Section: Tuesday, 12:00pm Name: ____________________

1. In each of the following parts, find the solution to the given differential equation using integrating factors.

(a) $y' - y = 2te^{2t}$, $y(0) = 1$.

Solution: Our differential equation is already in the form $y' + p(t)y = g(t)$; in this case, $p(t) = -1$, $g(t) = 2te^{2t}$. Our integrating factor is thus

$$\mu(t) =$$
$$\exp\left(\int p(t)\right) =$$
$$\exp(-t),$$

so that our integrating factor is e^{-t}. Thus we obtain

$$e^{-t}y' - e^{-t}y = 2te^{t},$$

which is equivalent to

$$\frac{d}{dt}\left[e^{-t}y(t)\right] = 2te^{t}$$
$$e^{-t}y(t) = \int 2te^{t}dt.$$

This integral on the RHS may be solved by integration by parts. Let $u = 2t, dv = e^{t}dt$. Then our integral is

$$\int 2te^{t}dt =$$
$$2te^{t} - \int 2e^{t}dt =$$
$$2te^{t} - 2e^{t} + C,$$

so our general solution is

$$e^{-t}y(t) = 2te^{t} - 2e^{t} + C$$
$$y(t) = 2t - 2 + Ce^{-t}.$$

Plugging $y(0) = 1$ into our DE gives $1 = C - 2$, so $C = 3$, and our solution is $y(t) = 2t - 2 + 3e^{-t}$.

(b) $y' + 2y = te^{-2t}$, $y(1) = 0$.

Solution: The setup is $p(t) = 2$, $g(t) = te^{-2t}$. Thus our integrating factor is

$$\mu(t) =$$
$$\exp\left(\int 2dt\right) =$$
$$\exp(2t),$$

so our differential equation is now

$$e^{2t}y' + 2e^{2t}y = t.$$

This is equivalent to

$$\frac{d}{dt}\left[e^{2t}y\right] = t,$$

so our general solution is

$$e^{2t}y = t^2/2 + C$$
$$y = (1/2)t^2e^{-2t} + Ce^{-2t}.$$

Plugging in our initial condition $y(1) = 0$ gives

$$0 = (1/2)Ce^{-2},$$

meaning that $C = 0$. Thus our solution is $y(t) = (1/2)t^2e^{-2t}$.

(c) $ty' + 2y = t^2 - t + 1$, $y(1) = 1/2, t > 0$.

Solution: Rewrite in the form $y' + p(t)y = g(t)$ by dividing by t:

$$y' + (2/t)y = t - 1 + 1/t.$$

Then the integrating factor is

$$\exp\left[\int (2/t)\right] = e^{2\ln|t|} = |t|^2 = t^2.$$

This gives the equation

$$t^2y' + 2ty = t^3 - t^2 + t$$
$$\frac{d}{dt}\left[t^2y\right] = t^3 - t^2 + t$$
$$t^2y = t^4/4 - t^3/3 + t^2/2 + C$$
$$y = t^2/4 - t/3 + 1/2 + Ct^{-2}.$$

Plugging in $y(1) = 1/2$ gives $1/2 = 1/4 - 1/3 + 1/2 + C$, so that $C = 1/12$, and our final answer is $y(t) = t^2/4 - t/3 + 1/2 + t^{-2}/12$.

(d) $y' + (2/t)y = (\cos t)/t^2$, $y(\pi) = 0, t > 0$.

Solution: Our integrating factor is

$$\exp\left[\int 2/t\right] = t^2,$$

so our new differential equation is

$$t^2y' + 2ty = \cos t,$$

giving

$$t^2y = \sin t + C$$
$$y = t^{-2}\sin t + Ct^{-2},$$

which when $y(\pi) = 0$ implies $0 = C\pi^{-2}$, so $C = 0$, and our solution is $y(t) = t^{-2}\sin t$.

(e) $y' - 2y = e^{2t}$, $y(0) = 2$.

Solution: Our integrating factor is e^{-2t}, so our new differential equation is

$$\begin{aligned} e^{-2t}y' - 2e^{-2t}y &= 1 \\ \frac{d}{dt}\left[e^{-2t}y\right] &= 1 \\ e^{-2t}y &= t + C \\ y &= te^{2t} + Ce^{2t}, \end{aligned}$$

which when $y(0) = 2$ means $2 = C$, so $y = te^{2t} + 2e^{2t}$.

(f) $ty' + 2y = \sin t, y(\pi/2) = 1,\ t > 0$.

Solution: Dividing by t gives

$$y' + (2/t)y = \sin t/t,$$

so that our integrating factor is t^2, and our new equation is

$$\frac{d}{dt}\left[t^2 y\right] = t \sin t.$$

Integrating $t \sin t$ is done by parts. Let $u = t$ and $dv = \sin t dt$; this gives

$$\begin{aligned} \int t \sin t dt = \\ -t\cos t + \int \cos t = \\ -t\cos t + \sin t + C, \end{aligned}$$

so our solution is

$$\begin{aligned} t^2 y &= \sin t - t\cos t + C \\ y &= t^{-2}\sin t - t^{-1}\cos t + Ct^{-2}, \end{aligned}$$

so using the initial condition $y(\pi/2) = 1$ gives

$$\begin{aligned} 1 = (\pi/2)^{-2} + C(\pi/2)^{-2} \\ (\pi/2)^2 = 1 + C \\ (\pi/2)^2 - 1 = C, \end{aligned}$$

so that the solution is

$$y = t^{-2}\sin t - t^{-1}\cos t + (\pi^2/4 - 1)t^{-2}$$

(g) $t^3 y' + 4t^2 y = e^{-t}$, $y(-1) = 0, t < 0$.

Solution: Dividing by t^3 gives

$$y' + (4/t)y = t^{-3}e^{-t},$$

which means the integrating factor is t^4, so that we have

$$\frac{d}{dt}\left[t^4 y\right] = te^{-t}.$$

Integrating the right hand side is an integral by parts. Let $u = t, dv = e^{-t}dt$. Then

$$\int te^{-t} = -te^{-t} + \int e^{-t}dt = -(t+1)e^{-t} + C,$$

so that our general solution

$$t^4 y = -(t+1)e^{-t} + C$$
$$y = \frac{-t-1}{t^4}e^{-t} + Ct^{-4},$$

so that using the initial conditions gives $0 = C(-1)^{-4} = C$, so our solution is $y = \frac{-t-1}{t^4}e^{-t}$.

(h) $ty' + (t+1)y = t$, $y(\ln 2) = 1$, $t > 0$.

Solution: Dividing gives

$$y' + \frac{t+1}{t}y = 1,$$

so our integrating factor is

$$\begin{aligned} \exp\left[\int (t+1)/t dt\right] &= \\ \exp\left(t + \ln|t|\right) &= \\ |t|e^t &= && \text{(when } t > 0) \\ te^t. \end{aligned}$$

Thus our new differential equation is

$$te^t y' + (t+1)e^t y = te^t$$
$$\frac{d}{dt}\left[te^t y\right] = te^t,$$

so that we must integrate te^t. By parts, if $u = t$ and $dv = e^t dt$, we have

$$\begin{aligned} \int te^t dt &= \\ te^t - \int e^t &= \\ te^t - e^t + C, \end{aligned}$$

so that our general solution is

$$te^t y = te^t - e^t + C$$
$$y = 1 - 1/t + Ce^{-t}/t,$$

and when $y(\ln 2) = 1$, we get $1 = 1 - 1/\ln 2 + C/2\ln 2$, so $C = 2$. Thus our solution is $y = 1 - 1/t + 2e^{-t}/t$.