1. **Tensor product of measures and Fubini theorem.** Let $(\mathcal{A}_j, \Omega_j, \mu_j)$, $j = 1, 2$, be two measure spaces. Recall that the new σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ with the unit element $\Omega_1 \times \Omega_2$ is the σ-algebra generated by the direct products of elements of $\mathcal{A}_{1,2}$. That is,

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma_{\Omega_1 \times \Omega_2}\left(\{E_1 \times E_2 \colon E_{1,2} \in \mathcal{A}_{1,2}\}\right).$$

   Our goal first will be to define a *product measure* on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

2. A measure ν defined on the σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$, such that

$$\nu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2) \quad \forall E_{1,2} \in \mathcal{A}_{1,2}$$

   is called the *(tensor) product of* μ_1 *and* μ_2. A somewhat surprising fact is that in such general situation the product ν is not unique, although always exists. We will denote *any* product measure by $\mu_1 \otimes \mu_2$. Thus

$$\nu = \mu_1 \otimes \mu_2$$

   makes sence, but $\mu_1 \otimes \mu_2 = \nu$ is a uniqueness statement (which might be wrong).

3. We shall establish the existence of $\mu_1 \otimes \mu_2$ by running the Caratheodory maschine. We need to prepare the setting for it. In the following notice the strong similarities with the construction of λ^n on $\mathbf{R}^n$.

4. A (measurable) *rectangle* is the set of the form $X \times Y$ with $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$. Notice that the intersection of rectangles is a rectangle

$$(A \times B) \cap (X \times Y) = (A \cap X) \times (B \cap Y),$$

   and that the complement of a rectangle is the disjoint union of two rectangles

$$(A \times B)^c = (A^c \times \Omega_2) \cup (A \times B^c).$$

   These observations will be useful later.

5. For a rectangle $X \times Y$ define

$$\pi(X \times Y) = \mu_1(X)\mu_2(Y).$$

   The following lemma will be the main tool in the later proofs.

   Lemma 1 *Let* $A, A_{1,2\ldots} \in \mathcal{A}_1$, $B, B_{1,2,\ldots} \in \mathcal{A}_2$ *be such that all* $A_j \times B_j$ *are disjoint, and*

$$A \times B = \bigcup_j A_j \times B_j.$$

   Then

$$\pi(A \times B) = \sum_j \pi(A_j \times B_j).$$

Let rectangles $C_1 \times D_1$, $C_2 \times D_2$, $\ldots$, $C_j \in \mathcal{A}_1$, $D_j \in \mathcal{A}_2$, *cover* $A \times B$,

$$A \times B \subset \bigcup_j C_j \times D_j.$$

Then

$$\pi(A \times B) \leq \sum_j \pi(C_j \times D_j).$$

Proof. **1.** For both parts of the theorem the key is the following observation. In the first case the formula

$$\chi_{A \times B}(x, y) = \chi_A(x)\chi_B(y) = \sum_j \chi_{A_j}(x)\chi_{B_j}(y)$$

holds for all $(x, y) \in \Omega_1 \times \Omega_2$. In the second case

$$\chi_{A \times B}(x, y) = \chi_A(x)\chi_B(y) \leq \sum_j \chi_{A_j}(x)\chi_{B_j}(y).$$

2. Let us prove, for example, the case of equality in the lemma. The other case is proved the same way. Fix $x \in \Omega_1$ and think of the functions in the formula as maps from Ω_2 . Apply the monotone convergence theorem on the measure space $(\mathcal{A}_2, \Omega_2, \mu_2)$. Deduce that

$$\begin{aligned}
\chi_A(x)\mu_2(B) &= \chi_A(x) \int_{\Omega_2} \chi_B(y)\, d\mu_2(y) \\
&= \sum_j \chi_{A_j}(x) \int_{\Omega_2} \chi_{B_j}(y)\, d\mu_2(y) \\
&= \sum_j \chi_{A_j}(x)\mu_2(B_j).
\end{aligned}$$

Next, think of the last formula as the identity for maps from Ω_1 . Application of the monotone convergence on $(\mathcal{A}_1, \Omega_1, \mu_1)$ completes the proof. □

6. For *any* $E \subset \Omega_1 \times \Omega_2$ define the outer measure

$$\pi^*(E) = \inf \left(\sum_j \pi(A_j \times B_j) \right),$$

where the infimum is taken over all coverings of E by at most countable number of measurable rectangles $A_j \times B_j$. Similarly to the construction of the Lebesgue measure in $\mathbf{R}^n$ we establish (by a different proof) the following proposition.

