Mathematical Quasicrystals

The Taming of the Hullabaloo

Kyle Hansen

University of California, Santa Barbara

February 5, 2025

The earth was without form and void, and darkness was over the face of the deep. And the Spirit of God was hovering over the face of the waters...

And God saw everything that he had made, and behold, it was very good.

– Genesis 1:2,31

As for you, you meant evil against me, but God meant it for good, to bring it about that many people should be kept alive, as they are today.

– Genesis 50:20

Then the Lord answered Job... "Who shut in the sea with doors...and prescribed limits for it and set bars and doors, and said, 'Thus far shall you come, and no farther, and here shall your proud waves be stayed'?"

– Job 38:1, 8, 10-11

For God is not a God of confusion but of peace.

– I Cor. 14:33

Crystals and Disorder




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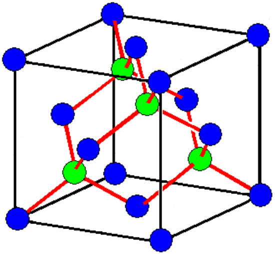

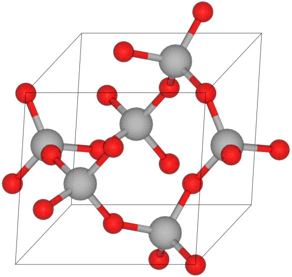

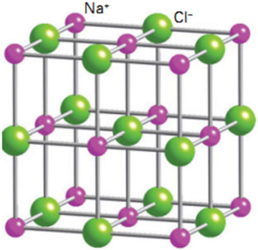

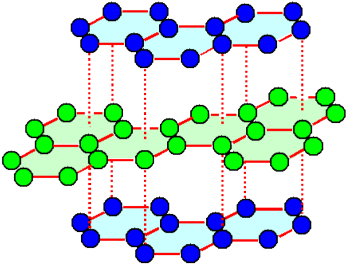

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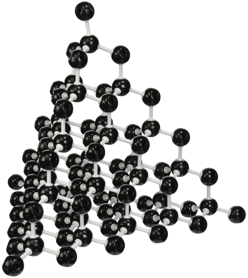

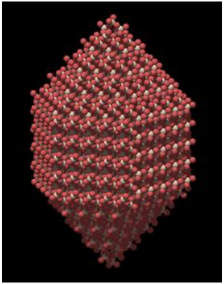

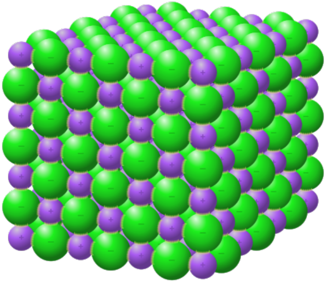

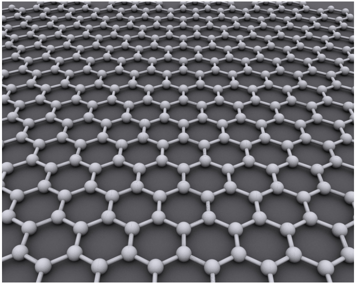

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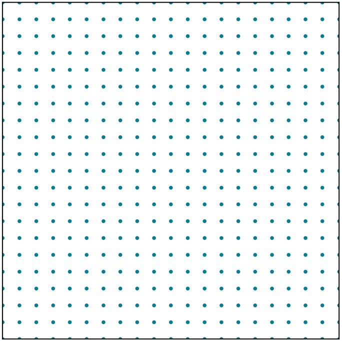

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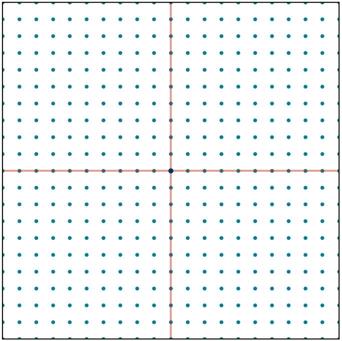

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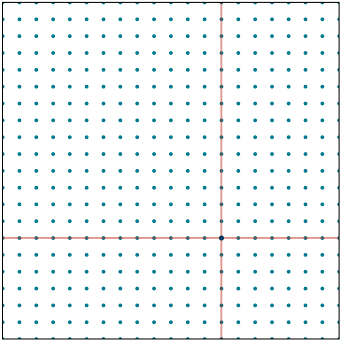

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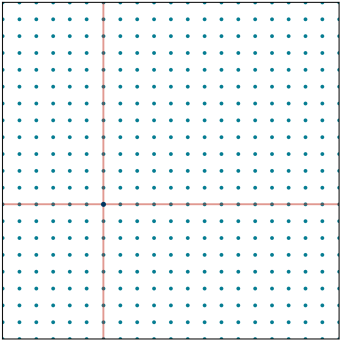

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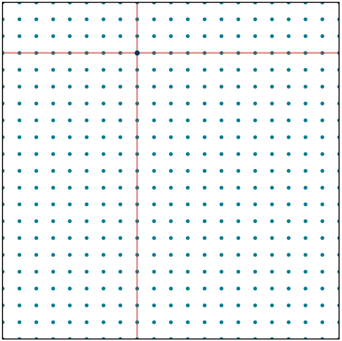

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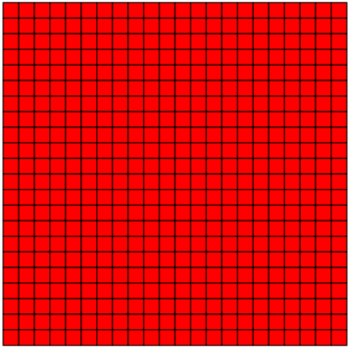

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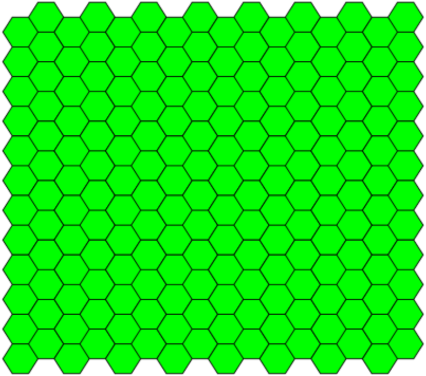

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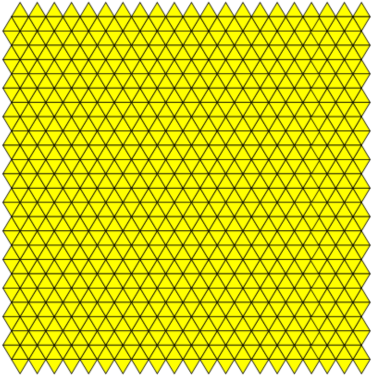

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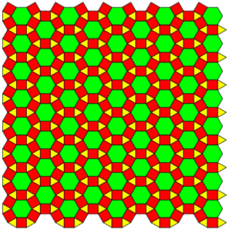
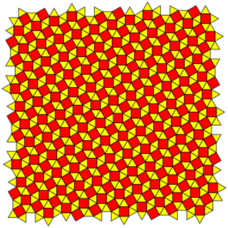
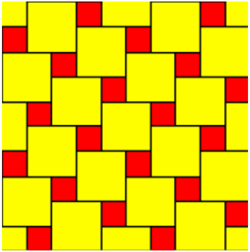
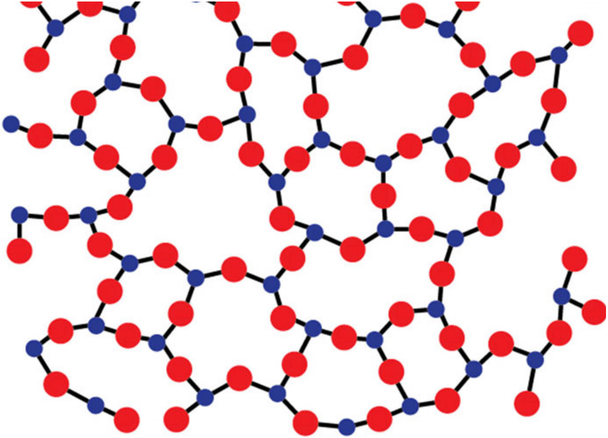
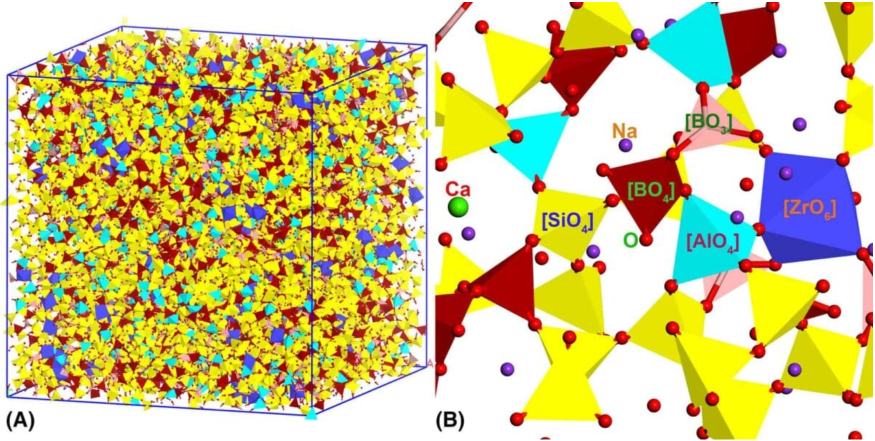


