201B Real Analysis
Assignment 6

1. Let Y be a closed subspace of a normed space X, and let $x_0 \notin Y$. Prove that there exists $\phi \in X^*$ such that $\langle \phi, x_0 \rangle = 1$ and $\phi|_Y = 0$.

2. Prove the Fredholm theorem: suppose $A \in \mathcal{B}(X, Y)$. Then $f \in \overline{\mathrm{im} A}$ if and only if $\langle \phi, f \rangle = 0$ for any $\phi \in \ker A^*$.

3. Let X be a normed space, $E \subset X$. Prove that E is bounded if and only if the set $\phi(E) \subset \mathbf{R}$ is bounded for every $\phi \in X^*$.

4. Use the classical dualities (cf. Kolmogorov-Fomin for the proofs) to prove that c_0 and l^1 are not reflexive. One way to do it, is first to prove that isomorphic normed spaces are separable only simultaneously, and that X^* is separable $\Rightarrow X$ is separable.

5. Let X be a normed space.

   Fix linear independent vectors $e_1, \ldots, e_n$ in X. Prove that there exist $\phi^1, \ldots \phi^n \in X^*$ (called *dual, or biorthogonal, to* e) such that $\langle \phi^k, e_j \rangle = \delta^k_j$. Are the functionals linear independent?

   Fix linear independent $\psi_1, \ldots, \psi_n \in X^*$. Prove that there exist biorthogonal $u_1, \ldots, u_n \in X$. Are the vectors linear independent?

6. Prove that a finite dimensional space is reflexive (you might find the biorthogonal basis useful for this). Suppose that a linear transformation of a finite dimensional complex space V is given in the fixed basis by a matrix A. Find the matrix for A^* in the dual basis of V^*.

7. Suppose X is complete. Show then that X is reflexive $\Leftrightarrow X^*$ is reflexive. Hint: to establish $\Leftarrow$ notice that $i_{can}(X)$ is now a *closed* subspace of X^{**}.