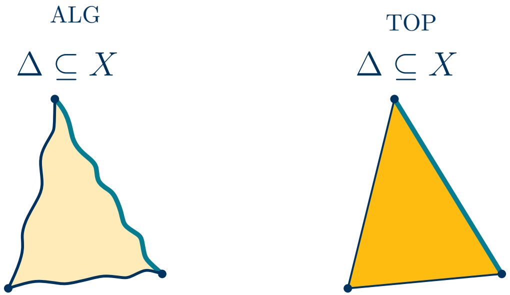


Goal

Make $\|[F|_E]_z\| \lesssim \|\phi\|$ and take $F|_E$ a minimal representative of $[F|_E]_z$.

Bounding $\|[F|_E]_z\|$

Attempt #1: Use ϕ Exactly



Bounding $\|[F|_E]_z\|$

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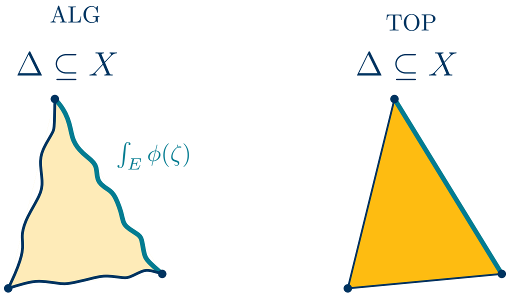
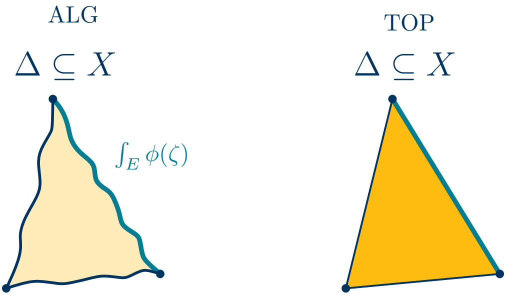


Bounding $\|[F|_E]_z\|$

Attempt #1: Use ϕ Exactly

- Given $\int_\square \phi(\zeta) : C_1(X) \to \mathbb{R}$
- Find $[F|_\square]_z : C_1(X) \to \mathbb{Z}$

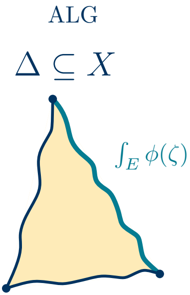


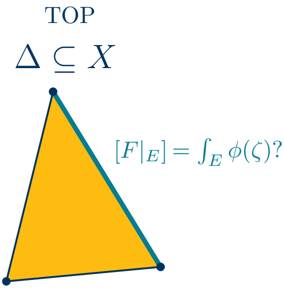
Bounding $\|[F|_E]_z\|$

Attempt #1: Use ϕ Exactly

- Given $\int_{\square} \phi(\zeta) : C_1(X) \to \mathbb{R}$
- Find $[F|_{\square}]_z : C_1(X) \to \mathbb{Z}$

Define $F|_E$ so that $[F|_E]_z = \int_E \phi(\zeta)$





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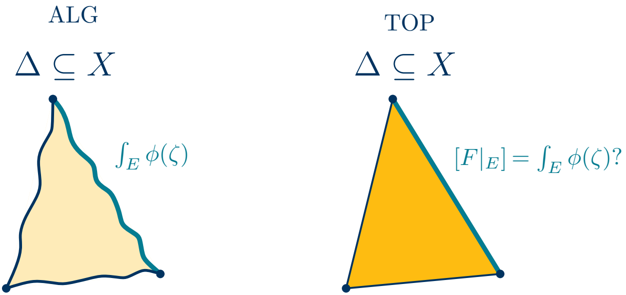
Issue



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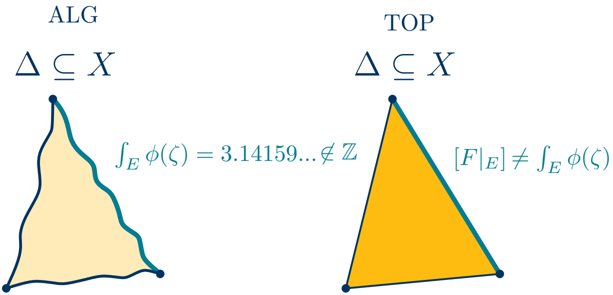
Issue

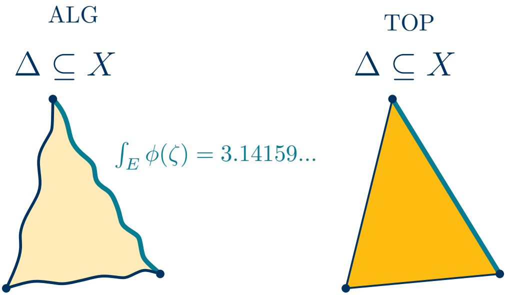
$\mathbb{Z} \neq \mathbb{R}$.