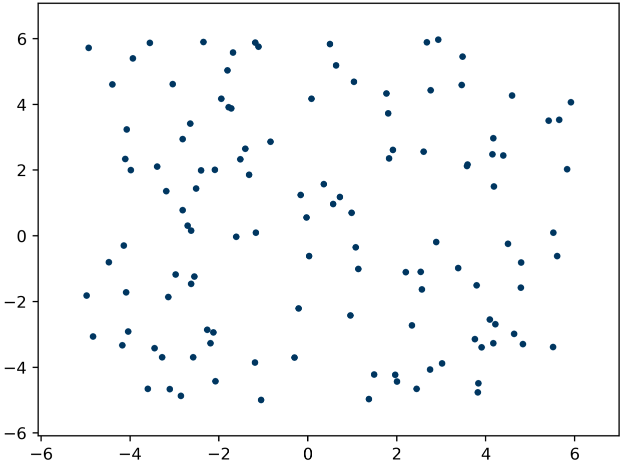








From [Du and Rimsza, 2017]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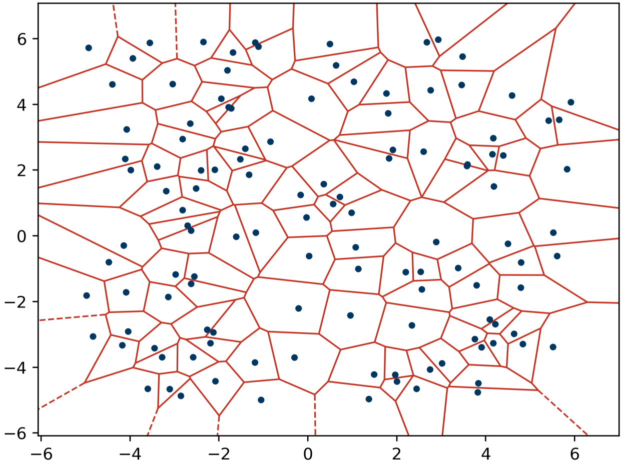
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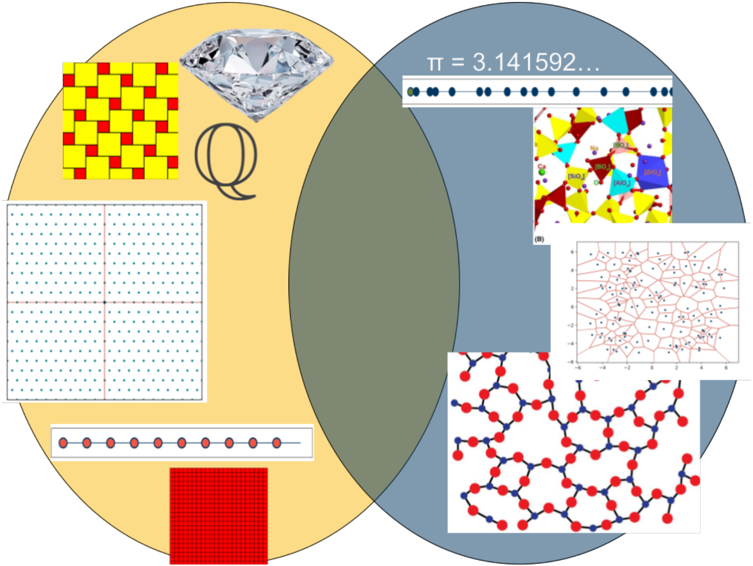
Quasicrystals and UALs



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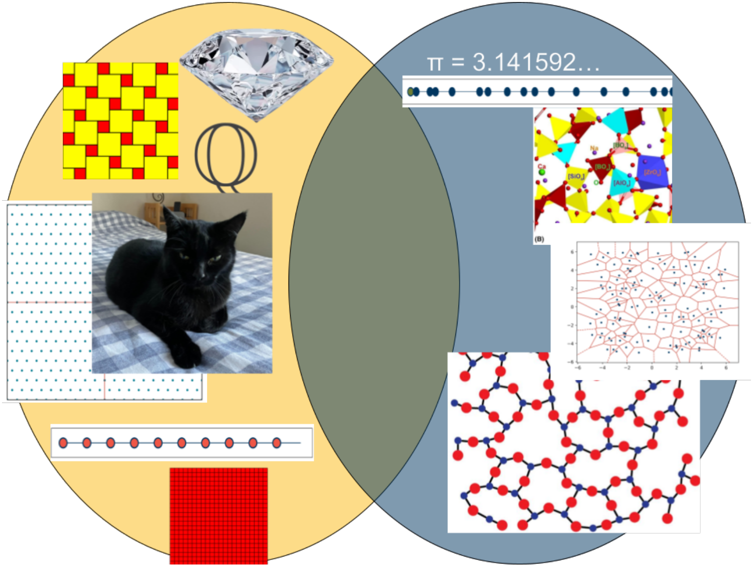

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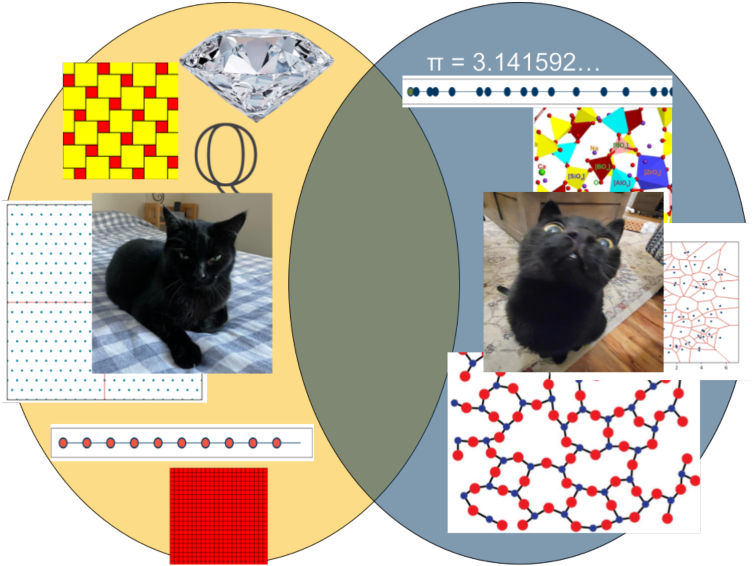

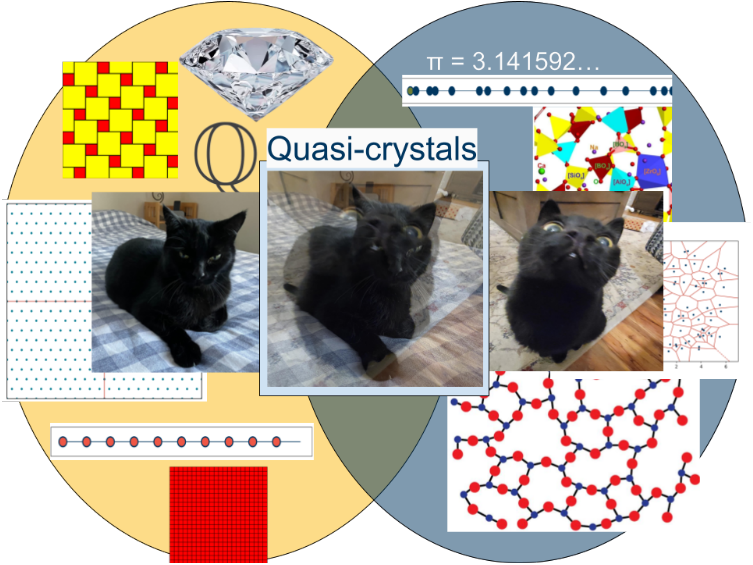
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Quasi-crystals

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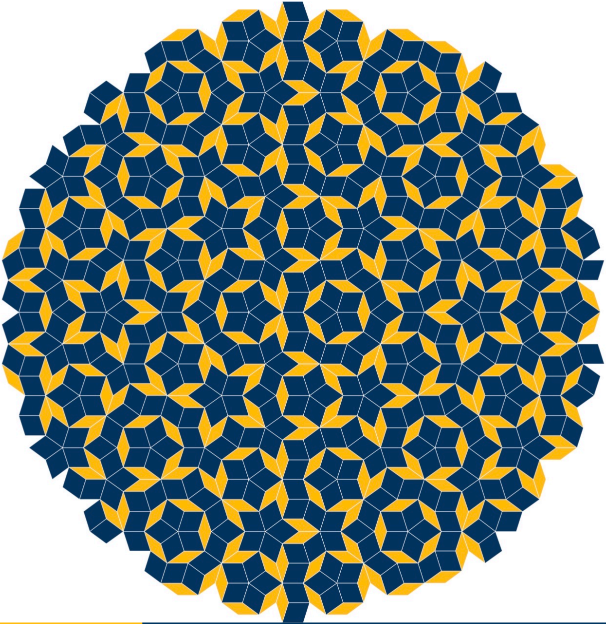

