4 Sequences of measurable functions

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space (complete, after a possible application of the completion theorem). In this chapter we investigate relations between various (nonequivalent) convergences of sequences of $\mathcal{A}$-measurable functions $\{f_n\}$ on Ω.

2. Let us recall the various notions of convergence. Suppose f, $f_1, f_2, \ldots$ are measurable.

   (a) Convergence μ-almost everywhere (in particular the pointwise convergence): $f_n \to f$ μ-a.e. in Ω as $n \to \infty$ iff for some $E \subset \Omega$ with $\mu(E) = 0$, we have $f_n(x) \to f(x)$ for any $x \in \Omega \setminus E$ as $n \to \infty$.

   (b) Convergence uniform on E: $f_n \to f$ uniformly on $E \subset \Omega$ as $n \to \infty$ iff
   $$\lim_{n\to\infty} \left(\sup_{x \in E} |f_n(x) - f(x)| \right) = 0.$$

   (c) Convergence in the Banach space L^p, $1 \le p \le \infty$: $f_n \to f$ in L^p as $n \to \infty$ iff $f, f_n \in L^p$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

   (d) *Convergence in measure:* $f_n \to f$ *in measure* as $n \to \infty$ (we write $f_n \to f$ in μ as $n \to \infty$) iff
   $$\forall t > 0 \qquad \lim_{n\to\infty} \mu\left(\{|f_n - f| > t\}\right) = 0.$$

   Except the convergence in measure, we have encountered these notions before. An important type of convergence is missing in the above list– later in the course we will also encounter the *weak convergence*.

   All these convergences are extremely important in applications such as PDEs, applied mathematics, stochastics, and others.

3. In contrast with the basic integration theory, it is inconvenient to allow the functions to be infinite on sets of positive measure, when dealing with convergences.

   For example, if $f_n \to f$ in μ as $n \to \infty$, then f must be μ-a.e. finite. Indeed, suppose $f|_E = \infty$ with $\mu(E) = m_0 > 0$. If all f_n are μ-a.e. finite on E, then $\mu(\{|f_n - f| > t\}) \ge m_0$ for all n and any $t > 0$, which contradicts the convergence in μ. The only way to save the convergence is to allow f_n be infinite on a set of positive measure. But then we have to clarify what do we mean by
   $$f_n - f = \infty - \infty$$
   on a set of positive measure. Unfortunately there is no useful universal agreement for " $\infty - \infty$ ", even though in some particular problems it is possible to assign a meaning to the equality.

   To make the exposition concise we make the standing assumption *in all statements in this chapter all functions are assumed to be measurable* **and finite** *almost everywhere.*

4. The a.e. convergence (say, the pointwise convergence) is the most elementary notion. It happens to be the most subtle and the most difficult to deal with. For example, the theory for λ^1-a.e. convergence of the e^{ikx}-Fourier series is substantially more complicated than, say, the L^2-convergence.

5. Some relations between different convergences are straightforward. Prove that the convergence uniform on Ω implies all other convergences, assuming that $\mu(\Omega) < \infty$. Prove that the convergence in L^p, $1 \le p \le \infty$ implies the convergence in measure (use the Tchebyshev's inequality for a finite p).

6. Let $\mathcal{F} = \mathcal{F}(\Omega)$ be the set of all measurable functions which are finite μ-a.e. in Ω. Suppose $\mu(\Omega) < \infty$. For $f, g \in \mathcal{F}$ define

$$d(f,g) = d_\mu(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|}\, d\mu.$$

Prove that:

(a) $\mathcal{F}$ is a vector space;

(b) $(\mathcal{F}, d)$ is a metric space with the translation invariant metric, $d(f,g) = d(f+h, g+h)$;

(c) $d(\cdot,\cdot)$ metrises the convergence in measure, that is $f_n \to f$ in μ as $n \to \infty$ if and only if $d(f_n, f) \to 0$, $n \to \infty$;

(d) $(\mathcal{F}, d)$ is a *complete* metric space (this is a more difficult problem, do it last).

It is possible to prove (we will not do it) that there does not exist a norm $\|\cdot\|_\mu$ on $\mathcal{F}$ such that the convergence in μ is equivalent to the convergence in the normed space $(\mathcal{F}, \|\cdot\|_\mu)$.

7. First we investigate the relation between the μ-a.e. convergence and the convergence in μ. The crucial role will be played by the following simple observations. Recall, that for a sequence of sets $\{S_n\}$

$$\limsup_{n\to\infty} S_n = \{\text{points which belong to infinitely many } S_n\}.$$

For a sequence of measurable functions $\{\varphi_j\}$, $\varphi_j \ge 0$,

$$\limsup_{j\to\infty} (\{\varphi_j \ge t\}) \subset \left\{ x \in \Omega\colon \limsup_{j\to\infty} \varphi_j(x) \ge t \right\}.$$

Prove this. Can we replace the inclusion by the equality here? (Prove or give an example.) In the opposite direction, prove that

$$\limsup_{j\to\infty} (\{\varphi_j > t\}) \supset \left\{ x \in \Omega\colon \limsup_{j\to\infty} \varphi_j(x) > t \right\}.$$

Can we replace the inclusion by the equality here? (Prove or give an example.)

8. **Theorem 1** *Let* $\{f_j\}$ *be a sequence, such that* $f_j \to f$ μ*-a.e.,* $j \to \infty$*. Suppose that* $\mu(\Omega) < \infty$*. Then* $f_j \to f$ *in* μ *as* $j \to \infty$*.*

Proof. **1.** Set $g_j = |f_j - f|$. Then $\{g_j\}$ is a sequence of measurable functions, $g_j \geq 0$, such that $g_j \to 0$ μ-a.e. in Ω as $j \to \infty$. For any $t > 0$ define

$$E_j^t = \{x \in \Omega \colon g_j(x) \geq t\}. \tag{4.1}$$

We must show that for any $t > 0$

$$\mu(E_j^t) \to 0 \quad \text{as} \quad j \to \infty.$$

2. For any $t > 0$ the inclusion

$$\limsup_j \left(E_j^t\right) \subset \left\{x \in \Omega \colon \limsup_{j \to \infty} g_j(x) \geq t\right\} \stackrel{\text{def}}{=} L_t$$

holds. At the same time clearly

$$\left\{g_j \to 0 \ \mu\text{–a.e., } j \to \infty\right\} \Longleftrightarrow \left\{\mu(L_t) = 0 \ \forall t > 0\right\}.$$

To use this fact, recall the expression

$$\limsup_j \left(E_j^t\right) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j^t.$$

Hence by the continuity of μ along the monotone sequences of sets

$$\begin{aligned} 0 = \mu(L_t) &\geq \mu\left(\limsup_j E_j^t\right) \\ &= \lim_{k \to \infty} \mu\left(\bigcup_{j=k}^{\infty} E_j^t\right) \\ &\geq \limsup_{k \to \infty} \mu(E_k^t). \end{aligned}$$

□

9. Where in the proof of Theorem 1 we used the assumption $\mu(\Omega) < \infty$? Prove that the Theorem 1 does not hold in general if $\mu(\Omega) = \infty$. Prove that the Theorem 1 does not hold in general if f is allowed to be infinite on a set of positive measure.

10. In general, convergence in measure (or convergence in L^p for a finite p) does not imply the a.e. convergence . Show this by example of the sequence of functions on $[0,1]$ given by the $\chi_{\Delta_{j_N}^{(N)}}$ for suitable j_N and N. Here $\Delta_j^{(N)} = [(j-1)2^{-N}, j2^{-N})$ is the dyadic interval. However, the implication holds if we pass to a subsequence.

11. **Theorem 2** *Let* $\mu(\Omega) < \infty$ *, and let* $\{f_j\}$ *be a sequence, such that* $f_j \to f$ *in* μ *as* $j \to \infty$ *. Then there exists a subsequence* $\{j_k\}$ *such that* $f_{j_k} \to f$ μ*-a.e when* $k \to \infty$ *.*

Proof. **1.** Set $g_j = |f_j - f|$. For any $\varepsilon > 0$ define

$$E_j^\varepsilon = \{g_j > \varepsilon\}.$$

(Notice the strict inequality here.) Then $\{g_j\}$ is a sequence of measurable functions, $g_j \geq 0$, such that

$$\mu(E_j^\varepsilon) \to 0 \quad \text{as} \quad j \to \infty. \tag{4.2}$$

We must find a subsequence $\{j_k\}$ such that $g_{j_k} \to 0$ μ-a.e. in Ω as $k \to \infty$.

2. Take any vanishing sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$ as $k \to \infty$ (for example $\varepsilon_k = 1/k$). Observe that for any subsequence $\{j_k\}$

$$\left\{x \in \Omega \colon \limsup_{k\to\infty} g_{j_k}(x) > 0\right\} \subset \limsup_{k\to\infty} E_{j_k}^{\varepsilon_k}. \tag{4.3}$$

The set in the left hand side is exactly the set where the sequence $\{g_{j_k}\}$ does not limit to 0 pointwisely. Our goal is to control its size.

3. We will choose the desired subsequence $\{j_k\}$. Test (4.2) with $\varepsilon = 1$ to find j_1 such that $\mu(E_{j_1}^1) < 1/2$. Then by induction test (4.2) with $\varepsilon = 1/k$ to find j_k such that

$$j_{k-1} < j_k, \quad \mu(E_{j_k}^{1/k}) < 2^{-k}, \quad k = 2, 3, \ldots.$$

Our goal is to estimate the measure of the set on which the convergence of $\{g_{j_k}\}$ to 0 fails. By (4.3) it is enough to show that

$$\mu\left(\limsup_k E_{j_k}^{1/k}\right) = 0.$$

But according to our choice

$$\sum_{k=1}^\infty \mu(E_{j_k}^{1/k}) \leq \sum_{k=1}^\infty 2^{-k} < \infty.$$

One of the earlier home work problems showed that for any set sequence $\{F_k\}$

$$\sum_k \mu(F_k) < \infty \Longrightarrow \mu\left(\limsup_k F_k\right) = 0.$$

This general statement is called Borel-Cantelli's lemma. It proves the theorem.

4. In order to make the exposition independent of the home assignments, let us prove the Borel-Cantelli's lemma. By the definition

$$\limsup_{k\to\infty} F_k = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} F_n .$$

From the continuity of μ along the monotone sequences of sets, derive that

$$\begin{aligned} \mu\left(\limsup_{k\to\infty} F_k\right) &= \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} F_n\right) \\ &\leq \lim_{k\to\infty} \left(\sum_{n=k}^{\infty} \mu\left(F_n\right)\right) \\ &= 0, \end{aligned}$$

where the last step holds due to the convergence of the series $\sum \mu(F_n)$.
□

12. Where in the proof did we use $\mu(\Omega) < \infty$? Prove that Theorem 2 holds for $\mu = \lambda^1$ and $\Omega = \mathbf{R}$ despite $\lambda^1(\mathbf{R}) = \infty$.

    Suppose $f_n \to f$ in $L^1(\mathbf{R})$ as $n \to \infty$, and let $f_n \geq 0$ for all n. Show that $f \geq 0$ almost everywhere.

13. The Egorov's theorem is important in applications. It asserts that the μ-a.e. convergence is uniform provided we throw away a set of arbitrary small measure.

Theorem 3 *Let* $\mu(\Omega) < \infty$, *and let* $f_k \to f$ μ*-a.e.. Then for any* $\varepsilon > 0$ *there exists a set* $F = F_\varepsilon \subset \Omega$ *such that:*

(i) $\mu(F) < \varepsilon$;

(ii) $f_k \to f$ *uniformly on* $\Omega \setminus F$.

Proof. **1.** We start as in the proof of Theorem 1. Set $g_j = |f_j - f|$. Then $\{g_j\}$ is a sequence of measurable functions, $g_j \geq 0$, such that $g_j \to 0$ μ-a.e. in Ω as $j \to \infty$. For any $t > 0$ define

$$E_j^t = \{x \in \Omega\colon g_j(x) \geq t\} .$$

Then

$$\limsup_j \left(E_j^t\right) \subset \left\{x \in \Omega\colon \limsup_{j\to\infty} g_j(x) \geq t\right\} .$$

The μ-a.e. convergence to 0 of $\{g_j\}$ implies that the set in the right hand side has measure 0. Hence

$$0 = \mu\left(\limsup_j E_j^t\right) = \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j^t\right) .$$

Consequently, the continuity of μ along the monotone sequences of sets implies

$$\lim_{k\to\infty} \mu\left(F_k^t\right) = 0, \tag{4.4}$$

where we defined

$$F_k^t = \bigcup_{j=k}^{\infty} E_j^t.$$

The sets F_k^t can be linked with the uniform convergence of g_k. Namely, on one hand

$$\sup_{\Omega\setminus F_k^t} |g_j| \le t \quad \forall j \ge k. \tag{4.5}$$

On the other hand, we must find the set F of small measure such that

$$\sup_{\Omega\setminus F} |g_j| \to 0 \quad \text{as} \quad j\to\infty.$$

2. Fix any $\varepsilon > 0$. Test (4.4) with $t = 1/m$ for $m = 1, 2, \ldots$. For each m utilise (4.4) to find an integer N_m such that

$$\mu\left(F_{N_m}^{1/m}\right) < \varepsilon 2^{-m}.$$

Then for

$$F = \bigcup_{m=1}^{\infty} F_{N_m}^{1/m}$$

we derive that

$$\mu(F) \le \sum_{m=1}^{\infty} \mu\left(F_{N_m}^{1/m}\right) < \varepsilon.$$

At the same time we claim that

$$\sup_{F^c} |g_j| \to 0 \quad \text{as} \quad j\to\infty.$$

Indeed, by the definitions $F^c \subset \left(F_{N_m}^{1/m}\right)^c$ for any m. Therefore by (4.5) for any $m = 1, 2, \ldots$ we have

$$\sup_{\Omega\setminus F} |g_j| \le 1/m \quad \forall j \ge N_m.$$

This implies $g_j \to 0$ uniformly on $\Omega\setminus F$. □

14. Where in the proof did we use that μ is finite? Prove that Egorov's theorem does not hold in general if $\mu(\Omega) = \infty$. Prove that Egorov's theorem does not hold in general if f is not finite a.e..

15. An application of Egorov's theorem is the criterion for the L^1-convergence of a sequence converging μ-a.e.. Recall, that for a general μ-a.e converging sequence we cannot pass to the limit under the integral sign. (Why?)

To state the criterion we need the notion of the *uniform integrability.* A family of measurable functions $\{\phi_\alpha\}_{\alpha\in\mathbf{A}}$, $\phi_\alpha : \Omega \to \overline{\mathbf{C}}$, is called uniformly integrable if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_E |\phi_\alpha| \, d\mu < \varepsilon \quad \forall E \text{ with } \mu(E) < \delta \quad \forall \alpha \in \mathbf{A}. \tag{4.6}$$

Here the "uniform" means that the same δ works simultaneously for all $\alpha \in \mathbf{A}$.

Theorem 4 *Let* $\mu(\Omega) < +\infty$, $f_n \in L^1$, $n = 1, 2, \ldots,$ *and let* $f_n \to f$ μ*-a.e. as* $n \to \infty$. *Then:*

$$\Big\{\{f_n\}_{n\in\mathbf{N}} \text{ is uniformly integrable}\Big\} \Longleftrightarrow \Big\{f \in L^1 \text{ and } \lim_{n\to\infty} \|f_n - f\|_1 = 0\Big\}.$$

Informally, the theorem states that the μ-a.e. convergence implies the L^1 convergence if and only if there is *no concentration.*

16. Construct an L^1-sequence (f_j) converging a.e. to some $f \in L^1$, for which $f_j \not\to f$ in L^1. Show directly (without using the previous theorem) that the family $\{f_j\}$ is not uniformly integrable. How would you describe the concentration phenomenon in your example?

17. *Proof of Theorem 4.* **1.** Proof of $\Rightarrow$. Fix any $\varepsilon > 0$. By the uniform integrability of the family $\{f_n\}_{n\in\mathbf{N}}$ find $\delta = \delta_\varepsilon$ such that (4.6) holds. Since μ is finite the Egorov's theorem can be applied. Consequently for this $\delta > 0$ we find $F_\delta \subset \Omega$ such that $\mu(F_\delta) < \delta$ and $f_n \to f$ uniformly on $\Omega \setminus F_\delta$ as $n \to \infty$. By the Fatou's lemma we deduce that

$$\begin{aligned} \int_{F_\delta} |f| \, d\mu &= \int_{F_\delta} \lim_{n\to\infty} |f_n(x)| \, d\mu(x) \\ &\le \liminf_{n\to\infty} \int_{F_\delta} |f_n| \, d\mu \\ &\le \varepsilon. \end{aligned}$$

On the set $\Omega \setminus F_\delta$ (which has a finite measure) the uniform convergence of $\{f_n\}$ implies the L^1-convergence. Hence

$$\begin{aligned} \int_\Omega |f_n - f| \, d\mu &= \int_{F_\delta} |f_n - f| \, d\mu + \int_{\Omega\setminus F_\delta} |f_n - f| \, d\mu \\ &\le \int_{F_\delta} |f_n| \, d\mu + \int_{F_\delta} |f| \, d\mu + \int_{\Omega\setminus F_\delta} |f_n - f| \, d\mu \\ &\le 2\varepsilon + \|f_n - f\|_{L^1(\Omega\setminus F_\delta)}. \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \|f_n - f\|_1 \le 2\varepsilon,$$

which shows that $f_n \to f$ in L^1 as $n \to \infty$.

2. Proof of $\Leftarrow$. Fix any $\varepsilon > 0$. Find N_ε such that

$$\|f_n - f\|_1 < \varepsilon \quad \forall\, n > N_\varepsilon.$$

Next, use the absolute continuity of the integral to find $\delta = \delta_\varepsilon$, such that for any $E \subset \Omega$ with $\mu(E) < \delta$ the estimate

$$\int_E (|f| + |f_1| + \cdots + |f_{N_\varepsilon}|)\, d\mu < \varepsilon \tag{4.7}$$

holds. Then the implication

$$\mu(E) < \delta \Longrightarrow \int_E |f_n|\, d\mu < \varepsilon$$

holds for $n = 1, \ldots, N_\varepsilon$. On the other hand, for any $n > N_\varepsilon$ and any E with $\mu(E) < \delta$ we use (4.7) to derive

$$\begin{aligned}
\int_E |f_n|\, d\mu &= \int_E |f_n - f + f|\, d\mu \\
&\le \int_E (|f_n - f| + |f|)\, d\mu \\
&\le \|f_n - f\|_1 + \int_E |f|\, d\mu \\
&< 2\varepsilon.
\end{aligned}$$

□