6 Classical dualities and reflexivity

1. **Classical dualities.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We will describe the duals for the Banach spaces $L^p(\Omega)$.

   First, notice that any $f \in L^{p'}$, $1 \le p \le \infty$, generates the linear functional F_f on L^p by the rule

$$F_f(u) = \int_\Omega uf\, d\mu, \quad u \in L^p.$$

   Indeed, by the Hölder inequality

$$\begin{aligned} |F_f(u)| &= \left| \int_\Omega fu\, d\mu \right| \\ &\le \|f\|_{p'} \|u\|_p. \end{aligned}$$

   Thus $\|F_f\|_* \le \|f\|_{p'}$. Consequently the mapping

$$\begin{aligned} I_p \colon L^{p'} &\longrightarrow L^{p*} \\ f &\longmapsto F_f, \end{aligned}$$

   $1 \le p \le \infty$, is a one to one bounded linear operator with

$$\|I_p\|_{\mathcal{B}(L^{p'}, L^{p*})} \le 1.$$

   The classical spaces l^p, $1 \le p \le \infty$, fit in this picture since

$$\|f\|_{l^p} = \|f\|_{L^p(\mathbf{N}, \mu_c)}$$

   for any sequence $f \colon \mathbf{N} \to \mathbf{C}$. Here μ_c is the counting measure. The classical duality theorem due to Riesz states that under some additional (rather general) assumptions, I_p is a linear bijective isometry for $p < \infty$.

2. **Theorem 1** *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Then for $1 \le p < \infty$ the map I_p is a linear bijective isometry for $L^{p'} \cong L^{p*}$. That is, for any $\phi \in L^{p*}$ there exist $f \in L^{p'}$, such that*

$$\langle \phi, u \rangle = \int_\Omega uf\, d\mu \quad \forall u \in L^p,$$

   and $\|\phi\|_* = \|f\|_{p'}$.

   The Riesz representation theorem does not hold for $p = \infty$. The relations

$$\begin{aligned} \langle I_p f, u \rangle &= \int_\Omega uf\, d\mu, \quad \forall f \in L^{p'}, u \in L^p, \\ \langle \phi, u \rangle &= \int_\Omega u I_p^{-1}(\phi)\, d\mu \quad \forall \phi \in L^{p*}, u \in L^p, \end{aligned}$$

   are useful for the problem solving.

3. For the classical spaces of sequences we have a substitute for the Riesz representation theorem in the case $p=\infty$. Every sequence in l^1 generates a bounded linear functional on l^∞. For the narrower space $c_0 \subset l^\infty$ this procedure gives all linear functionals.

   Theorem 2 *The map* $I_0\colon l^1 \to c_0^*$ *given by*

$$\langle I_0 f, u\rangle = \sum_{n=1}^{\infty} u_n f_n$$

   is a linear bijective isometry.

4. **Reflexivity.** Let $X=(X,\|\cdot\|)$ *an arbitrary* normed space over $\mathbf{C}$ (or $\mathbf{R}$). For a fixed $x\in X$ consider the map

$$\begin{aligned} F_x\colon X^* &\longrightarrow \mathbf{C} \\ \phi &\longmapsto \langle \phi, x\rangle. \end{aligned}$$

   Clearly $F_x \in (X^*)'$. Moreover, by the dual expression of the norm

$$\begin{aligned} \|F_x\|_{X^{**}} &= \sup_{\|\phi\|_*=1} F_x(\phi) \\ &= \sup_{\|\phi\|_*=1} \langle \phi, x\rangle \\ &= \|x\|. \end{aligned}$$

   Hence the mapping

$$\begin{aligned} i_{\mathrm{can}}\colon X &\longrightarrow X^{**} \\ x &\longmapsto F_x \end{aligned}$$

   is a linear isometry, $\|i_{\mathrm{can}} x\|_{**} = \|x\|$. Thus we obtain a canonical isometric embedding of X into X^{**}.

   The standard (somewhat confusing) way of stating this is

   " X is a subspace of X^{**} ".

   What is really meant by this, is the canonical identification $X \cong i_{\mathrm{can}}(X)$ with the subspace $i_{\mathrm{can}}(X) \subset X^{**}$.

5. A Banach space X is called *reflexive* if the canonical isometry i_{can} is an onto mapping (and hence i_{can} is a linear bijective isometry $X \cong X^{**}$).

   In general

$$i_{\mathrm{can}}(X) \subset X^{**}, \quad i_{\mathrm{can}}(X) \neq X^{**},$$

   is a proper algebraic subspace. Indeed, X^{**} is complete as a dual of a normed space. Thus

$$i_{\mathrm{can}}(X) = X^{**} \Longrightarrow X \ \text{ is complete.}$$

However, the completeness is not sufficient for the reflexivity. For many Banach spaces arising in the applications the inclusion is strict. Observe that

$$X \text{ is complete} \Longrightarrow i_{\text{can}}(X) \text{ is a } \textit{closed} \text{ subspace of } X$$

(why?).

6. The reflexivity is a property of the map i_{can}. According to the definition a mere existence of a linear bijective isometry $X \cong X^{**}$ is not enough for reflexivity. Therefore the standard notation for the reflexivity,

$$X = X^{**},$$

is somewhat confusing. To prove that X is reflexive one must in fact prove that

$$\forall F \in X^{**} \qquad \text{there is } x_F \in X \text{ such that } F(\omega) = \langle \omega, x_F \rangle \ \forall \omega \in X^*.$$

Reflexivity happens to be equivalent to other important properties of Banach spaces.

7. For problem solving the following formulae are useful. The isometric inclusion

$$i_{\text{can}} \colon X \longrightarrow X^{**}$$

is defined for any normed X, and

$$\langle \omega, x \rangle = i_{\text{can}}(x)(\omega) \quad \forall \omega \in X^*, \ \forall x \in X. \tag{6.1}$$

If X is reflexive then i_{can}^{-1} is defined everywhere on X^{**}, and

$$F(\omega) = \langle \omega, i_{\text{can}}^{-1}(F) \rangle \quad \forall \omega \in X^*, \ \forall F \in X^{**}. \tag{6.2}$$

8. To illustrate typical arguments for dealing with the reflexivity let us prove that for a Banach space X

$$X \text{ is reflexive} \Longleftrightarrow X^* \text{ is reflexive.}$$

We denote vectors in X by $u, v, \ldots$, functionals in X^* by $\phi, \omega, \ldots$, functionals in X^{**} by $F, G, \ldots$, and functionals from X^{***} by $\Omega, \Psi, \ldots$.

$\Rightarrow$ Fix $\Omega_0 \in X^{***}$. Let us find $\omega_0 \in X^*$ such that

$$\Omega_0(F) = F(\omega_0) \quad \forall F \in X^{**}.$$

In fact, consider the map $\Omega_0 \circ i_{\text{can}} \colon X \to \mathbf{C}$ and set $\omega_0 = \Omega_0 \circ i_{\text{can}}$. Then $\omega_0 \in X^*$ and

$$\langle \omega_0, u \rangle = \Omega_0(i_{\text{can}} u) \quad \forall u \in X.$$

But i_{can} is a bijection. Hence we can test the last formula with $u = i_{\text{can}}^{-1}(F)$ to derive that

$$\langle \omega_0, i_{\text{can}}^{-1}(F) \rangle = \Omega_0(F) \quad \forall F \in X^{**}.$$

By (6.2)

$$\langle \omega_0, i_{\text{can}}^{-1}(F) \rangle = F(\omega_0),$$

and ω_0 is the desired functional.

$\Leftarrow$ Seeking a contradiction suppose that $i_{\text{can}}(X) \neq X^{**}$. Then since X is Banach, $i_{\text{can}}(X)$ is a *proper* closed subspace of X^{**}. Use the Hahn-Banach theorem to find $\Omega_0 \in X^{***}$ such that

$$\Omega_0|_{i_{\text{can}}(X)} = 0, \quad \Omega_0 \neq 0.$$

Utilise the reflexivity of X^* to find $\omega_0 \in X^*$, such that

$$\Omega_0(F) = F(\omega_0) \quad \forall F \in X^{**},$$

and therefore $\omega_0 \neq 0$. Test the last formula with $F = i_{\text{can}}(u)$, $u \in X$, to derive that

$$\begin{aligned} 0 &= \Omega_0(i_{\text{can}} u) \\ &= (i_{\text{can}} u)(\omega_0) \\ &= \langle \omega_0, u \rangle \end{aligned}$$

for all $u \in X$. Thus $\omega_0 = 0$, which is a contradiction.

9. Suppose $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. Let us prove that $L^p = L^p(\Omega, \mu)$ is reflexive provided $1 < p < \infty$.

   In fact, fix any $F \in (L^{p*})^*$. We must find $f \in L^p$, $f = f_F$, such that

$$F(\xi) = \langle \xi, f \rangle \quad \forall \xi \in L^{p*}.$$

   Recall the isometric identification $L^{p'} \cong L^{p*}$ through the map $I_p \colon L^{p'} \to L^{p*}$,

$$\langle \xi, f \rangle = \int_\Omega I_p^{-1}(\xi)\, g \, d\mu \quad \forall g \in L^p,\ \xi \in L^{p*}.$$

   Hence $F \circ I_p \colon L^{p'} \to \mathbf{C}$ is linear and bounded. But then we can utilise the identification $L^p \cong (L^{p'})^*$. Therefore for our $F \circ I_p \in L^{p'*}$ we find a unique $f \in L^p$ such that

$$(F \circ I_p)(u) = \int_\Omega u f \, d\mu \quad \forall u \in L^{p'}.$$

Since I_p is a bijection we can test the last formula with $u = I_p^{-1}(\xi)$, $\xi \in L^{p*}$. We derive that

$$\begin{aligned} F(\xi) &= \int_\Omega I_p^{-1}(\xi)\, f\, d\mu \\ &= \langle \xi, f \rangle \end{aligned}$$

for all $\xi \in L^{p*}$. $\square$

10. Let us show that the space c_0 is not reflexive.That is, we must show that the map $i_{\text{can}} \colon c_0 \to c_0^{**}$ is not a bijective isometry.

    What is the easiest way to see that $i_{\text{can}} \colon X \to X^{**}$ fails to be an isometric bijection? This failure clearly holds if for some reason there are *no linear bijective isometries* between X and X^{**}. Separability is one of the simplest invariants of Banach spaces with respect to linear bijective isometries. Hence, if X is separable but X^{**} is not, there can be no linear bijective isometry between them. In particular i_{can} cannot be one.

    Now, seeking a contradiction suppose that c_0 is reflexive. It is separable. Therefore c_0^{**} must be separable. At the same time $c_0^* \cong l^1$ and $l^{1*} \cong l^\infty$. It is easy to see that $X \cong Y \;\Rightarrow\; X^* \cong Y^*$ (indeed, if the isometry m provides $X \cong Y$, then m^* provides $Y^* \cong X^*$). Thus $c_0^{**} \cong l^\infty$ (write the isometry map explicitly using the classical duality maps I_p). But then c_0^{**} is not separable since l^∞ is not separable. $\square$

7 Analysis in $C(K)$

1. In this section we will deal with measures and functions on a *compact topological space* K.

   The results that we establish, admit generalisations (in a suitably corrected form) to *locally compact* spaces. We shall not describe these extensions here.

2. **Stone-Weierstrass and related theorems.** Let K be a compact space.

3. **Riesz representation theorem.** Let us first describe the motivating problem.

   Suppose K is a compact space. Let $C(K)$ be the space of all *real valued* continuous functions on K. We know that $C(K)$ is a Banach space over $\mathbf{R}$.

   As any topological space, K has a natural σ-algebra. Namely, it is the Borel σ-algebra generated by the topology of K,

$$\mathcal{B}(K) = \sigma(\{\text{all open subsets of } K\}).$$

Consider the Banach space $\mathfrak{M} = \mathfrak{M}(K, \mathcal{B}(K))$ of all charges on $(K, \mathcal{B}(K))$ with the total variation norm

$$\|\phi\| = |\phi|(K), \quad \phi \in \mathfrak{M}.$$

For example, the Dirac mass $\delta_p \in \mathfrak{M}$ for any $p \in K$.

Any $\nu \in \mathfrak{M}$ generates a bounded linear functional on $C(K)$ via the integration. Indeed, let

$$\nu = \nu^+ - \nu^-, \quad \nu^\pm \geq 0,$$

be the Jordan decomposition of ν. That is $\nu^+ = \nu \llcorner P$, $\nu^- = -\nu \llcorner N$, where P, N are the positive and the negative sets in the Hahn decomposition of K with respect to ν. Since any $u \in C(K)$ is $\mathcal{B}(K)$-measurable (why?), we may write

$$F_\nu(u) = \int_K u \, d\nu^+ - \int_K u \, d\nu^-. \tag{7.1}$$

The inequality

$$\begin{aligned} |F_\nu(u)| &\leq \left| \int_K u \, d\nu^+ \right| + \left| \int_K u \, d\nu^- \right| \\ &\leq \max_K |u| \left(\nu^+(K) + \nu^-(K) \right) \\ &= \|u\|_{C(K)} \|\nu\| \end{aligned}$$

implies that $F_\nu \in C(K)^*$ with $\|F_\nu\|_* \leq \|\nu\|$.

It is easy to see that $\|F_\nu\|_* = \|\nu\|$ for $\nu \in \mathfrak{M}$, $\nu \geq 0$ (why?).

These considerations raise the natural questions: is it true that $\|F_\nu\|_* = \|\nu\|$? does any functional in $C(K)^*$ come from some Borel charge on K?

4. The *classical* Riesz representation theorem provides the complete answers in the case of $K = [a, b]$, $a, b \in \mathbf{R}$, or more generally for a compact set $K \subset \mathbf{R}^n$.

5. Let K be a compact space. A linear functional $\Phi \in C(K)'$ is called *positive* if

$$f \in C(K), f \geq 0 \text{ on } K \Longrightarrow \langle \Phi, f \rangle \geq 0.$$

Positive functionals are continuous (why?).

6. The classical Riesz representation theorem is the following statement.

Theorem 3 *Let $K \subset \mathbf{R}^n$ be a compact set. Let $\mathfrak{M} = \mathfrak{M}(K, \mathcal{B}_n)$ be the space of Borel charges on K. The map*

$$\begin{aligned} i \colon \mathfrak{M} &\longrightarrow C(K)^* \\ \phi &\longmapsto F_\phi \end{aligned}$$

is a linear bijective isometry. If ϕ is a positive functional, then $i^{-1}\phi$ is a finite positive measure.

In particular, the theorem states that for any positive functional $\Phi \in C(K)^*$ there exists a positive Borel measure μ on K such that

$$\mu(K) = \|\Phi\|_{C(K)^*},$$

and

$$\langle \Phi, u \rangle = \int_K u\, d\mu \quad \forall u \in C(K).$$

Even in the case $K = [a, b]$ the proof of the theorem is nontrivial. We will prove the more general theorem below.

7. Let ν be a positive Borel measure in $\mathbf{R}^n$. By the Riesz theorem the following dual expression for the measure of a compactum $K \subset \mathbf{R}^n$ holds:

$$\nu(K) = \sup \left\{ \int_K u\, d\nu\colon\ u \in C(\mathbf{R}^n),\ \|u\|_C \le 1 \right\}.$$

   As any dual expression it can be useful in applications.

8. The extension of the classical Riesz theorem to general compact spaces reveals some subtle issues for Borel measures on (compact) topological spaces.

   We will prove the general Riesz representation theorem for *Hausdorff compact space* K. This is an additional restriction on K. The basic example of Hausdorff compact spaces is a compact metric space. Many applications require only the metric space case. We will see that the general case is subtler.

9. Let K be a compact space. A positive Borel measure μ is called *Borel regular* (or regular with respect to the topology of K) if for every $E \in \mathcal{B}(K)$

$$\begin{aligned} \mu(E) &= \inf\{\mu(O)\colon\ E \subset O - \text{open}\} \\ &= \sup\{\mu(F)\colon\ E \supset F - \text{closed}\} \end{aligned}$$

   A charge $\phi \in \mathfrak{M}(K, \mathcal{B}(K))$ is Borel regular if its total variation $|\phi|$ is a Borel regular measure. This is equivalent to $\phi^{\pm}$ being regular in the Jordan decomposition $\phi = \phi^+ - \phi^-$, $\phi^{\pm} \ge 0$.

   For example, if $K \subset \mathbf{R}^n$ then $\lambda^n \llcorner K$ is Borel regular. We proved that during our study of the Lebesgue measure.

   We define

$$\mathfrak{M}_0 = \{\phi \in \mathfrak{M}\colon\ \phi \text{ is regular}\}.$$

10. Surprisingly, there exist compact Hausdorff spaces K for which $\mathfrak{M}_0 \ne \mathfrak{M}$. On the positive side the following statement holds.

Theorem 4 *Let* K *be a compact space. Then* $\mathfrak{M}_0(K)$ *is a closed subspace of the Banach space* $\mathfrak{M}(K)$ *.*
Let M *be a compact metric space. Then* $\mathfrak{M}_0(M) = \mathfrak{M}(M)$ *.*

Thus, every Borel measure on a compact metric space is Borel regular. This is not true in a general compact Hausdorff space.

11. The exact generalisation of the classical Riesz theorem holds for a compact metric space.

Theorem 5 *Let* M *be a compact metric space. The map*

$$\begin{aligned} i\colon \mathfrak{M} &\longrightarrow C(M)^* \\ \phi &\longmapsto F_\phi \end{aligned}$$

is a linear bijective isometry. If ϕ *is a positive functional, then* $i^{-1}\phi$ *is a finite positive measure on* M *.*

Thus, for a compact metric space, the dual of the space of continuous functions can be identified with the space of Borel charges.

12. We will prove the following version of the Riesz theorem, which implies the previous statements.

Theorem 6 *Let* K *be a compact Hausdorff space.*

(a) For any positive $\Lambda \in C(K)^*$ *there exists a unique Borel regular measure* $\mu \in \mathfrak{M}_0$ *such that*

$$\langle \Lambda, u\rangle = \int_K u\,d\mu \quad \forall u \in C(K). \tag{7.2}$$

Moreover, $\|\Lambda\|_{C(K)^*} = \|\mu\|$ *.*

(b) The map

$$\begin{aligned} i\colon \mathfrak{M}_0 &\longrightarrow C(K)^* \\ \nu &\longmapsto F_\nu \end{aligned}$$

defined by (7.1) is a linear bijective isometry.

Thus, for compact Hausdorff spaces, the dual of the space of continuous functions can be identified with the space of *Borel regular* charges.

13. Notice the following interesting consequence of the theorem. Take any non-regular measure $\nu \in \mathfrak{M}(K)$. It still generates the positive functional $F_\nu \in C(K)^*$ according to (7.1). Apply now the Riesz representation

theorem to F_ν . We derive, that for any Borel measure ν on the compact Hausdorff K , there exists a *regular* measure ν_0 on K , such that

$$\int_K u\,d\nu = \int_K u\,d\nu_0 \quad \forall u \in C(K).$$

Thus, as far as the action on continuous functions is concerned, all Borel measures on a compact Hausdorff space can be assumed to be regular.

14. The proof of Theorem 6 will require Urysohn's theorem (and related statements) for compact Hausddorff spaces. In the case of Stone-Weierstrass theorem we needed the Hausdorff axiom only for the case of *locally* compact spaces. For the Riesz representation theorem the Hausdorff axiom is used even for the compact spaces.

In what follows K denotes a fixed compact Hausdorff space.

Let $u \in C(K)$. Define the *support of* u by writing

$$\mathrm{supp} u = \overline{\{x \in K :\ u(x) \neq 0\}}.$$

Thus $\mathrm{supp} u$ is a closed set. For $u \in C(K)$ we write

$$u \prec O$$

if $O \subset K$ is open, $0 \leq u \leq 1$, and $\mathrm{supp} u \subset O$. We write

$$F \prec u$$

if $F \subset K$ is closed, $0 \leq u \leq 1$, and $u|_F = 1$.

15. The *Urysohn's lemma* asserts that for $F \subset O \subset K$, F closed, O open, there exists $\varphi \in C(K)$ such that

$$F \prec \varphi \prec O.$$

16. We will need a slightly more general statement called *the partition of unity lemma*. For a closed (and hence compact) $F \subset K$ and open sets $O_{1,\dots,N}$, such that

$$F \subset O_1 \cup \dots \cup O_N,$$

there exist N functions $\varphi_i \in C(K)$, such that

$$\varphi_i \prec O_i \quad i = 1, \dots, N,$$

and

$$F \prec \varphi_1 + \dots + \varphi_N.$$

The functions $\varphi_{1,\dots,N}$ are called the *partition of unity* on F corresponding to the cover $O_{1,\dots,N}$.

If $N = 1$ then the statement is just the original Urysohn's lemma. The general case is derived from it by an inductive argument.

The partition of the unity will be the main technical tool in the proof of the Riesz theorem.

17. *Proof of Theorem 6, part (a).* **1.** Fix $\Lambda \in C(K)^*$. We use Λ to define an outer measure on 2^K. First, for an open $O \subset K$ set

$$\lambda(O) \stackrel{\text{def}}{=} \sup \{\langle \Lambda, u\rangle : \ u \prec O\}.$$

Then, for any $A \subset K$ write

$$\lambda(A) \stackrel{\text{def}}{=} \inf \{\lambda(O) : \ A \subset O - \text{open}\}. \tag{7.3}$$

If A is open we get the same value since λ is monotone with respect to inclusion. Since K is compact

$$\lambda(A) \le \langle \Lambda, 1\rangle < \infty.$$

We shall prove that

$$\lambda \colon 2^K \longrightarrow [0, \infty)$$

enjoys the following properties

(a) $\lambda(\emptyset) = 0$ (triviality);

(b) $A_1 \subset A_2 \Longrightarrow \lambda(A_1) \le \lambda(A_2)$ (monotonicity);

(c) for an arbitrary sequence A_j, $j = 1, 2, \dots$,

$$\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \lambda(A_j) \tag{7.4}$$

(semiadditivity).

In other words, λ is an outer measure on K.

2. Indeed, the triviality and monotonicity clearly hold. Let us prove semiadditivity (7.4).

First, we claim that for open $O_{1,2}$

$$\lambda(O_1 \cup O_2) \le \lambda(O_1) + \lambda(O_2). \tag{7.5}$$

Indeed, take any $u \prec O_1 \cup O_2$. Thus the compactum $\operatorname{supp} u$ is covered by $O_{1,2}$. Let $\varphi_{1,2}$ be the partition of unity on $\operatorname{supp} u$ subordinated to this cover. Then $u = u\varphi_1 + u\varphi_2$ with $u\varphi_i \prec O_i$. Hence

$$\begin{aligned} \langle \Lambda, u\rangle &= \langle \Lambda, u\varphi_1\rangle + \langle \Lambda, u\varphi_2\rangle \\ &\le \lambda(O_1) + \lambda(O_2). \end{aligned}$$

Maximising over u we derive (7.5).

Next, we establish (7.4). Fix any $\varepsilon > 0$. For any A_j use (7.3) to find an open $O_j \supset A_j$ such that

$$\lambda(O_j) < \lambda(A_j) + \frac{\varepsilon}{2^j}.$$

Then

$$\bigcup_{j=1}^{\infty} A_j \subset O = \bigcup_{j=1}^{\infty} O_j,$$

and by the monotonicity

$$\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) \le \lambda(O).$$

To estimate $\lambda(O)$ take any $u \prec O$. By the compactness of $\mathrm{supp} u$, there exists a finite N, such that $O_{1,\dots,N}$ covers $\mathrm{supp} u$. Let $\varphi_{1,\dots,N}$ be the partition of unity on $\mathrm{supp} u$ subordinated to this cover. Then

$$\begin{aligned} &u = u\varphi_1 + \cdots + u\varphi_N, \\ &u\varphi_i \prec O_i. \end{aligned}$$

Consequently

$$\begin{aligned} \langle \Lambda, u \rangle &= \langle \Lambda, u\varphi_1 + \cdots + u\varphi_N \rangle \\ &= \langle \Lambda, u\varphi_1 \rangle + \cdots + \langle \Lambda, u\varphi_N \rangle \\ &\le \lambda(O_1) + \cdots + \lambda(O_N) \\ &< \sum_{j=1}^{\infty} \lambda(A_j) + \varepsilon. \end{aligned}$$

Maximising over u, we deduce that

$$\lambda(O) \le \sum_{j=1}^{\infty} \lambda(A_j) + \varepsilon$$

for any $\varepsilon > 0$. Hence (7.4) follows.

3. It is temting at this stage to launch the Caratheodory construction for the outer measure λ in order to obtain desired measure in (7.2). Such aproach works well in the case of a compact metric space K. The case of the compact Hausdorff space causes some difficulties. We approach it without explicitly appealing to the Caratheodory construction.

Define $\mathcal{A}$ to be the collection of all $E \subset K$ for which

$$\lambda(E) = \sup\{\lambda(F) :\ E \supset F - \text{closed}\}.$$

We shall prove that $\mathcal{A}$ is a σ-algebra. The proof is long, and we brake it into several steps.

4. We claim that all closed sets are in $\mathcal{A}$, and that

$$\lambda(F) = \inf\{\langle \Lambda, u\rangle : \ F \prec u\} \quad \text{for } F \text{ closed.} \tag{7.6}$$

Let F be a closed set. Clearly $F \in \mathcal{A}$ by the monotonicity of λ. To prove (7.6) take any $\varepsilon > 0$. Use (7.3) to find $O_\varepsilon \supset F$, such that

$$\lambda(F) > \lambda(O_\varepsilon) - \varepsilon.$$

Use Urysohn's lemma to find $f \in C(K)$ such that $F \prec f \prec O_\varepsilon$. Deduce at once that

$$\begin{aligned} \lambda(F) &> \langle \Lambda, f\rangle - \varepsilon \\ &\geq \inf\{\langle \Lambda, u\rangle : \ F \prec u\} - \varepsilon \end{aligned}$$

for any $\varepsilon > 0$.

To prove the opposite inequality fix any $u \in C(K)$, $F \prec u$. For any $\varepsilon > 0$ the set

$$O_\varepsilon = \{z \in K : \ u(z) > 1 - \varepsilon\}$$

is open, and $F \subset O_\varepsilon$. Observe that

$$v \prec O_\varepsilon \Longrightarrow v \leq u/(1-\varepsilon).$$

But then by the positivity of Λ

$$\begin{aligned} \lambda(F) &\leq \lambda(O_\varepsilon) \\ &= \sup\{\langle \Lambda, v\rangle : \ v \prec O_\varepsilon\} \\ &\leq \frac{1}{1-\varepsilon}\langle \Lambda, u\rangle \end{aligned}$$

Minimising over such u derive

$$\lambda(F) \leq \frac{1}{1-\varepsilon} \inf\{\langle \Lambda, u\rangle : \ F \prec u\}.$$

Now (7.6) follows since ε is arbitrary.

5. Let us show that all open sets are in $\mathcal{A}$.

Indeed, fix an open set O. By the monotonicity of λ

$$\lambda(O) \geq \sup\{\lambda(F) : \ O \supset F - \text{closed}\}.$$

To prove the opposite inequality, fix any $\varepsilon > 0$. Find $u_\varepsilon \prec O$, such that

$$\lambda(O) < \langle \Lambda, u_\varepsilon\rangle + \varepsilon.$$

Notice that the estimate

$$\langle \Lambda, u_\varepsilon \rangle \le \lambda(\mathrm{supp} u_\varepsilon)$$

holds. Indeed, for an arbitrary open V, $V \supset \mathrm{supp} u_\varepsilon$, we have $u_\varepsilon \prec V$. Therefore

$$\langle \Lambda, u_\varepsilon \rangle \le \lambda(V),$$

and the desired estimate follows by (7.3) after minimising over V. Thus

$$\begin{array}{rcl} \lambda(O) & < & \lambda(\mathrm{supp} u_\varepsilon) + \varepsilon \\ & \le & \sup\{\lambda(F) \colon \ O \supset F - \text{closed}\} + \varepsilon. \end{array}$$

Since ε is arbitrary, we conclude that $O \in \mathcal{A}$.

6. Let $F_{1,2}$ be compact sets, $F_1 \cap F_2 = \emptyset$. We claim that

$$\lambda(F_1 \cup F_2) = \lambda(F_1) + \lambda(F_2).$$

Due to semiadditivity (7.4) we just need to show that

$$\lambda(F_1 \cup F_2) \ge \lambda(F_1) + \lambda(F_2)$$

To prove the latter, fix any $\varepsilon > 0$. Utilise (7.6) to find $u_\varepsilon \in C(K)$, $F_1 \cup F_2 \prec u$, such that

$$\lambda(F_1 \cup F_2) \ge \langle \Lambda, u_\varepsilon \rangle - \varepsilon.$$

Next apply Urysohn's lemma to find $\varphi \in C(K)$ such that $\varphi|_{F_1} = 1$, $\varphi|_{F_2} = 0$. Therefore $F_1 \prec u_\varepsilon \varphi$, $F_2 \prec u_\varepsilon(1-\varphi)$, and by (7.6)

$$\begin{array}{rcl} \lambda(F_1 \cup F_2) & \ge & \langle \Lambda, u_\varepsilon[\varphi + (1-\varphi)] \rangle - \varepsilon \\ & = & \langle \Lambda, u_\varepsilon \varphi \rangle + \langle \Lambda, u_\varepsilon (1-\varphi) \rangle - \varepsilon \\ & \ge & \lambda(F_1) + \lambda(F_2) - \varepsilon. \end{array}$$

Letting $\varepsilon \to 0$ deduce that the finite additivity on the compact sets holds.

7. Let $E_j \in \mathcal{A}$, $j = 1, 2 \ldots$, be a disjoint sequence. We claim that

$$\bigcup_{j=1}^{\infty} E_j = E \in \mathcal{A}, \tag{7.7}$$

and

$$\lambda(E) = \sum_{j=1}^{\infty} \lambda(E_j). \tag{7.8}$$

Indeed, fix any $\varepsilon > 0$. For each E_j by the definition of $\mathcal{A}$ we find a compactum $F_j \subset E_j$, such that

$$\lambda(F_j) > \lambda(E_j) - \frac{\varepsilon}{2^j}.$$

Then for any finite N we use the additivity on compact sets to deduce

$$\begin{aligned} \lambda(E) &\geq \lambda(F_1 \cup \cdots \cup F_N) \\ &= \sum_{j=1}^{N} \lambda(F_j) \\ &\geq \sum_{j=1}^{N} \lambda(E_j) - \varepsilon. \end{aligned}$$

Since ε and N are arbitrary we derive that

$$\lambda(E) \geq \sum_{j=1}^{\infty} \lambda(E_j).$$

The opposite inequality holds always due to semiadditivity (7.4). Thus (7.8) holds. Moreover, taking in the previous argument $N = N_\varepsilon$ large enough we deduce that

$$\lambda(E) \geq \lambda(F_1 \cup \cdots \cup F_N) \geq \lambda(E) - 2\varepsilon.$$

A finite union of compact sets is compact. Thus $E \in \mathcal{A}$.

8. We claim that

$$A_{1,2} \in \mathcal{A} \Longrightarrow A_1 \cup A_2, A_1 \cap A_2, A_1 \setminus A_2 \in \mathcal{A}. \tag{7.9}$$

In other words, $\mathcal{A}$ is closed under a finite number of the set theoretic operations.

To prove this we first establish that for any $E \in \mathcal{A}$ and any $\varepsilon > 0$ there exist an open set O and a compact set F such that $F \subset E \subset O$, and

$$\lambda(O \setminus F) < \varepsilon. \tag{7.10}$$

Indeed, by the definition of $\mathcal{A}$ we find a compact F and an open O squeezing E, such that

$$\lambda(O) - \varepsilon < \lambda(E) < \lambda(F) + \varepsilon.$$

Since $O \setminus F$ is open we can use the additivity to deduce

$$\lambda(F) + \lambda(O \setminus F) = \lambda(O) < \lambda(F) + 2\varepsilon.$$

Hence (7.10) holds.

Let us prove (7.9). It is enough to show that $A_1 \setminus A_2 \in \mathcal{A}$. Fix any $\varepsilon > 0$. Squeeze A_i as above: $F_i \subset A_i \subset O_i$ $\lambda(O_i \setminus F_i) < \varepsilon$. Notice that $F_1 \setminus O_2$ is compact and

$$F_1 \setminus O_2 \subset A_1 \setminus A_2.$$

At the same time

$$\begin{aligned} A_1 \setminus A_2 &\subset O_1 \setminus F_2 \\ &= (O_1 \setminus F_1) \cup (F_1 \setminus O_2) \cup (O_2 \setminus F_2). \end{aligned}$$

Therefore by the semiadditivity

$$\lambda(A_1 \setminus A_2) < \lambda(F_1 \setminus O_2) + 2\varepsilon.$$

Hence (7.9) holds since ε is arbitrary.

9. Let us summarise what is proved.

(a) $(\mathcal{A}, K)$ is a σ-algebra. This follows at once from (7.7) and (7.9).

(b) $\mathcal{B}(K) \subset \mathcal{A}$ since all open and closed sets are in $\mathcal{A}$.

(c) λ is a complete measure on $\mathcal{A}$. This follows immediately from (7.8). The completeness is obvious from the definition of $\mathcal{A}$.

(d) Define

$$\mu = \lambda|_{\mathcal{B}(K)}.$$

Then μ is a finite Borel measure on K. Directly from the definitions

$$\begin{aligned} E \in \mathcal{A} \Longrightarrow \lambda(E) &= \inf\{\lambda(O) \colon\ E \subset O - \text{open}\} \\ &= \sup\{\lambda(F) \colon\ E \supset F - \text{compact}\}. \end{aligned}$$

Consequently μ is a regular Borel measure.

(e) The completeness of $\mathcal{A}$ immediately implies that $\mathcal{A}$ is the completion of $\mathcal{B}(K)$ with respect to μ.

10. We have the measure space $(K, \mathcal{B}(K), \mu)$, where μ is regular. Now we establish (7.2) for the integration on this measure space. By the linearity we just need to show that

$$\langle \Lambda, u \rangle \le \int_K u \, d\mu \tag{7.11}$$

for all $u \in C(K)$.

To prove (7.11) take any such u. Fix $\varepsilon > 0$. The set $u(K)$ is compact. Fix $a, b \in \mathbf{R}$ such that $u(K) \subset (a, b)$. Divide the interval (a, b) into N segments of length

$$|a - b|/N < \varepsilon.$$

Thus $a = t_0 < t_1 < \cdots < t_N = b$.

Define

$$E_j = \{t_{j-1} < u \le t_j\}, \quad j = 1, \ldots, N.$$

The sets $E_{1,\dots,N}$ form a disjoint partition of K. The set $\{u < t_j + \varepsilon\}$ is open. Hence using the regularity of μ we can find open sets $O_{1,\dots,N}$ such that

$$\begin{aligned} &u < t_j + \varepsilon \quad \text{on } O_j, \\ &\mu(O_j) < \mu(E_j) + \frac{\varepsilon}{N} \end{aligned}$$

for all $j = 1, \dots, N$. Sets $O_{1,\dots,N}$ form a finite open cover of K. Let $\varphi_{1,\dots,N}$ be the partition of unity on K subordinated to this cover.

Now, we proceed as follows:

$$\begin{aligned} \langle \Lambda, u \rangle &= \sum_{j=1}^{N} \langle \Lambda, u\varphi_j \rangle \qquad \text{since } \varphi_1 + \cdots + \varphi_N = 1 \\ &\le \sum_{j=1}^{N} (t_j + \varepsilon) \langle \Lambda, \varphi_j \rangle \qquad \text{since } u\varphi_j \le (t_j + \varepsilon)\varphi_j \\ &\le \sum_{j=1}^{N} (t_j + \varepsilon) \mu(O_j) \qquad \text{since } \varphi_j \prec O_j \\ &\le \sum_{j=1}^{N} (t_j + \varepsilon) \mu(E_j) + \sum_{j=1}^{N} (t_j + \varepsilon) \frac{\varepsilon}{N} \\ &\le \sum_{j=1}^{N} (t_j - \varepsilon + 2\varepsilon) \mu(E_j) + (|b| + 1)\varepsilon \\ &\le \sum_{j=1}^{N} (t_j - \varepsilon) \mu(E_j) + 2\varepsilon\mu(K) + (|b| + 1)\varepsilon \\ &\le \int_K u \, d\mu + \varepsilon(|b| + 1 + 2\mu(K)) \qquad \text{since } t_j - \varepsilon \le u \text{ on } E_j. \end{aligned}$$

Since ε is arbitrary we conclude that (7.11) holds.

11. Let us prove the uniqueness statement in part (a). Suppose the measures $\tilde{\mu}, \mu \in \mathfrak{M}_0$ both satisfy (7.2). Due to their regularity the equality $\tilde{\mu} = \mu$ follows as soon as we prove that

$$\tilde{\mu}(F) = \mu(F) \quad \forall F \text{ closed.} \tag{7.12}$$

To prove (7.12) take any closed $F \subset K$. Fix any $\varepsilon > 0$. Due to the regularity of μ there exists an open set $O_\varepsilon \supset F$, such that

$$\mu(O_\varepsilon) < \mu(F) + \varepsilon.$$

Apply the Urysohn's lemma to obtain $u \in C(K)$ such that $F \prec u \prec O_\varepsilon$.

But then

$$\begin{aligned} \tilde{\mu}(F) &= \int_K 1_F \, d\tilde{\mu} \\ &\le \int_K u \, d\tilde{\mu} \\ &= \langle \Lambda, u \rangle \\ &= \int_K u \, d\mu \\ &\le \int_K 1_{O_\varepsilon} \, d\mu \\ &= \mu(O_\varepsilon) \\ &\le \mu(F) + \varepsilon. \end{aligned}$$

Thus $\tilde{\mu}(F) \le \mu(F)$ for any closed F. Similarly the regularity of $\tilde{\mu}$ implies $\tilde{\mu}(F) \ge \mu(F)$ for closed F. Hence (7.12) follows.

12. It is left to prove the isometry property. For positive Λ this is easy. Indeed, on one hand by (7.2)

$$\begin{aligned} \|\Lambda\|_{C(K)^*} &= \sup\{\langle \Lambda, f \rangle : \ \|f\|_{C(K)} \le 1\} \\ &\le \int_K 1 \, d\mu \\ &= \|\mu\|. \end{aligned}$$

On the other hand by (7.2) $\langle \Lambda, 1 \rangle = \|\mu\|$. $\square$

18. The proof of the second part of the theorem heavily relies on the first part, but still requires some work.

19. *Proof of Theorem 6, part (b).* **1.** Let us prove that i is linear.

If $\mu_{1,2} \ge 0$, then

$$\int_K f \, d(\mu_1 + \mu_2) = \int_K f \, d\mu_1 + \int_K f \, d\mu_2$$

for any integrable f. Indeed, the equality obviously holds for simple f. Therefor, general case follows from the definition of the integral.

Next, let $\nu \in \mathfrak{M}$,

$$\begin{aligned} \nu &= \nu^+ - \nu^- \\ &= \nu_1 - \nu_2, \end{aligned}$$

$\nu^\pm, \nu_{1,2} \in \mathfrak{M}$, and $\nu^\pm, \nu_{1,2} \ge 0$. Consequently $\nu^+ + \nu_2 = \nu^- + \nu_1$, and for any integrable f

$$\int_K f \, d\nu^+ + \int_K f \, d\nu_2 = \int_K f \, d\nu^- + \int_K f \, d\nu_1.$$

Thus

$$\int_K f\, d\nu^+ - \int_K f\, d\nu^- = \int_K f\, d\nu_1 - \int_K f\, d\nu_2,$$

and the value F_ν in (7.1) does not depend on the representation of ν as the difference of two positive measures. The linearity of i now follows, since for $\mu, \nu \in \mathfrak{M}$ we can always write

$$\mu + \nu = (\mu^+ + \nu^+) - (\mu^- + \nu^-).$$

2. Let us prove that i is injective on $\mathfrak{M}_0$. By the linearity we need to show that its kernel is trivial.

Indeed, let $\nu \in \mathfrak{M}_0$ be such that $F_\nu = 0$ in (7.1). This means that

$$\int_K u\, d\nu^+ = \int_K u\, d\nu^- \quad \forall u \in C(K),$$

where the positive measures $\nu^\pm \geq 0$ are regular. Aply part (a) to the positive functionals $F_{\nu^\pm}$ and utilise the regularity of $\nu^\pm$ to discover that $\nu^+ = \nu^-$. Hence $\nu = 0$.

3. Let us prove that i is surjective. Fix any $\Lambda \in C(K)^*$. The desired surjectivity follows at once, provided we find positive functionals $\Lambda^\pm$ on $C(K)$, such that $\Lambda = \Lambda^+ - \Lambda^-$. In what follows we construct $\Lambda^\pm$.

For $u \in C(K)$, $u \geq 0$, define

$$\Lambda^+(u) = \sup\{\langle \Lambda, v\rangle : \ 0 \leq v \leq u, \ v \in C(K)\}.$$

It is easy to see that

$$|\Lambda^+(u)| \leq \|\Lambda\|_{C(K^*)} \|u\|_{C(K)},$$

and that

$$\Lambda^+(cu) = c\Lambda^+(u), \quad c \geq 0.$$

We also have

$$\Lambda^+(u_1 + u_2) = \Lambda^+(u_1) + \Lambda^+(u_2), \quad \forall u_{1,2} \in C(K), \ u_{1,2} \geq 0. \qquad (7.13)$$

In fact, for any $v_i \in C(K)$, $0 \leq v_i \leq u_i$, the linearity of Λ implies that

$$\begin{aligned} \langle \Lambda, v_1\rangle + \langle \Lambda, v_2\rangle &= \langle \Lambda, v_1 + v_2\rangle \\ &\leq \Lambda^+(u_1 + u_2). \end{aligned}$$

Maximising over $v_{1,2}$ gives $\Lambda^+(u_1) + \Lambda^+(u_2) \leq \Lambda^+(u_1 + u_2)$. On the other hand, for any $v \in C(K)$ such that $0 \leq v \leq u_1 + u_2$, we define

$$\begin{aligned} &v_1 = v \wedge u_1 \\ &v_2 = 0 \vee (v - u_1). \end{aligned}$$

Then $0 \le v_i \le u_i$, $v_i \in C(K)$, and $v = v_1 + v_2$. Hence

$$\begin{aligned} \langle \Lambda, v \rangle &= \langle \Lambda, v_1 \rangle + \langle \Lambda, v_2 \rangle \\ &\le \Lambda^+(u_1) + \Lambda^+(u_2). \end{aligned}$$

Maximising over v produces (7.13).

For an arbitrary $u \in C(K)$ write $u = u^+ - u^-$, $u^\pm \ge 0$, $u^\pm \in C(K)$, and set

$$\Lambda^+(u) = \Lambda^+(u^+) - \Lambda^+(u^-).$$

By (7.13) $\Lambda^+(u)$ does not depend on the decomposition. Hence Λ^+ is a well defined linear functional on $C(K)$.

If $u \ge 0$, $u \in C(K)$, we also define

$$\Lambda^-(u) = (-\Lambda)^+(u).$$

Thus Λ^- is also extended to a well defined linear functional on $C(K)$. The calculation gives

$$\Lambda^-(u) = -\inf\{\langle \Lambda, v \rangle \colon 0 \le v \le u,\ v \in C(K)\}, \quad u \in C(K),\ u \ge 0.$$

Thus for any $u \in C(K)$, $u \ge 0$ we derive

$$\begin{aligned} \Lambda^-(u) &= -\inf\{\langle \Lambda, u \rangle - \langle \Lambda, u - v \rangle \colon 0 \le v \le u,\ v \in C(K)\} \\ &= -\langle \Lambda, u \rangle + \sup\{\langle \Lambda, u - v \rangle \colon 0 \le u - v \le u,\ v \in C(K)\} \\ &= -\langle \Lambda, u \rangle + \Lambda^+(u). \end{aligned}$$

Hence $\Lambda = \Lambda^+ - \Lambda^-$ on positive continuous functions, and therefore on the entire $C(K)$.

4. Take any $\Lambda \in C(K)^*$. We know that i is bijective. Let $\nu = i^{-1}(\Lambda)$, $\nu \in \mathfrak{M}_0$. Let us prove that

$$\|\Lambda\|_{C(K)^*} = \|\nu\|. \tag{7.14}$$

Indeed, for any u with $\|u\|_{C(K)} = 1$ we derive that

$$|\langle \Lambda, u \rangle| \le \int_K |u|\, d\nu^+ + \int_K |u|\, d\nu^- \le |\nu|(K).$$

Thus $\|\Lambda\|_{C(K)^*} \le \|\nu\|$.

To prove the opposite inequality fix any $\varepsilon > 0$. Then by the property of the total variation we can partition K into a finite number of disjoint pieces $A_{1,\dots,N}$, $A_j \in \mathcal{B}(K)$, such that

$$|\nu|(K) < \varepsilon + \sum_{j=1}^{N} |\nu(A_j)|.$$

Choose the numbers $\sigma_j = \pm 1$ such that

$$\begin{aligned}\sum_{j=1}^{N} |\nu(A_j)| &= \sum_{j=1}^{N} \pm\nu(A_j) \\ &= \int_K \sum_{j=1}^{N} \sigma_j 1_{A_j} \, d\nu \\ &= \int_K f_\varepsilon \, d\nu\end{aligned}$$

Since ν is regular we can apply the approximation theorem (cf. below) to the integrable f_ε . Thus we find a function $u_\varepsilon \in C(K)$ such that

$$\begin{aligned}\|u_\varepsilon\|_{C(K)} &\leq \|f_\varepsilon\|_{C(K)} \\ &= 1,\end{aligned}$$

and

$$\|u_\varepsilon - f_\varepsilon\|_{L^1(K,\nu)} < \varepsilon.$$

Therefore

$$\begin{aligned}|\nu|(K) &< 2\varepsilon + \left| \int_K u_\varepsilon \, d\mu \right| \\ &= 2\varepsilon + |\langle \Lambda, u_\varepsilon \rangle| \\ &\leq 2\varepsilon + \|\Lambda\|_{C(K)^*}.\end{aligned}$$

Consequently $\|\Lambda\|_{C(K)^*} \geq \|\nu\|$ and (7.14) holds. □

20. **Theorem 7** *Let K be a compact Hausdorff space, and let μ be a Borel regular measure on it. Suppose $1 \leq p < \infty$. Then for any $\varepsilon > 0$ and any $f \in L^p(K,\mu)$, there exists a complex valued $f_\varepsilon \in C(K)$ such that*

$$\|f_\varepsilon\|_{C(K)} \leq \|f\|_{L^\infty} \quad \textit{and} \quad \|f - f_\varepsilon\|_{L^p} < \varepsilon.$$