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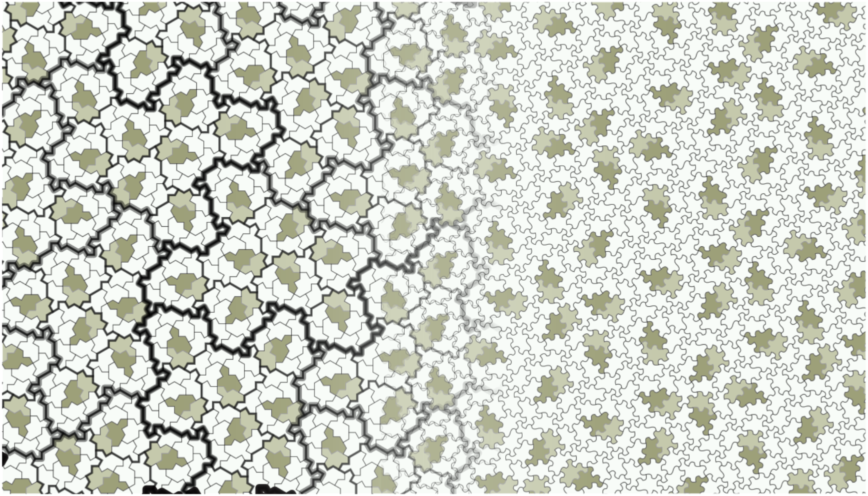
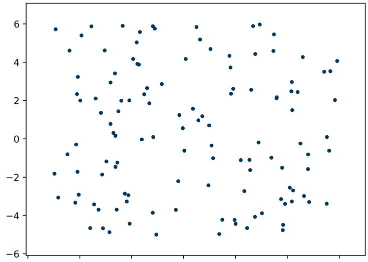
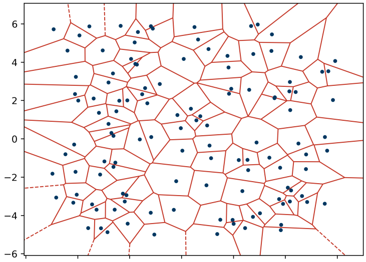


Two Main Perspectives

Point Patterns $\longleftrightarrow$ Tilings





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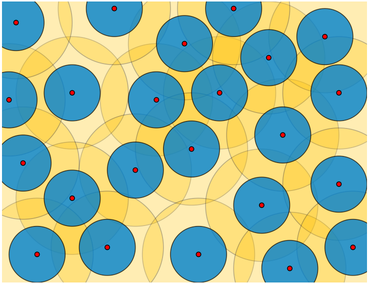
- ...**relatively dense** if
- ...**uniformly discrete** if
- ...a **Delaunay set** if



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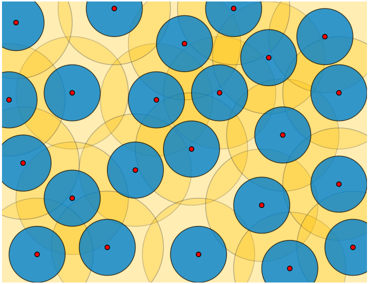

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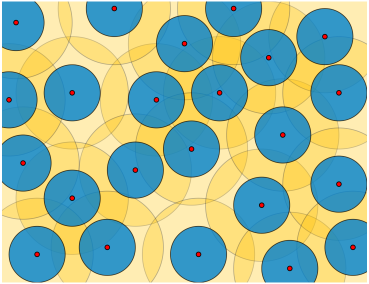

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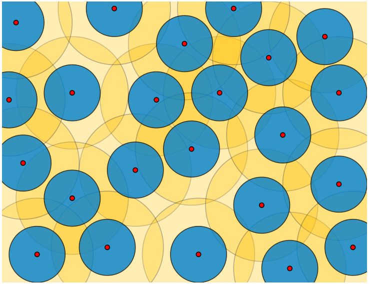

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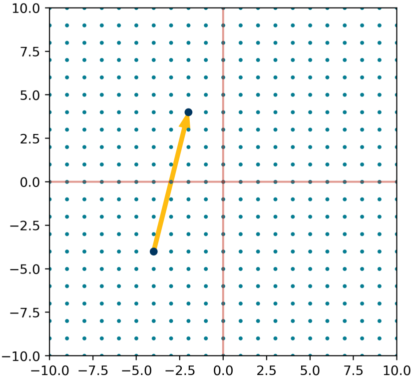

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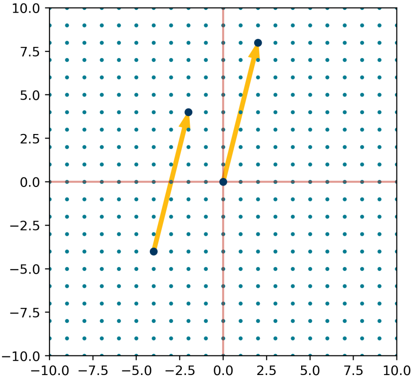

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A **(uniform) approximate lattice** is a Delaunay set $\Lambda \subseteq \mathbb{R}^n$ such that if $a, b \in \Lambda$, then $a - b \in \Lambda - F$ for some finite set $F \subseteq \mathbb{R}^n$.

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Note

Here $\Lambda - F := \{\lambda - f \mid \lambda \in \Lambda, f \in F\}$.

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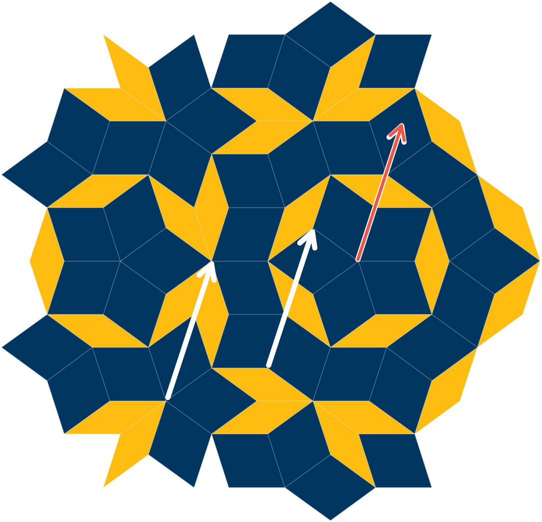

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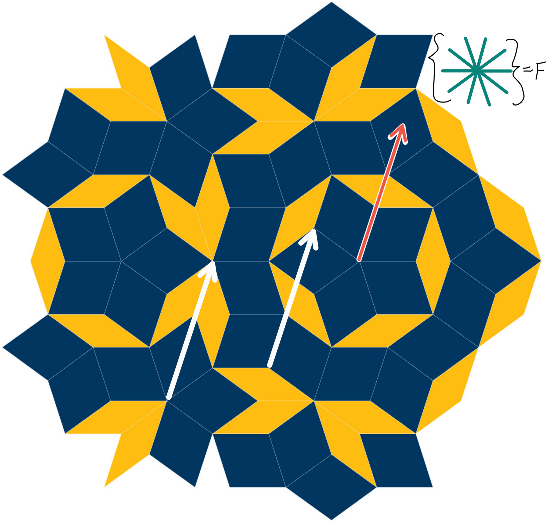

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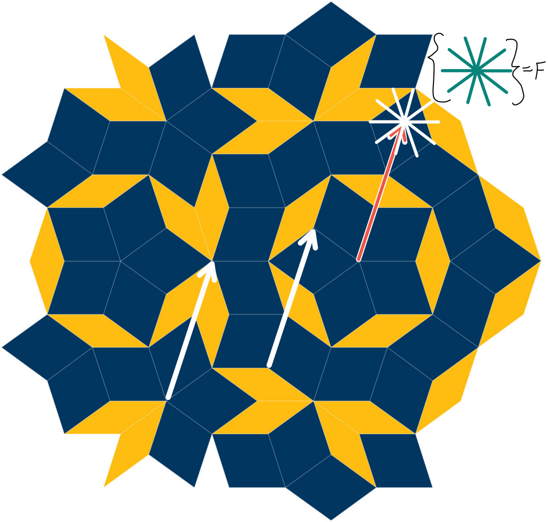