Proposition 2 *The outer measure* $\pi^*\colon 2^{\Omega_1\times\Omega_2} \to [0,+\infty]$ *enjoys the following properties:*

(i) $\pi^*(\emptyset) = 0$ *;*

(ii) $\pi^*(E_1) \le \pi^*(E_2)$ *if* $E_1 \subset E_2$ *;*

(iii) semiadditivity holds, that is

$$\pi^*\left(\bigcup_j E_j\right) \le \sum_j \pi^*(E_j);$$

(iv) $\pi^*(A\times B) = \pi(A\times B)$ *for any measurable rectangle* $A\times B$ *.*

Proof. **1.** The proofs of (i)–(iii) are exactly the same as in the corresponding proposition in part 1. However (iv) requires a completely different approach than in part 1 (why?).

2. On the one hand by the definition

$$\pi^*(A\times B) \le \pi(A\times B).$$

On the other hand for any disjoint cover of $A\times B$ by rectangles $\{A_j\times B_j\}$ we have

$$\begin{aligned} A\times B &\subset (A\times B)\cap\left(\bigcup_j A_j\times B_j\right) \\ &= \bigcup_j (A\cap A_j)\times(B\cap B_j). \end{aligned}$$

Hence by Lemma 1

$$\begin{aligned} \pi(A\times B) &\le \sum_j \pi((A\cap A_j)\times(B\cap B_j)) \\ &\le \sum_j \pi(A_j\times B_j). \end{aligned}$$

We conclude the proof of (iv) by taking the infimum over all covers in the latter formula. □

7. We see that π^* satisfies all axioms of the outer measure in the Caratheodory construction which we developed in part 1. As we emphasized there, the theory relies on nothing except the axioms. Apply the Caratheodory construction to the outer measure π^* and obtain a σ-algebra $(\mathcal{P}, \Omega_1\times\Omega_2)$ of π^*-measurable sets defined by

$$\begin{aligned} \mathcal{P} = \Big\{E\subset\Omega_1\times\Omega_2\colon\ \pi^*(X) &= \pi^*(X\cap E) \\ &\quad + \pi^*(X\cap E^c)\ \forall X\subset\Omega_1\times\Omega_2\Big\}. \end{aligned}$$

Moreover, according to the Caratheodory construction, the restriction $\pi^*|_{\mathcal{P}}$ is a complete measure on $\mathcal{P}$.

8. **Theorem 3** *Let* $(\mathcal{A}_j, \Omega_j, \mu_j)$, $j = 1, 2$, *be two measure spaces. Then any rectangle* $A \times B$, $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$, *is* π^**-measurable, and therefore*

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \subset \mathcal{P}.$$

Moreover

$$\pi^*(A \times B) = \mu_1(A)\mu_2(B).$$

Proof. **1.** The equality was proved in Proposition 2 (iv). We just need to show that any rectangle is π^*-measurable.

2. Set $S = A \times B$. To show the measurability of S we take any test set $X \subset \Omega_1 \times \Omega_2$. Fix any $\varepsilon > 0$. Cover this X by a family of rectangles $\{R_j\}$ so that

$$\sum_j \pi(R_j) \leq \pi^*(X) + \varepsilon.$$

For each j we write

$$R_j = (R_j \cap S) \cup (R_j \cap S^c),$$

where the union is disjoint. Every $R_j \cap S$ is a rectangle and $X \cap S$ is covered by $\{R_j \cap S\}$. Similarly $R_j \cap S^c$ is a disjoint union of at most two rectangles,

$$R_j \cap S^c = C_{1j} \cup C_{2j},$$

and $X \cap S^c$ is covered by $\{C_{1j}, C_{2j}\}$. For every j by Lemma 1

$$\pi(R_j) = \pi(R_j \cap S) + \pi(C_{1j}) + \pi(C_{2j}).$$

Therefore

$$\begin{aligned} \pi^*(X) + \varepsilon &\geq \sum_j \pi(R_j \cap S) + \pi(C_{1j}) + \pi(C_{2j}) \\ &\geq \pi^*(X \cap S) + \pi^*(X \cap S^c). \end{aligned}$$

We can take ε arbitrary small. Hence S is measurable. □

9. For any pair of the measure spaces $(\mathcal{A}_j, \Omega_j, \mu_j)$, $j = 1, 2$, we constructed $\pi^* = \mu_1 \otimes \mu_2$ defined on $\mathcal{A}_1 \otimes \mathcal{A}_2$. In general situation the tensor product of measures is not unique (cf. below). However, if the terms $\mu_{1,2}$ are σ-finite, then $\mu_1 \otimes \mu_2$ is unique as the following theorem states.