Bounding $\|[F|_E]_z\|$

Attempt #2: Round ϕ

- Given $\int_\square \phi(\zeta) : C_1(X) \to \mathbb{R}$
- Find $[F|_\square]_z : C_1(X) \to \mathbb{Z}$



Bounding $\|[F|_E]_z\|$

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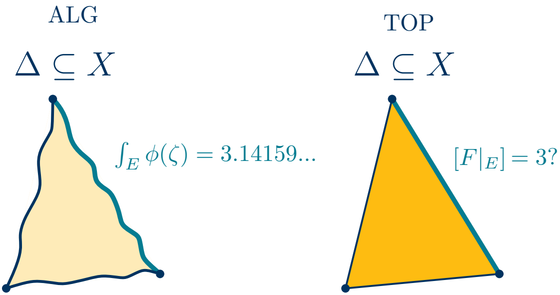
Define $F|_E$ so that $[F|_E]_z \approx \int_E \phi(\zeta)$



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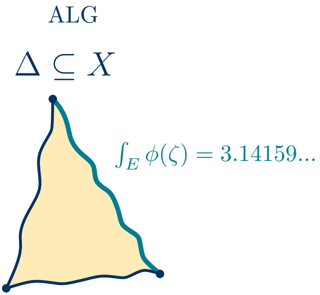
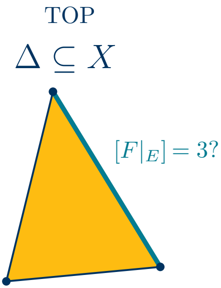
Issue





Bounding $\|[F|_E]_z\|$

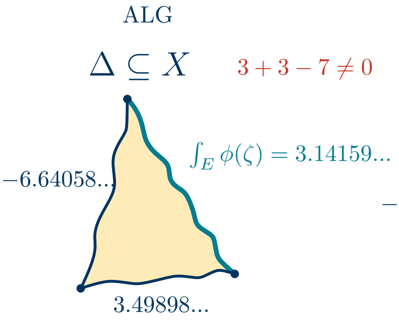
Attempt #2: Round ϕ

- Given $\int_\square \phi(\zeta) : C_1(X) \to \mathbb{R}$
- Find $[F|_\square]_z : C_1(X) \to \mathbb{Z}$

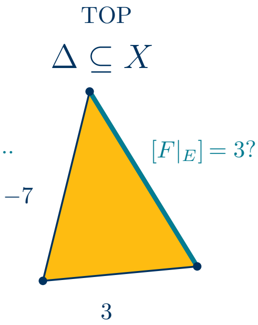
Define $F|_E$ so that $[F|_E]_z \approx \int_E \phi(\zeta)$

Issue

- Extending on higher cells



$3 + 3 - 7 \neq 0$



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Attempt #3: Round using $G^* m_Y \simeq \phi$

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- $\int_\square \phi(\zeta) : C_1(X) \to \mathbb{R}$
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- $\int_\square \phi(\zeta) : C_1(X) \to \mathbb{R}$
- $[G|_\square]_z : C_1(X) \to \mathbb{Z} \quad \Longrightarrow \int_\square G^* m_Y(\zeta) : C_1(X) \to \mathbb{Z} \subseteq \mathbb{R}$
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- Round $\Psi = \widetilde{\Psi} + \rho \implies G^*m_Y - \widetilde{\phi} = d\widetilde{\Psi} \rightsquigarrow \mathbb{Z}$

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- $[G|_\square]_z - [F|_\square]_z = d[\widetilde{\Psi}|_\square]_z \implies F \simeq G$ extends on 2-cells

Theorem

$\delta^1_{Heis}(L) = O(L^3)$ *using the Shadowing Principle*

Proof.