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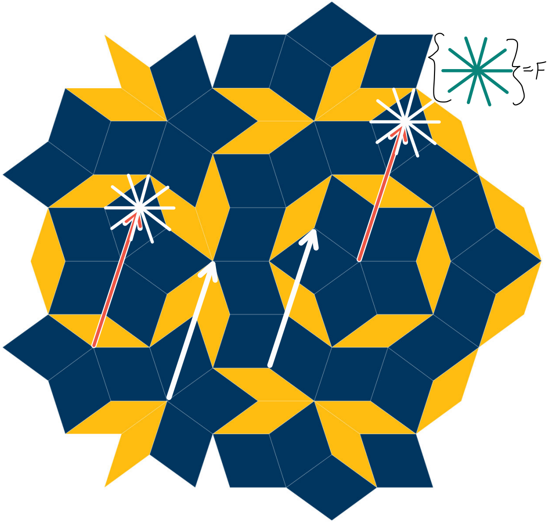

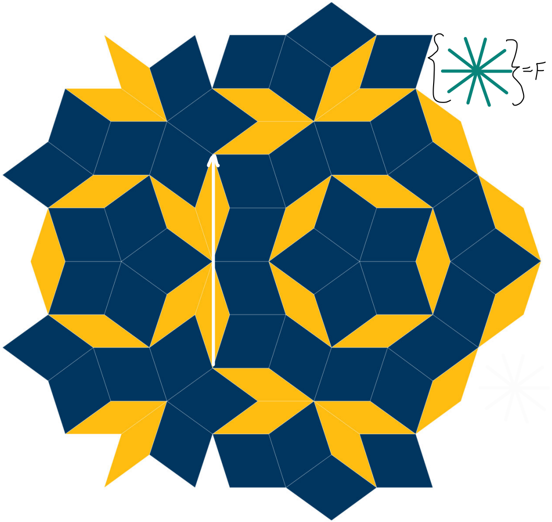
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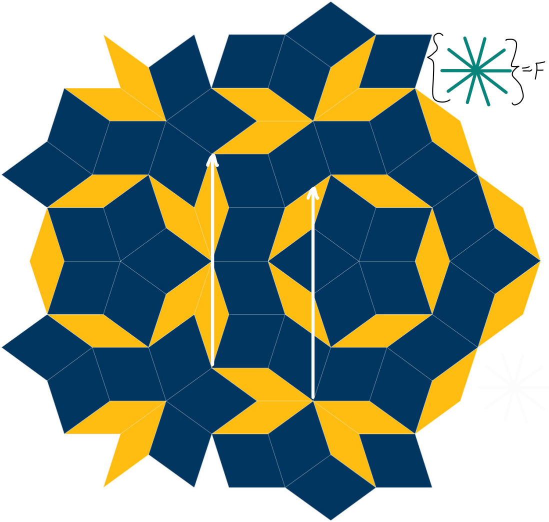

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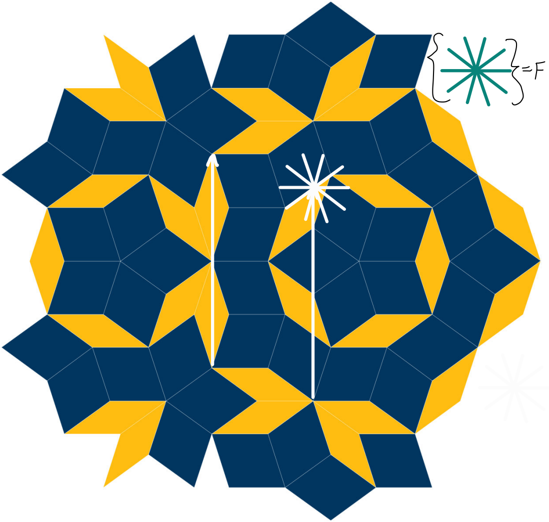

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"greater good" = "pretty pictures and cool math"

One-Dimensional Substitutions

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Let $\mathcal{A}$ be a finite set $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ called an **alphabet**, whose elements are called **letters**. A **word** in $\mathcal{A}$ is a (possibly bi-infinite) list of letters from $\mathcal{A}$. Denote the set of all words of $\mathcal{A}$ by $w(\mathcal{A})$.

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- `fibonacci`
- `catcatcatcatcat...`
- `...tactactactactac`
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Let $\mathcal{F} = \{\mathtt{a}, \mathtt{b}\}$. The **Fibonacci tiling** σ is defined by $\mathtt{a} \mapsto \mathtt{b}$ and $\mathtt{b} \mapsto \mathtt{ab}$.

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Represent "a" by a tile of length 1 and "b" by a tile of length $\varphi = \frac{1+\sqrt{5}}{2}$.

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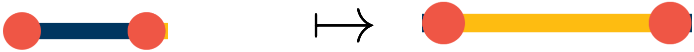

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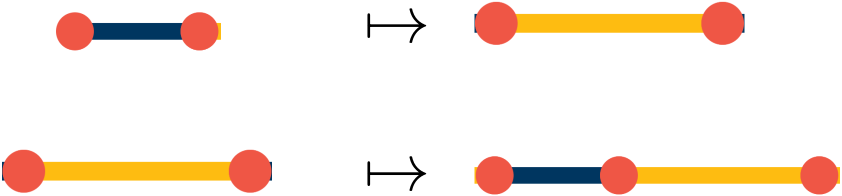

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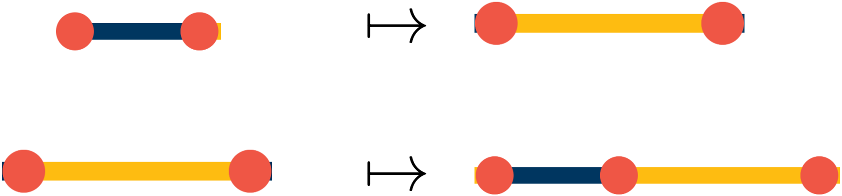
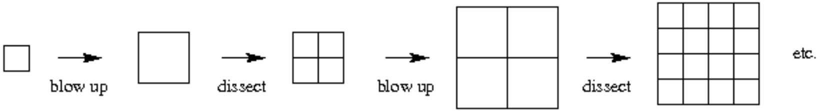


$$\sigma^5(\mathtt{b.a}) = \mathtt{abbabbababbab.bababbab}$$

Substitution Schemes



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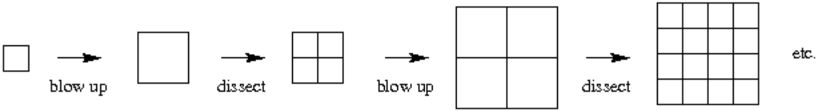

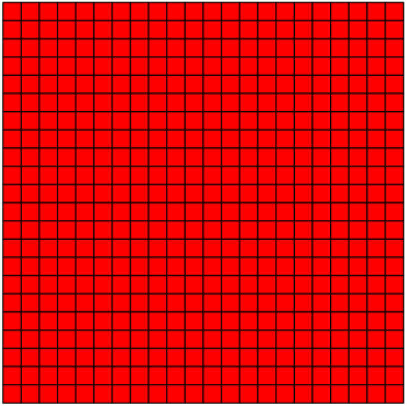

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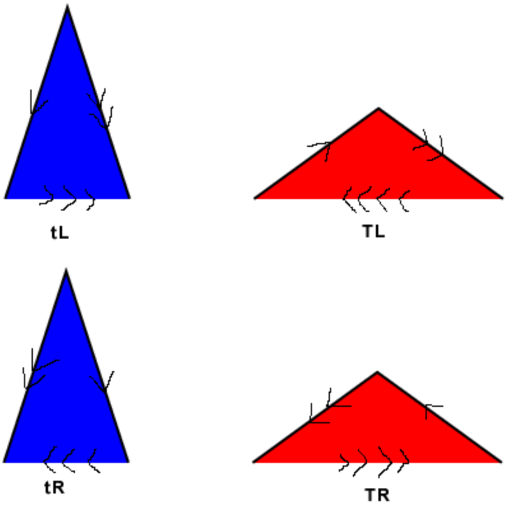

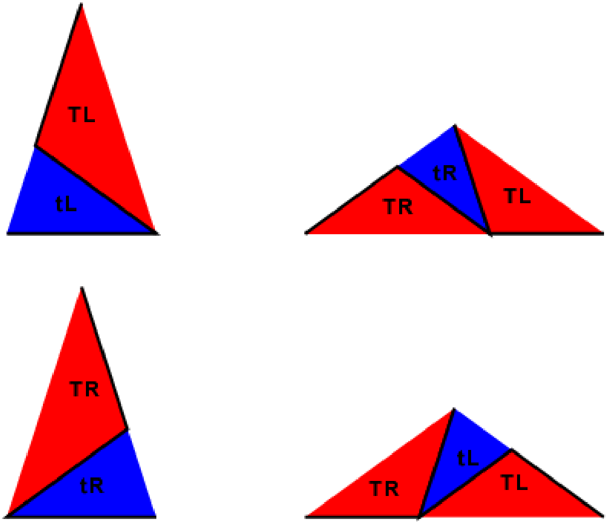
TL
tL
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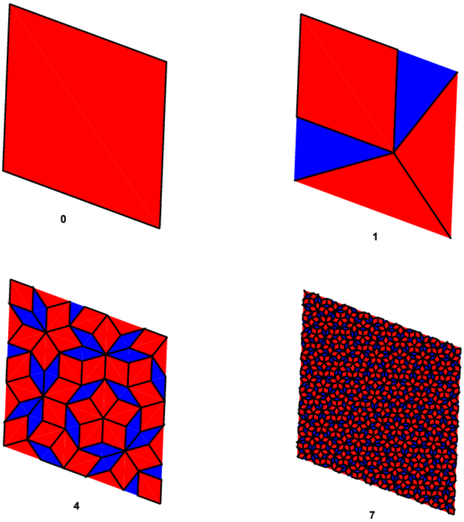
0
1
4
7

What We've Seen

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Under certain conditions on M_σ, *substitution* $\rightsquigarrow$ *self-similar.*

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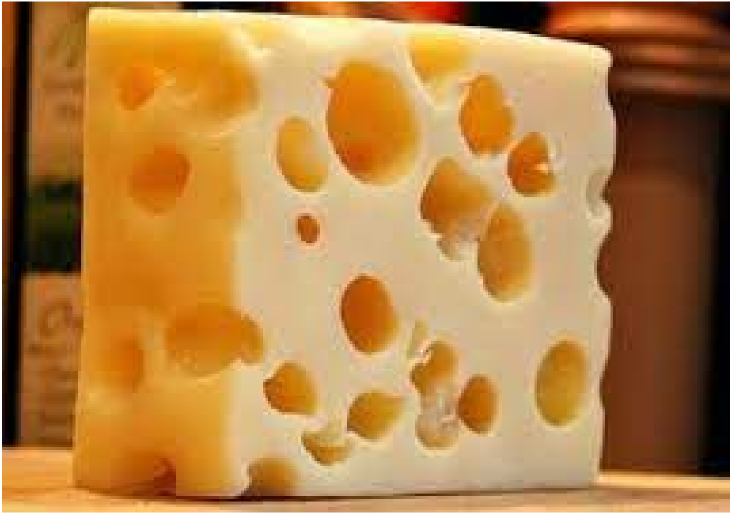
Under certain conditions, cut and project scheme $\rightsquigarrow$ UAL.

Cut-and-Project Cheese



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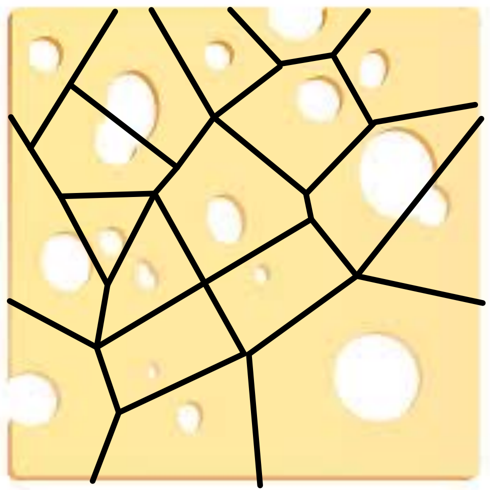

Cut-and-Project Schemes

- the **cheese block**

Cut-and-Project Schemes

- the **total space** $\mathbb{R}^N$
- the **cheese block**

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- the **cheese block**
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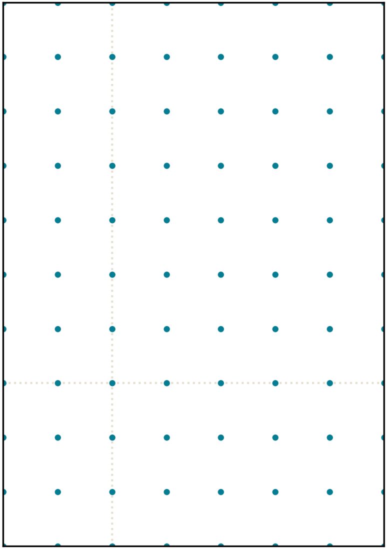

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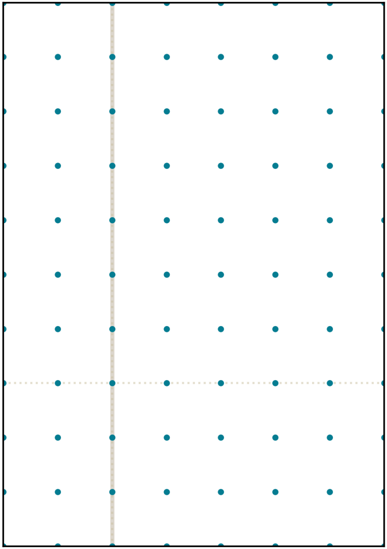

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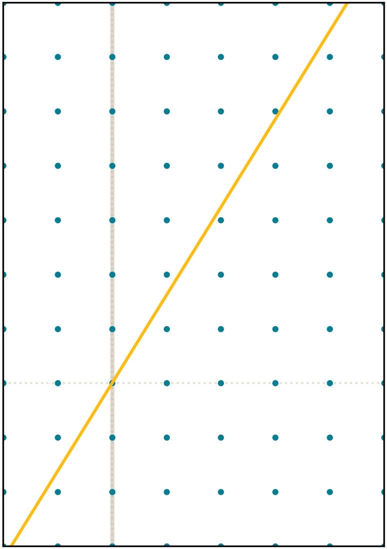

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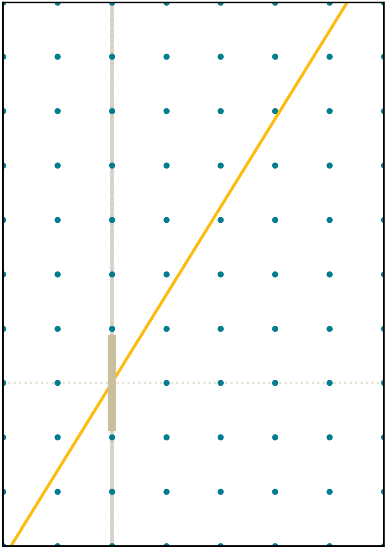

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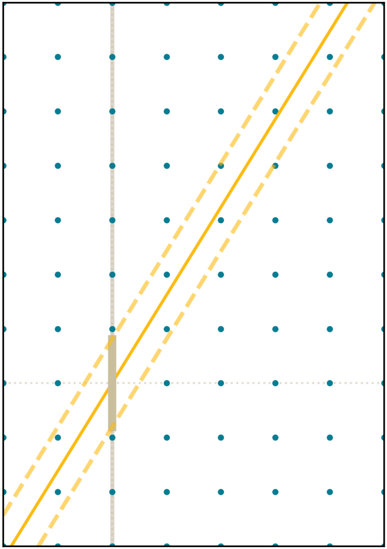

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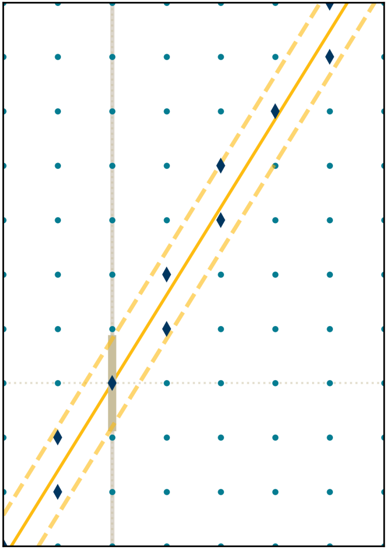

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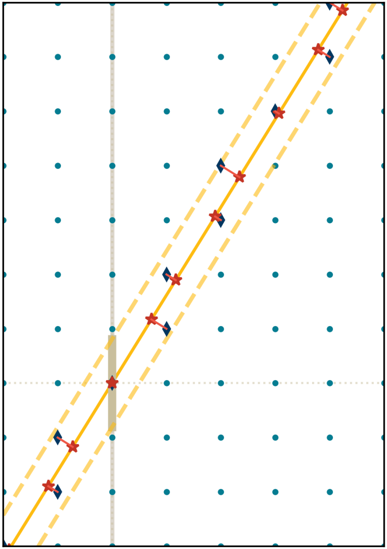
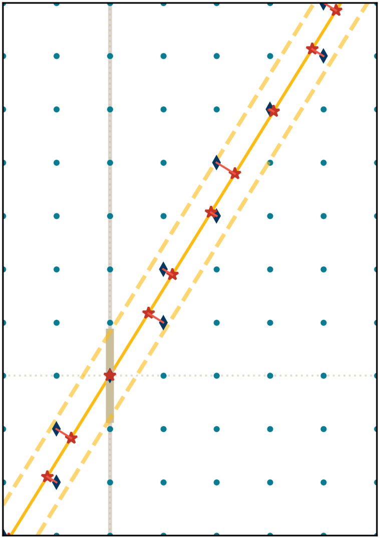
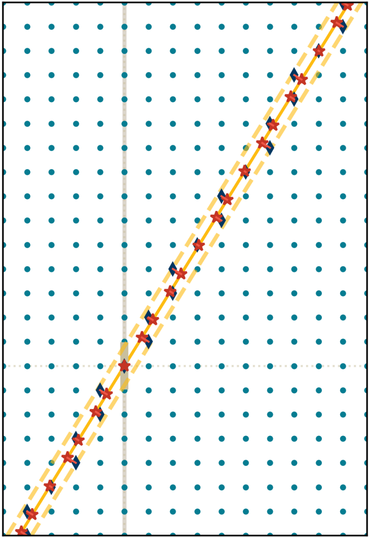