10. On the σ-algebra $(\mathbf{R}, 2^{\mathbf{R}})$ consider the measure μ,

$$\mu(E) = \begin{cases} 0, & E \text{ is at most countable} \\ \infty, & E \text{ is uncountable.} \end{cases}$$

Set $\mathcal{A} = 2^{\mathbf{R}} \otimes 2^{\mathbf{R}}$. Prove the following statements about $\mu \otimes \mu$.

(a) For $E \in \mathcal{A}$ define $\nu(E) = 0$, if $E = G \cup H$ for some $G, H \in \mathcal{A}$, such that $\mathrm{proj}_x(G)$ and $\mathrm{proj}_y(H)$ are both countable. Otherwise define $\nu(E) = \infty$. Prove that ν is a measure on $\mathcal{A}$, and that $\nu = \mu \otimes \mu$.

(b) Let $L = \{(x, y) \colon x = y\}$ be the bisector line in $\mathbf{R}^2$. For $E \in \mathcal{A}$ define $\rho(E) = 0$, if $E = G \cup H \cup K$ for some $G, H, K \in \mathcal{A}$, such that $\mathrm{proj}_x(G)$, $\mathrm{proj}_y(H)$, and $\mathrm{proj}_L(K)$ are all countable. Otherwise define $\rho(E) = \infty$. Prove that ρ is a measure on $\mathcal{A}$, and that $\rho = \mu \otimes \mu$.

(c) It will be helpful to prove that if $\nu(E) = 0$ or $\rho(E) = 0$, then E is contained in a countable set of lines.

(d) Define $E_0 = \{(x, y) \colon x = -y\}$. Prove that $E_0 \in \mathcal{A}$, and $\nu(E_0) \neq \rho(E_0)$.

11. We shall prove that any $\mu_1 \otimes \mu_2$ must be equal to π^* obtained from the Caratheodory construction, *provided* the terms are σ-finite.

Theorem 4 *Let $(\mathcal{A}_j, \Omega_j, \mu_j)$, $j = 1, 2$, be two measure spaces with σ-finite $\mu_{1,2}$. Let ν be any measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that*

$$\nu(X_1 \times X_2) = \mu_1(X_1)\mu_2(X_2)$$

for all $X_{1,2} \in \mathcal{A}_{1,2}$. Then $\nu = \pi^*$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Proof. **1.** First suppose $\|\mu_{1,2}\| < \infty$. Then both ν and π^* are also finite. Take any $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$. For any sequence of rectangles $\{R_j\}$ covering E we derive

$$\begin{aligned} \nu(E) &\leq \nu\left(\bigcup_j R_j\right) \\ &\leq \sum_j \nu(R_j) \\ &= \pi(R_j). \end{aligned}$$

Take the infimum over all such coverings to discover

$$\nu(E) \leq \pi^*(E) \quad \forall E \in \mathcal{A}_1 \otimes \mathcal{A}_2. \tag{0.1}$$

On the other hand in the formula

$$\nu(E) + \nu(E^c) = \mu_1(\Omega_1)\mu_2(\Omega_2) = \pi^*(E) + \pi^*(E^c)$$

we utilise the finiteness of all terms to deduce

$$\begin{aligned} \nu(E) &= \pi * (E) + \pi^*(E^c) - \nu(E^c) \\ &\geq \pi^*(E) \end{aligned}$$

since $\pi^*(E^c) - \nu(E^c) \geq 0$ by (0.1).

2. Write down a limit argument from the finite to the σ-finite case to complete the proof of the theorem. □

12. Now we turn to the proerties of the integral with respect to $\mu_1 \otimes \mu_2$. Our goal is to establish relation between the "double" and "repeated" integrals. This circle of ideas is usually called the Fubini's theorem, despite there are more than one theorem there.

We need to introduce the following objects. For any $E \subset \Omega_1 \times \Omega_2$ define its *sections* by writing

$$E_x \stackrel{\text{def}}{=} \{y \in \Omega_2 : \ (x, y) \in E\},$$

and

$$E^y \stackrel{\text{def}}{=} \{x \in \Omega_1 : \ (x, y) \in E\},$$

Similarly for a function $f : \Omega_1 \times \Omega_2 \to \mathbf{R}$ define its *sections* by

$$\begin{array}{rcl} f_x : \Omega_2 & \longrightarrow & \mathbf{R} \\ y & \longmapsto & f(x, y), \end{array}$$

and

$$\begin{array}{rcl} f^y : \Omega_1 & \longrightarrow & \mathbf{R} \\ x & \longmapsto & f(x, y). \end{array}$$

Thus $f_x = f(x, \cdot)$, $f^y = f(\cdot, y)$.