✓1) *Replace* $f : S^1 \xrightarrow{L-\mathrm{Lip}} Heis$ *nullhomotopic with* $f_L : LS^1 \xrightarrow{1-\mathrm{Lip}} Heis$

✓2) *Find bounded nullhomotopy* $\phi : \mathcal{M}^*_{Heis} \to \Omega^*(LS^1 \times [0, L^2])$ *of* $f_L^* m_{Heis}$

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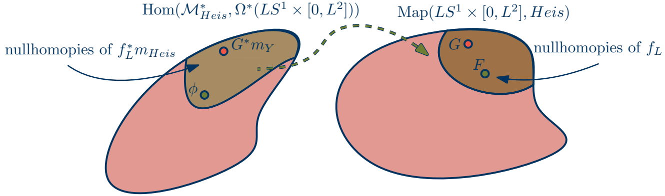
4) *Adjust* G *into a bounded* F *using* ϕ

5) $\mathrm{vol}(F \circ \mathrm{rescale}) = O(L) \cdot O(L^2)$ *nullhomotopy of* f □



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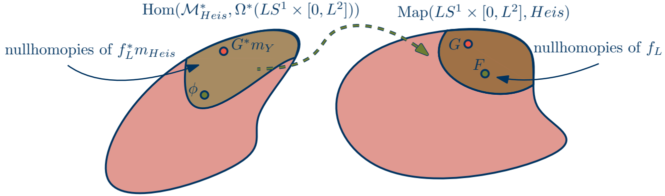

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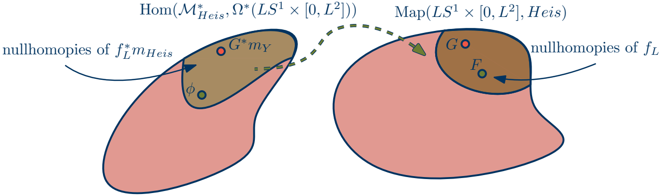

Y is...	*Simply Connected*	*c-Step Nilpotent*
Lower Bounds on $\delta^n_Y(L)$	$\Omega(L^{2(n-1)})$ [CMW18]	
Upper Bounds on $\delta^n_Y(L)$	$O(L^{2n})$ [Man19]	$O(L^{(4c-1)n})$ [H. 2025]

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Theorem (Chambers, Manin, Weinberger 2018)

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$\alpha^{(2)}$

$\Downarrow [\,\cdot\,, \beta]$

$\alpha^{(3)}$

$\Downarrow [\,\cdot\,, \beta]$

$\alpha^{(4)}$

$\vdots$

$\alpha^{(n)}$

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Note

$$\alpha^{(n)} = [\alpha^{(n-1)}, \beta]$$

$$\leftrightsquigarrow d\alpha^{(n)} = \alpha^{(n-1)} \wedge \beta + \Omega$$

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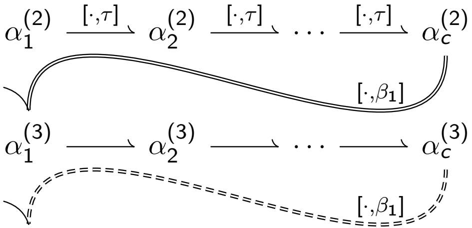

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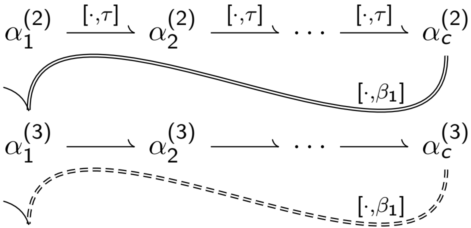

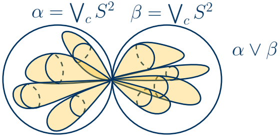

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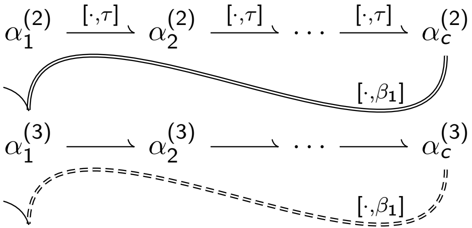

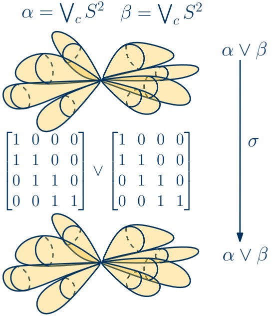

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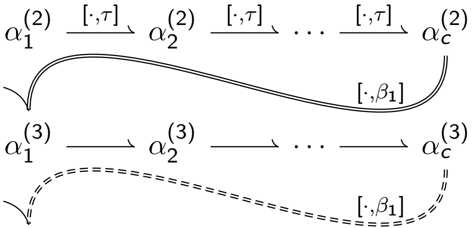