Cut-and-Project Schemes



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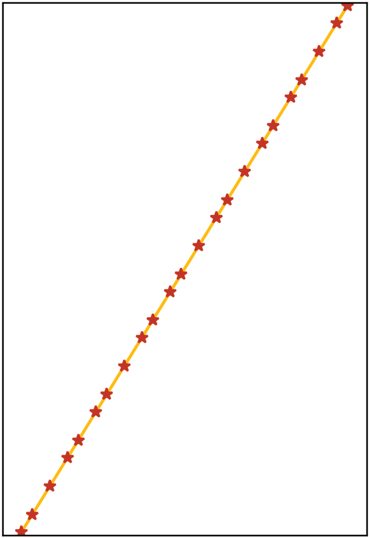

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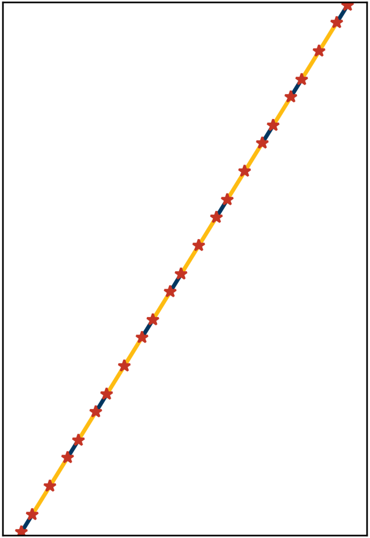

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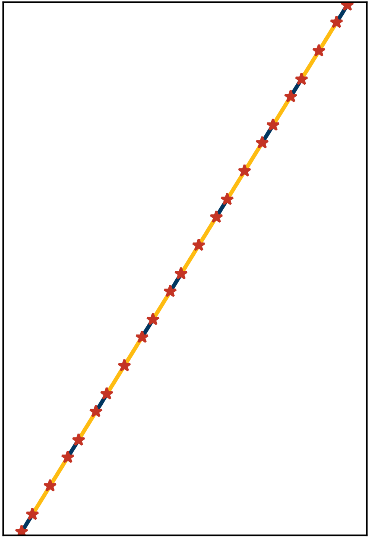
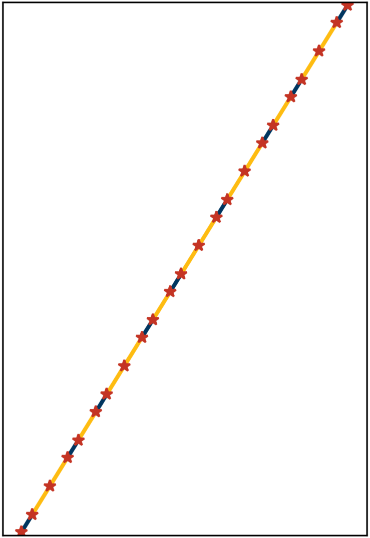
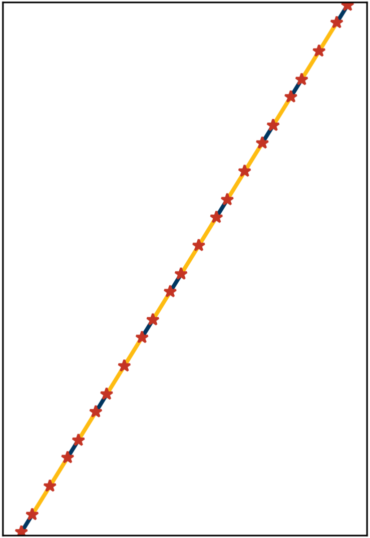




Facts



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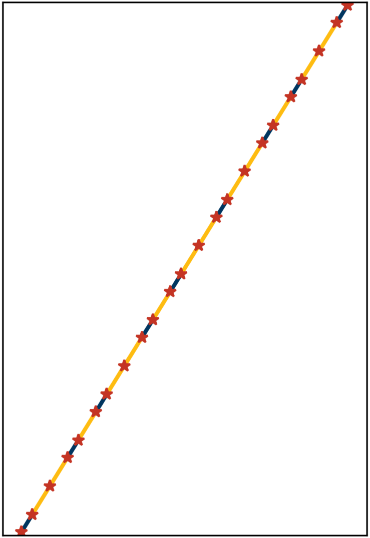

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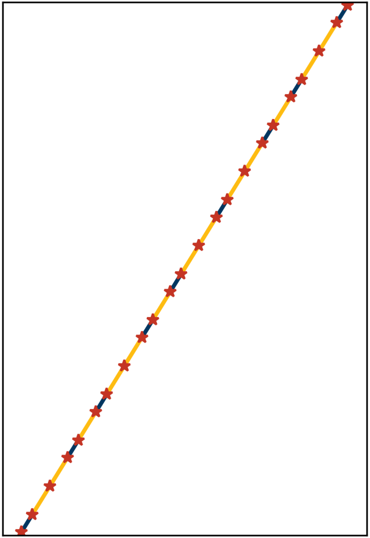

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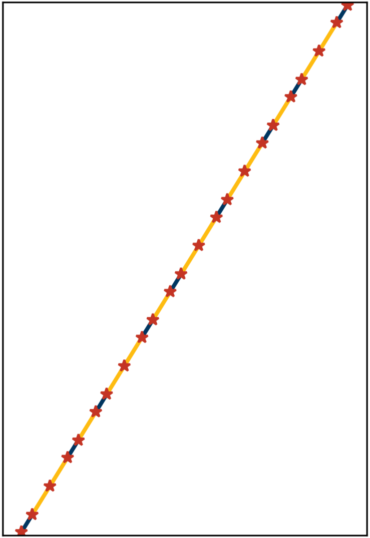

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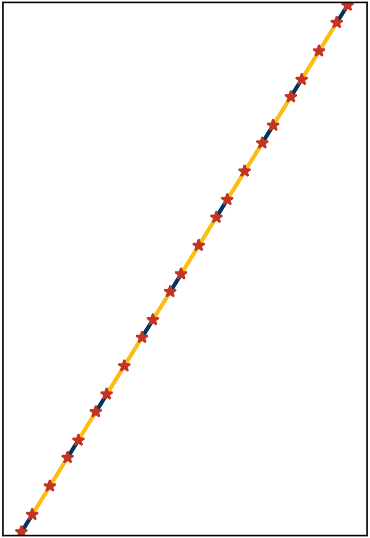

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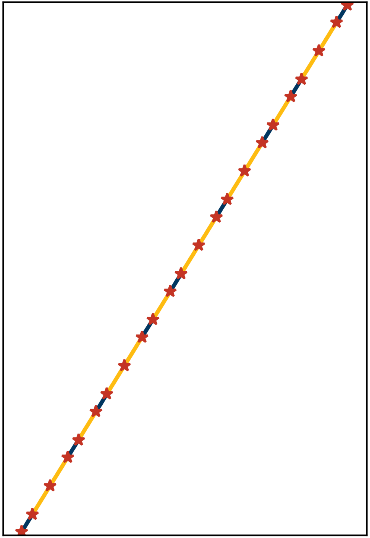

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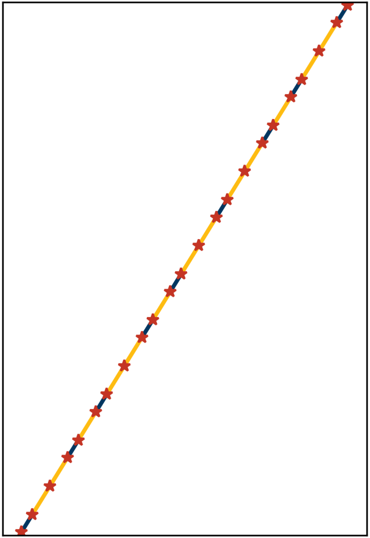

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The Big Result(s)

Theorem ([Meyer, 1972])

Every UAL is the model set of some cut and project scheme.

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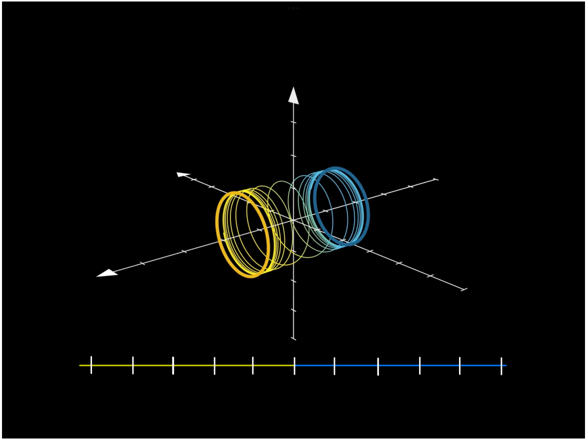
Interpretation

Even with another type of limited chaos, the same result holds.