13. **Lemma 5** *(i) For any $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and any $x \in \Omega_1$, $y \in \Omega_2$ the sections are measurable: $E_x \in \mathcal{A}_2$, $E^y \in \mathcal{A}_1$.*

(ii) For any $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable function f the sections are measurable: f_x is $\mathcal{A}_2$ -measurable, f^y is $\mathcal{A}_1$ -measurable.

Proof. **1.** Prove (i) using the principle of good sets from the part 1. Namely, prove that the collection of all sets $E \subset \Omega_1 \times \Omega_2$, such that all E_x, E^y are measurable, is a σ -algebra. Then prove that all rectangles are in this collection. Conclude then that the entire $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also a part of this collection.

2. Fix any $x \in \Omega_1$ Let us show, for example, that $f_x^{-1}((t, +\infty)) \in \mathcal{A}_2$ for any $t \in \mathbf{R}$. Indeed, clearly

$$\begin{aligned} f_x^{-1}((t, +\infty)) &= \{y \in \Omega_2 : f(x, y) > t\} \\ &= (\{(x, y) \in \Omega_1 \times \Omega_2 : f(x, y) > t\})_x . \end{aligned}$$

Hence the statement follows from the measurability of f and part (i). $\square$

14. The main technical part of the Fubini's theory is the following lemma.

Lemma 6 *Let $(\mathcal{A}_j, \Omega_j, \mu_j)$, $j = 1, 2$, be two measure spaces with σ - finite $\mu_{1,2}$. For any $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$ define $f_{1,2} : \Omega_{1,2} \to \bar{\mathbf{R}}$ by writing*

$$\begin{aligned} f_1(x) &= \mu_2(E_x), \quad x \in \Omega_1 \\ f_2(y) &= \mu_1(E^y), \quad y \in \Omega_2. \end{aligned}$$

Then $f_{1,2}$ *are* $\mathcal{A}_{1,2}$ *-measurable, and*

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_1} f_1 \, d\mu_1 = \int_{\Omega_2} f_2 \, d\mu_2.$$

15. Accepting this lemma let us prove the main theorems. The following *Tonelli's theorem* claims that for *positive* measurable functions the finiteness of the "double" integral is equivalent to the finiteness of the "repeated". We will see later that for sign changing functions this is not the case.

Theorem 7 *Let* $(\mathcal{A}_j, \Omega_j, \mu_j)$ *,* $j = 1, 2$ *, be two measure spaces with* σ *-finite* $\mu_{1,2}$ *. Let*

$$F \colon \Omega_1 \times \Omega_2 \to \bar{\mathbf{R}}, \quad F \geq 0,$$

be $\mathcal{A}_1 \otimes \mathcal{A}_2$ *-measurable. Then the functions*

$$f(x) = \int_{\Omega_2} F_x \, d\mu_2, \quad x \in \Omega_1,$$

and

$$g(y) = \int_{\Omega_1} F^y \, d\mu_1, \quad y \in \Omega_2,$$

are measurable, and

$$\int_{\Omega_1} f \, d\mu_1 = \int_{\Omega_2} g \, d\mu_2 = \int_{\Omega_1 \times \Omega_2} F \, d(\mu_1 \otimes \mu_2).$$

In other symbols,

$$\begin{aligned} \int_{\Omega_1} \left(\int_{\Omega_2} F(x,y) \, d\mu_2(y) \right) d\mu_1(x) &= \int_{\Omega_2} \left(\int_{\Omega_1} F(x,y) \, d\mu_1(x) \right) d\mu_2(y) \\ &= \int_{\Omega_1 \times \Omega_2} F(x,y) \, d(\mu_1 \otimes \mu_2)(x,y). \end{aligned}$$

In this theorem we require only that F is $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable. The integrals can be finite or infinite.

16. *Proof of Theorem 7.* **1.** By Lemma 6 the theorem holds for the case

$$F = \chi_E, \quad \forall E \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

Then by linearity we at once conclude that the theorem holds for any *simple* F . The general case will follow by a careful approximation procedure.