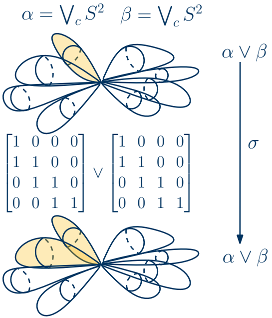

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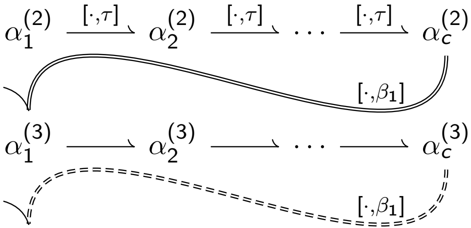

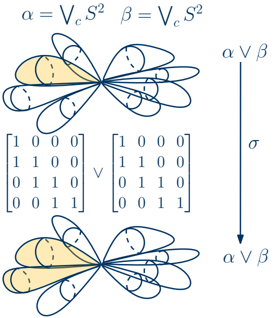

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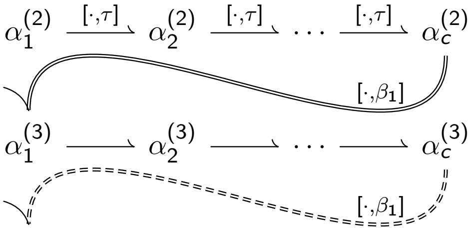

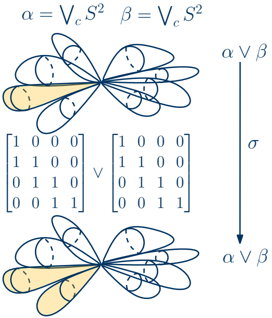

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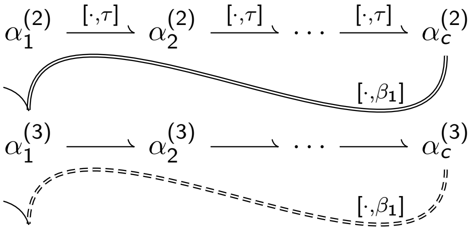

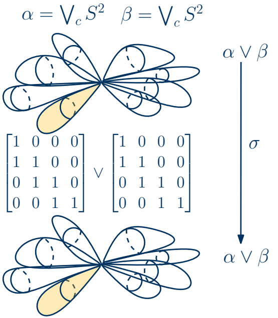

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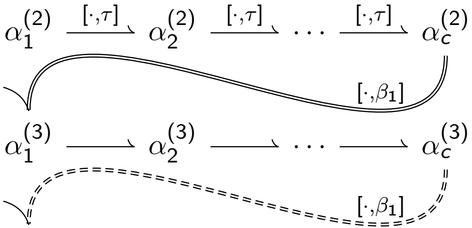

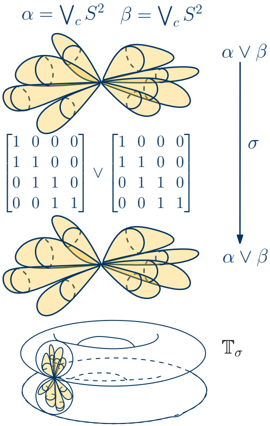

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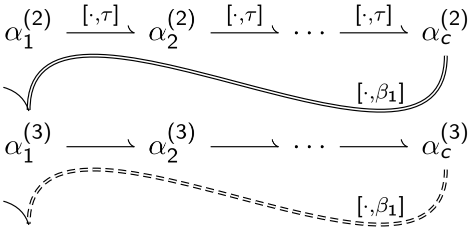

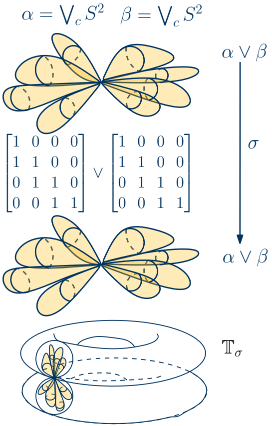

Note

$$\alpha_q^{(n)} = [\alpha_{q-1}^{(n)}, \tau]$$

$$\leftrightsquigarrow d\alpha_q^{(n)} = \alpha_{q-1}^{(n)} \wedge \tau + \Omega$$

Theorem (H. 2025)

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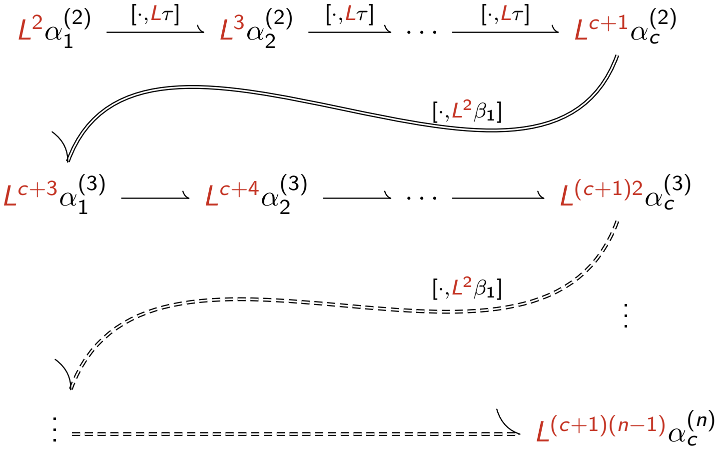
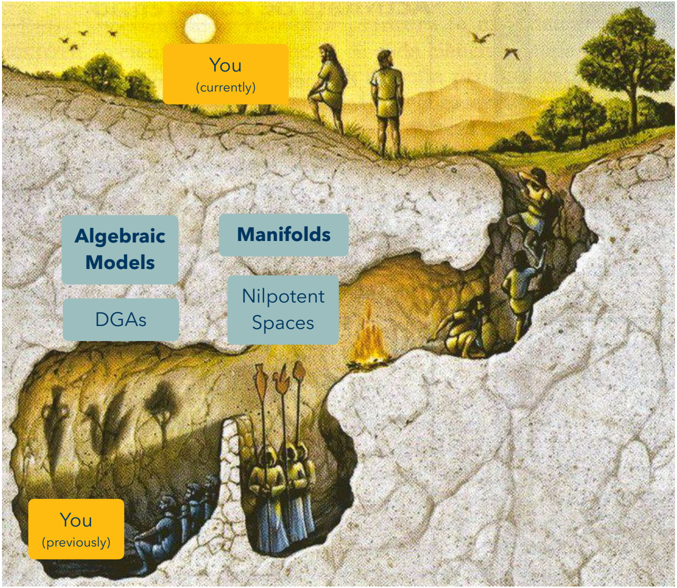


Y is...	*Simply Connected*	*Simple*	*c-Step Nilpotent*	*Coformal + c-Step Nilpotent*
Lower Bounds on $\delta^n_Y(L)$	$\Omega(L^{2(n-1)})$ [CMW18]	?	?	$\Omega(L^{(c+1)(n-1)})$ [H. 2025]
Upper Bounds on $\delta^n_Y(L)$	$O(L^{2n})$ [Man19]	$O(L^{2n+1})$ [H. 2025]	$O(L^{(4c-1)n})$ [H. 2025]	$O(L^{(c+1)n})$ [H. 2025]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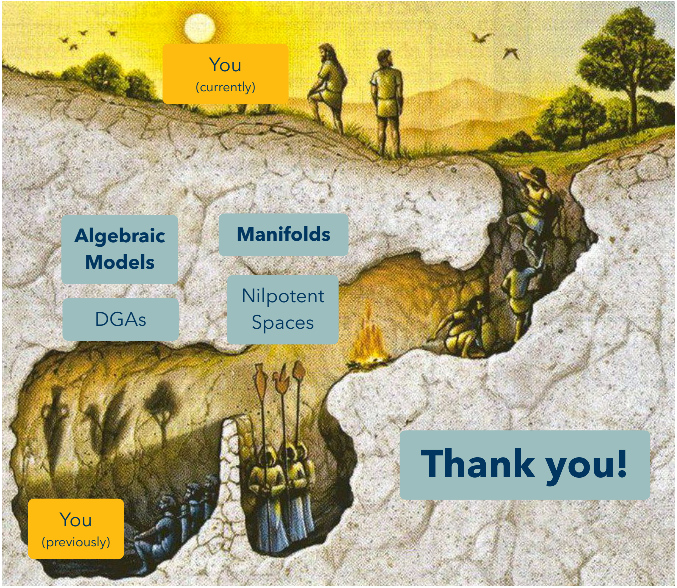
Algebraic Models
Manifolds
DGAs
Nilpotent Spaces
You
(previously)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Thank you!
You
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