Definition (Heuristic)

The **hull** of a tiling T is the collection Ω_T of all tilings that can't be distinguished from some translate of T at any finite scale.



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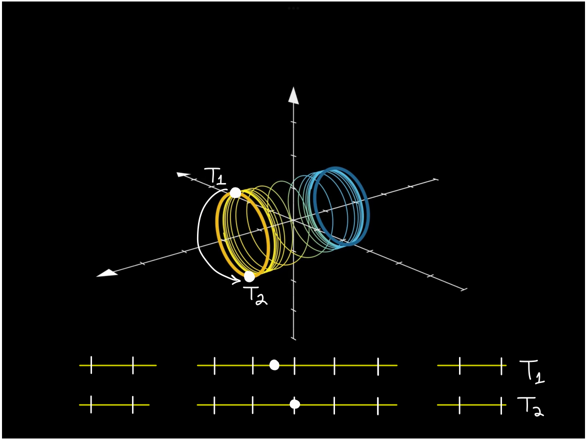

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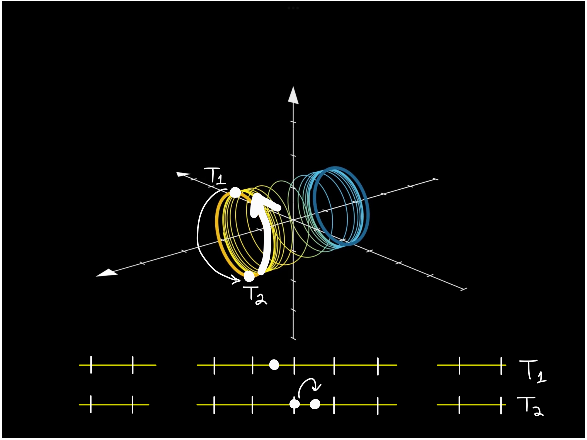

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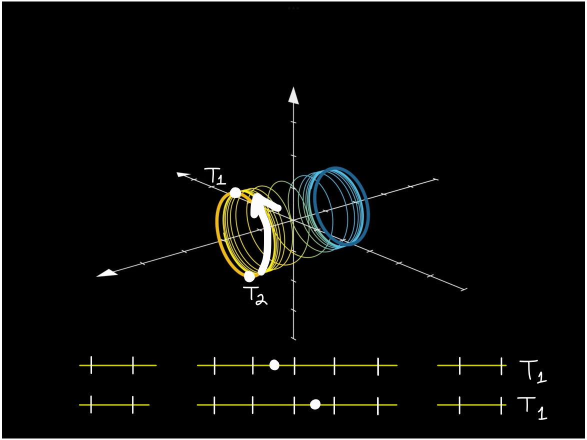

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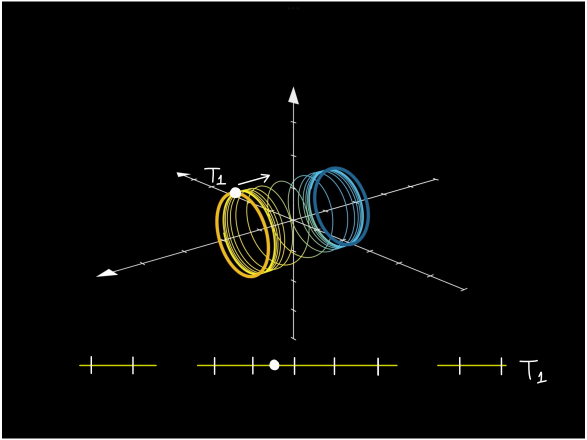

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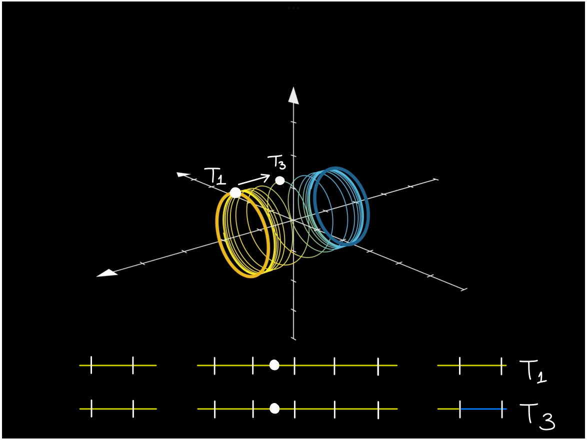

Questions for Investigation

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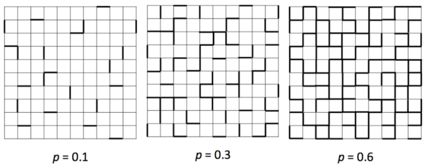
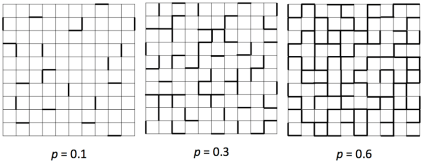


- Percolation on tilings

Questions for Investigation



- Percolation on tilings
    - Substitution?

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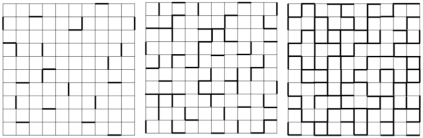

$p = 0.1$ $p = 0.3$ $p = 0.6$

- Percolation on tilings
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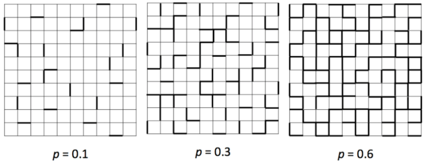

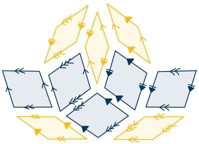

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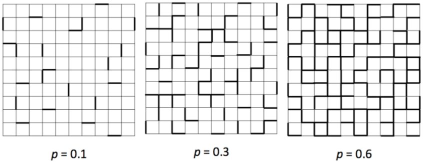

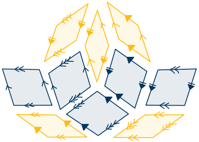

- Percolation on tilings
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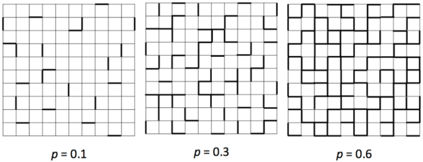

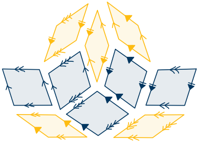

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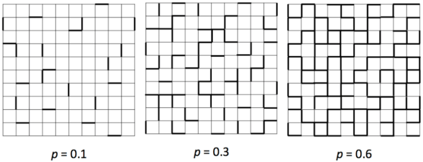

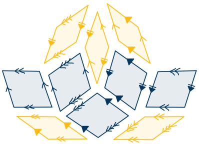

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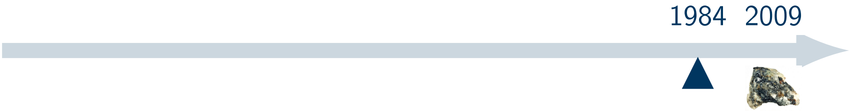
    - Topologically?

Quasicrystals Beyond Mathematics

2009

- 2009: Discovery of natural quasicrystal icosahedrite ($Al_{63}Cu_{24}Fe_{13}$)



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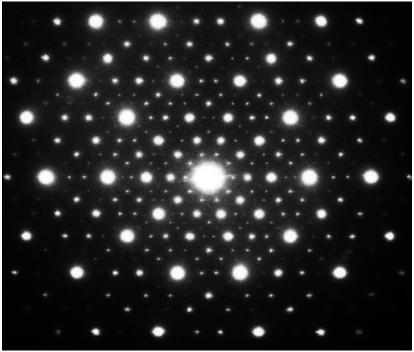
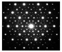
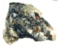


1974 1984 2009





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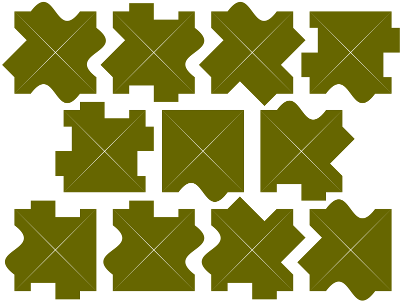
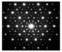


Quasicrystals Beyond Mathematics

1420s | 1963 | 1974 | 1984 | 2009



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Fibonacci Tiling $\longleftarrow$ 1-dimensional UAL $\longleftarrow$ 2-dimensional lattice

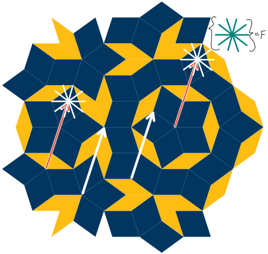
Fibonacci Tiling ⟵ 1-dimensional UAL ⟵ 2-dimensional lattice

Penrose Tiling

Quasicrystals Beyond Mathematics

Fibonacci Tiling ⟵ 1-dimensional UAL ⟵ 2-dimensional lattice

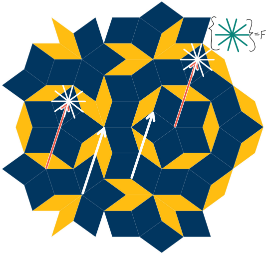
Penrose Tiling ⟵ 2-dimensional UAL

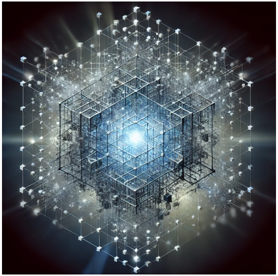


Quasicrystals Beyond Mathematics

Fibonacci Tiling $\longleftarrow$ 1-dimensional UAL $\longleftarrow$ 2-dimensional lattice

Penrose Tiling $\longleftarrow$ 2-dimensional UAL $\longleftarrow$ 5-dimensional lattice

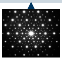
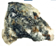




Quasicrystals Beyond Mathematics

1420s 1963 1974 1984 2009



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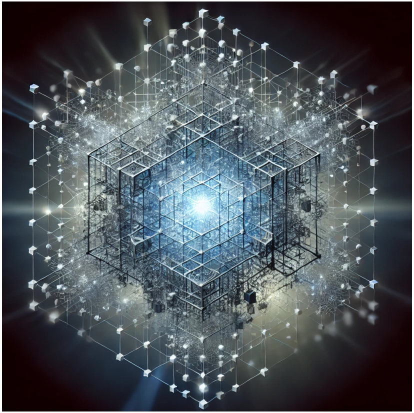
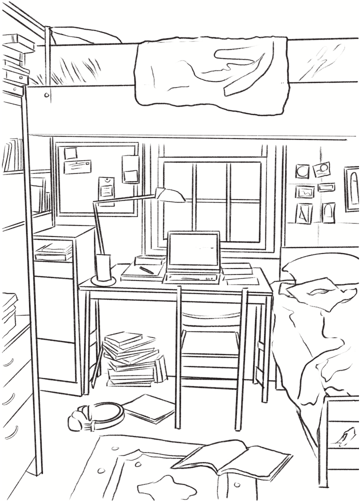
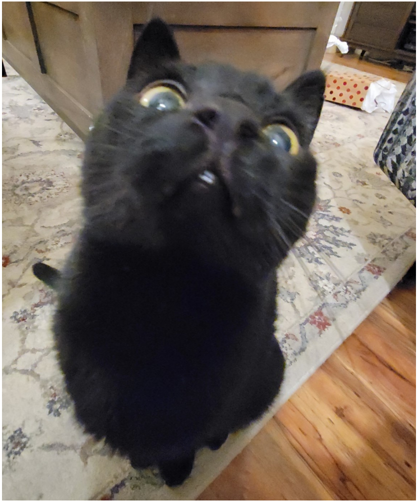


Quasicrystals Beyond Mathematics



Artwork by Maddy Cowan

Quasicrystals Beyond Mathematics



Quasicrystals Beyond Mathematics

Joseph Thrown into a Pit, 1644, Colyn

Quasicrystals Beyond Mathematics

The Storm on the Sea of Galilee, 1632, Rembrandt

Quasicrystals Beyond Mathematics

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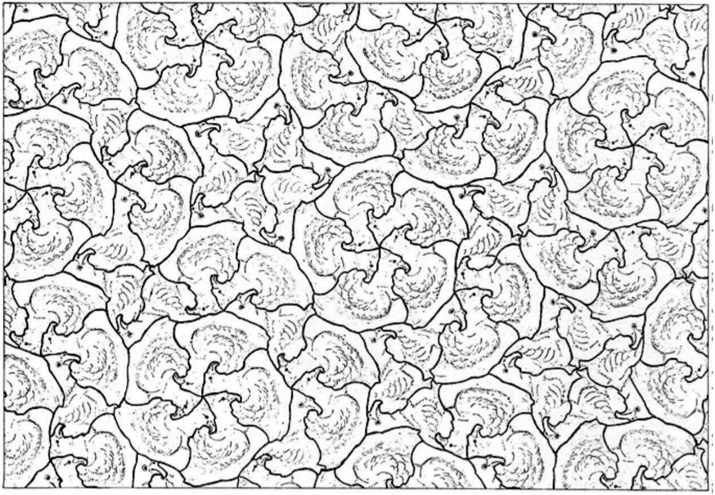

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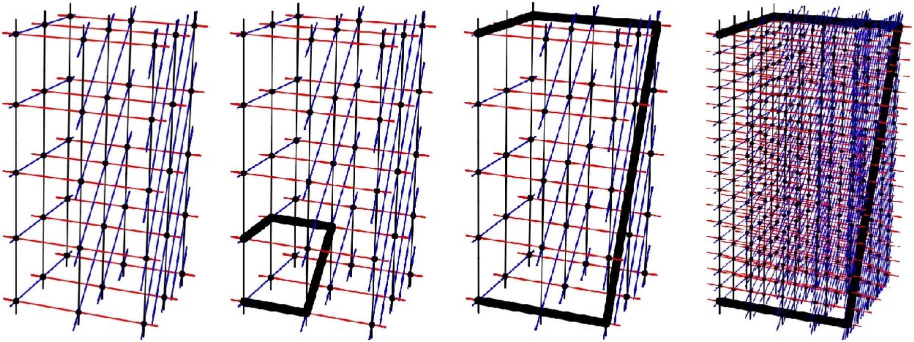
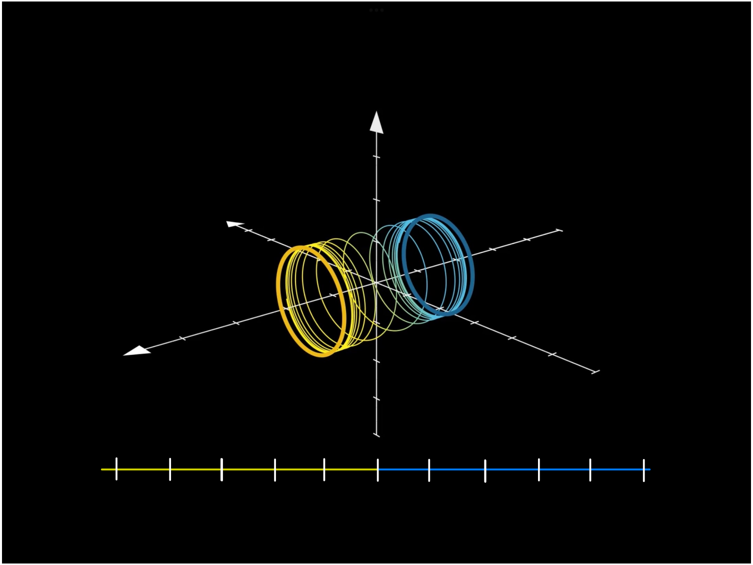
Definition (Heuristic)

An *n*-dimensional **connected, simply connected, nilpotent Lie group** is a copy of $\mathbb{R}^n$, but with distored ("almost commutative") group law.





Theorem

- *There is an equivalence for tilings of* $\mathbb{R}^n$*:*

$$\textit{Tiling} \to \textit{Delaunay set} \to \textit{"Simple" Tiling} \qquad \text{([Frank, 2000])}$$

Let T *be a simple tiling of* $\mathbb{R}^n$ *(think Penrose tiling,* $n = 2$*)*

- Ω_T *is an inverse limit of "compact branched manifolds" ([Sadun, 2003])*
- Ω_T *is a fiber bundle over* $\mathbb{T}^n$ *([Sadun, 2003])*
- $\check{H}^*(\Omega_T; \mathbb{R}) \approx H^*_{PE}(T; \mathbb{R})$ *([Kellendonk and Putnam, 2006])*

Theorem

- *There is an equivalence for tilings of CSCNL groups:*

$$\textit{Tiling} \to \textit{Delaunay set} \to \textit{"Simple" Tiling} \qquad \text{(H. 2024)}$$

Let T be a simple tiling of a CSCNL group N.

- *Ω_T is an inverse limit of "compact branched manifolds" ([Sadun, 2003])*
- *Ω_T is a "fiber bundle" over N/Λ for some lattice $\Lambda \leq N$, as long as N is a "rational" CSCNL group (H. 2024)*
- *$\check{H}^*(\Omega_T; \mathbb{R}) \approx H^*_{PE}(T; \mathbb{R})$ (H. 2024)*