2. Fix any nonnegative measurable F . By the approximation theorem there exists a monotone sequence of positive simple functions $\{\Phi_n\}$,

$$\Phi_n \leq \Phi_{n+1} \quad \text{on} \quad \Omega_1 \times \Omega_2,$$

such that

$$\Phi_n \longrightarrow F \text{ as } n \longrightarrow \infty,$$

pointwisely on $\Omega_1 \times \Omega_2$. For each n define

$$\phi_n(x) = \int_{\Omega_2} (\Phi_n)_x \, d\mu_2, \quad x \in \Omega_1.$$

From Lemma 6 it follows that for any n the function ϕ_n is $\mathcal{A}_1$-measurable and

$$\int_{\Omega_1 \times \Omega_2} \Phi_n \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \phi_n \, d\mu_1. \tag{0.2}$$

We would like to pass in (0.2) to the limit as $n \to \infty$. The left hand side there is easy: the monotone convergence gives at once that

$$\int_{\Omega_1 \times \Omega_2} \Phi_n \, d(\mu_1 \otimes \mu_2) \longrightarrow \int_{\Omega_1 \times \Omega_2} F \, d(\mu_1 \otimes \mu_2), \; n \to \infty.$$

In order to pass to the limit in the right hand side of (0.2) notice first that for any fixed $x \in \Omega_1$

$$(\Phi_n)_x \leq (\Phi_{n+1})_x \text{ on } \Omega_2,$$

and that

$$(\Phi_n)_x \longrightarrow F_x, \; n \to \infty, \text{ on } \Omega_2.$$

All functions here are measurable as the sections of measurable functions. Hence by the monotone convergence we discover that for any fixed $x \in \Omega_1$

$$\begin{aligned} \phi_n(x) &= \int_{\Omega_2} (\Phi_n)_x \, d\mu_2 \\ &\longrightarrow \int_{\Omega_2} F_x \, d\mu_2, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Monotonicity of $\{\Phi_n\}$ implies that

$$\phi_n \leq \phi_{n+1} \quad \text{on} \quad \Omega_1.$$

Once more apply the monotone convergence on $(\Omega_1, \mathcal{A}_1, \mu_1)$ to derive

$$\int_{\Omega_1} \phi_n(x) \, d\mu_1(x) \longrightarrow \int_{\Omega_1} \left(\int_{\Omega_2} F_x \, d\mu_2 \right) d\mu_1(x), \quad \text{as} \quad n \to \infty.$$

Thus passing to the limit as $n \to \infty$ in (0.2) we deduce

$$\int_{\Omega_1 \times \Omega_2} F \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} F_x \, d\mu_2 \right) d\mu_1(x) = \int_{\Omega_1} f \, d\mu_1.$$

3. The proof of

$$\int_{\Omega_1 \times \Omega_2} F \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_2} \left(\int_{\Omega_1} F^y \, d\mu_1 \right) d\mu_2(y) = \int_{\Omega_2} g \, d\mu_2$$

proceeds verbatim. □

17. The following theorem due to Fubini covers the case of sign changing functions.

Theorem 8 *Let* $(\mathcal{A}_j, \Omega_j, \mu_j)$ *,* $j = 1, 2$ *, be two measure spaces with* σ *-finite* $\mu_{1,2}$ *. Suppose*

$$\int_{\Omega_1 \times \Omega_2} |F| \, d(\mu_1 \otimes \mu_2) < \infty$$

(that is $F \in L^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ *). Define the functions*

$$f(x) = \int_{\Omega_2} F_x \, d\mu_2, \quad x \in \Omega_1,$$

and

$$g(y) = \int_{\Omega_1} F^y \, d\mu_1, \quad y \in \Omega_2.$$

Then $f \in L^1(\Omega_1, \mu_1)$ *,* $g \in L^1(\Omega_2, \mu_2)$ *, and*

$$\int_{\Omega_1} f \, d\mu_1 = \int_{\Omega_2} g \, d\mu_2 = \int_{\Omega_1 \times \Omega_2} F \, d(\mu_1 \otimes \mu_2).$$

In other symbols,

$$\begin{aligned} \int_{\Omega_1} \left(\int_{\Omega_2} F(x,y) \, d\mu_2(y) \right) d\mu_1(x) &= \int_{\Omega_2} \left(\int_{\Omega_1} F(x,y) \, d\mu_1(x) \right) d\mu_2(y) \\ &= \int_{\Omega_1 \times \Omega_2} F(x,y) \, d(\mu_1 \otimes \mu_2)(x,y). \end{aligned}$$

Proof. Prove the theorem. Hint: split F into a positive and negative part, $F = F^+ - F^-$, and derive the statement from Tonelli's theorem. Indicate clearly the step when the assumption $F \in L^1$ is crucial. □

18. Give an example showing that the Fubini's theorem fails if F is only $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable.