Section: Tuesday, 12:00pm Name: ______________________

1. This next question applies the techniques of separable equations to a new kind of differential equation called a Homogeneous Equation. If the differential equation $dy/dx = f(x, y)$ can be rewritten as $dy/dx = g(y/x)$, then we say that this differential equation is said to be homogeneous. We can solve these homogeneous equations by transforming them into separable equations via a change of variable.

   (a) Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

   Show that this equation may be rewritten as a homogeneous equation.

   Solution: Multiply both numerator and denominator of the RHS by the expression $1/x$. This gives

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)},$$

   which is of the form

$$\frac{dy}{dx} = g(y/x)$$

   when

$$g(v) = \frac{v - 4}{1 - v}.$$

   (b) If $v = y/x$, express both y and dy/dx in terms of x, v, and dv/dx.

   Solution: If $v = y/x$, then $y = vx$, and the product rule gives

$$\frac{dy}{dx} = v + \frac{dv}{dx}x.$$

   (c) Replace $y, dy/dx$ in the equation you wrote in the first part of this question. Solve the separable equation.

   Solution: Replacing gives

$$v + \frac{dv}{dx}x = \frac{v - 4}{1 - v}.$$

   This is a separable equation in v:

$$\begin{aligned}
\frac{dv}{dx}x &= \frac{v - 4}{1 - v} - v \\
\frac{dv}{dx}x &= \frac{v - 4}{1 - v} - \frac{v - v^2}{1 - v} \\
\frac{dv}{dx}x &= \frac{v^2 - 4}{1 - v} \\
\int \frac{1 - v}{v^2 - 4}dv &= \int \frac{dx}{x}.
\end{aligned}$$

The rational function on the LHS can be split into factors via the method of Partial Fraction Decomposition:

$$
\begin{aligned}
\frac{1-v}{v^2-4} &= \frac{A}{v-2} + \frac{B}{v+2} \\
-v+1 &= Av + 2A + Bv - 2B \\
-1 = A+B \qquad & \qquad 1 = 2A - 2B \\
-2 = 2A+2B \qquad & \qquad 1 = 2A - 2B \\
-1 = 4A \qquad & \qquad -3 = 4B \\
\frac{-1}{4} = A \qquad & \qquad \frac{-3}{4} = B \\
\frac{1-v}{v^2-4} &= \frac{-1}{4(v-2)} + \frac{-3}{4(v+2)},
\end{aligned}
$$

which means that the integrals become

$$
\begin{aligned}
\int \frac{1-v}{v^2-4} dv &= \int \frac{dx}{x} \\
\int \frac{-dv}{4(v-2)} + \frac{-3dv}{4(v+2)} &= \int \frac{dx}{x} \\
\frac{-1}{4} \ln|v-2| + \frac{-3}{4} \ln|v+2| &= \ln|x| + C \\
\ln\left(|v-2|^{-1/4}|v+2|^{-3/4}\right) &= \ln|x| + C \\
|v-2|^{-1/4}|v+2|^{-3/4} &= C|x| \\
|y/x-2|^{-1/4}|y/x+2|^{-3/4} &= C|x|.
\end{aligned}
$$

2. In each case, solve the differential equation after multiplying through by the appropriate integrating factor.

   (a) $x^2y^3 + x(1+y^2)y' = 0$, $\mu(x,y) = 1/xy^3$.

Solution: Multiplying through by the integrating factor gives

$$x + (1+y^2)/y^3y' = 0,$$

which is an exact equation (in fact, it's separable), since if $M(x,y) = x$ and $N(x,y) = (1+y^2)/y^3$, then

$$N_x = 0 = M_y.$$

Thus, we seek a ψ such that $\psi_x = M$ and
$psi_y = N$. Since M and N involve only x and y respectively, we suspect that

$$\psi =$$
$$\int M(x,y)dx + \int N(x,y)dy =$$
$$\int xdx + \int (1+y^2)/y^3dy =$$
$$x^2/2 + -3/y^2 + \ln|y|$$

would work. This gives that

$$\frac{d}{dx}\left[x^2/2 - 3/y^2 + \ln|y|\right] = 0$$
$$x^2 - 3/y^2 + \ln|y| = C$$
$$x^2 + C = 3/y^2 - \ln|y|.$$

   (b) $ydx + (2x - ye^y)dy = 0$, $\mu(x,y) = y$.

Solution: Multiplying through gives

$$y^2dx + (2xy - y^2e^y)dy = 0.$$

If $M(x,y) = y^2$ and $N(x,y) = (2xy - y^2e^y)$, then

$$M_y(x,y) = 2y = N_x,$$

so this is an exact equation. Choosing $\psi(x,y) = xy^2 - \int y^2e^ydy$ gives $\psi_x = y^2$ and $\psi_y = 2yx - y^2e^y$, so let us compute this last integral:

$$\int y^2e^ydy = \qquad (u = y^2, dv = e^ydy)$$
$$y^2e^y - \int 2ye^ydy = \qquad (u = y, dv = e^ydy)$$
$$y^2e^y - 2\left[ye^y - \int e^y\right] =$$
$$y^2e^y - 2ye^y + 2e^y,$$

so that we have

$$\frac{d}{dx}\left[xy^2 - y^2e^y + 2ye^y - 2e^y\right] = 0$$
$$xy^2 - y^2e^y + 2ye^y - 2e^y = 0.$$

(c) $(x+2)\sin y dx + x\cos y dy = 0$, $\mu(x,y) = xe^x$.

Solution: Multiplication gives

$$(x^2e^x + 2xe^x)\sin y + x^2e^y\cos y y' = 0,$$

which is visibly the derivative of $x^2e^x\sin y$ with respect to x, giving as a solution

$$\begin{aligned} \frac{d}{dx}\left[x^2e^x\sin y\right] &= 0 \\ x^2e^x\sin y &= C \\ \sin y &= Cx^{-2}e^{-x} \\ y &= \arcsin\left[Cx^{-2}e^{-x}\right]. \end{aligned}$$