Notes on the center of $U_q sl(2,\mathbb{C})$

1 Introduction

This note is about the structure of the center of the small quantum group $U_q sl(2)$ over $\mathbb{C}$. In particular, we give a proof of Kerler's conjecture in [Ker1].

In 1994, Lyubashenko and Majid [LM] found an action of $SL(2,\mathbb{Z})$ on a factorizable ribbon Hopf algebra H. That is there are $\mathcal{S},\mathcal{T}: H \to H$ and some $\lambda \in \mathbb{C}$ such that

$$(\mathcal{S}\mathcal{T})^3 = \lambda \mathcal{S}^2, \quad \mathcal{S}^2 = S^{-1}.$$

Here S is the antipode of H. $\mathcal{S}$ and $\mathcal{T}$ are given by, for all $x \in H$,

$$\mathcal{S}(x) = (id \otimes \mu)(R^{-1}(1 \otimes x)R_{21}^{-1}), \quad \mathcal{T}x = \theta x$$

where μ is a right integral, R is the universal R-matrix and θ is the ribbon element of H. When restricted to the center $\mathcal{Z}$, we have $\mathcal{S}^4 = S^{-2} = id|_{\mathcal{Z}}$ for S^2 is a conjugation induced by the balancing element. So we obtain a projective representation of $SL(2,\mathbb{Z})$ on the center $\mathcal{Z}$.

In [Ker1], Kerler studied this representation in the case of the small quantum group $U_q sl(2)$ with q a p-th root of unity. He stated a conjecture linking this projective representation of $SL(2,\mathbb{Z})$ with that obtained from RT TQFT and checked explicitly the case of $p = 3, 5$. In 1995, this conjecture was stated as a theorem in [Ker2] based on some observation but no details of proof.

Theorem 1 (Kerler)**.** *Let $p = 2h+1$ be an odd number and q be a p-th primitive root of unity. The $SL(2,\mathbb{Z})$ representation on the center $\mathcal{Z}$ of $U_q sl(2)$ decomposes as*

$$\mathcal{Z} = \mathcal{P}_{h+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_h$$

$\mathcal{P}_{h+1}$ is an $(h+1)$ dimensional representation and $\mathbb{C}^2$ is the standard representation of $SL(2,\mathbb{Z})$. $\mathcal{P}_h$ is an h dimensional representation when restricted on which the matrices $\mathcal{S}$ and $\mathcal{T}$ are the same as those obtained by RT TQFT.

In [FGST], a similar theorem was proven for the restricted quantum groups which can be viewed as a cousin of the small $U_q(sl_2)$ when choosing q an even root of unity. In the following, we will apply the idea of [FGST] and check in details that similar calculation works for the case of q being an odd root of unity. Moreover, The Verlinde Formula for projective indecomposable modules will be given in the last section.

2 Small quantum group $U_q sl(2)$ at root of unity

Throughout this paper, $p = 2h + 1$ is an odd natural number and q is a primitive p^{th} root of unity. The small quantum group $U_q sl(2)$ is a Hopf algebra generated by E, F, and K with the relations

$$E^p = F^p = 0, \quad K^p = 1$$

and the Hopf algebra structure given by

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

The PBW-basis of $U_q sl(2)$ are given by $\{F^m E^n K^l\}$ for $0 \leq m, n, l \leq p - 1$. So its dimension is p^3. By induction,

$$\Delta(F^m E^n K^k) = \sum_{r=0}^{m} \sum_{s=0}^{n} q^{2(n-s)(r-m)+r(m-r)+s(n-s)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} F^r E^{n-s} K^{r-m+l} \otimes E^{m-r} E^s K^{n-s+l}.$$

The integral and cointegral are given by

$$\mu(F^m E^n K^l) = \frac{1}{\zeta} \delta_{m,p-1} \delta_{n,p-1} \delta_{l,1},$$

and

$$c = \zeta F^{p-1} E^{p-1} \sum_{j=0}^{p-1} K^j,$$

where $\zeta = \frac{\sqrt{p}}{([p-1]!)^2}$ is a normalization for future convenience.

$U_q sl(2)$ is a ribbon Hopf algebra with an universal R-matrix

$$R = \frac{1}{p} \sum_{0 \le m,i,j \le p-1} \frac{(q-q^{-1})^m}{[m]!} q^{\frac{m(m-1)}{2}+2m(i-j)-2ij} E^m K^i \otimes F^m K^j,$$

and a ribbon element

$$\theta = \frac{1}{p}(\sum_{r=0}^{p-1} q^{hr^2})(\sum_{0 \le m,j \le p-1} \frac{(q-q^{-1})^m}{[m]!}(-1)^m q^{-\frac{1}{2}m+mj+\frac{1}{2}(j+1)^2} F^m E^m K^j)$$

The M-matrix $M = R_{21}R_{12}$ is

$$M = \frac{1}{p} \sum_{0 \le m,n,i,j \le p-1} \frac{(q-q^{-1})^{m+n}}{[m]![n]!} q^{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-m^2-mj+mi+2nj-2ni-ij} F^m E^n K^j \otimes E^m F^n K^i.$$

When M is represented as $M = \sum_I m_I \otimes n_I$, $\{m_I\}$ and $\{n_I\}$ are two bases in $U_q sl(2)$. So $U_q sl(2)$ is a factorizable Hopf algebra.

3 The center of $U_q(sl_2)$

Let $\mathcal{Z}$ denote the center of $U_q sl(2)$. In [Ker1], Kerler constructed a canonical base for $\mathcal{Z}$ by study the Casimir element of $U_q sl(2)$, which is

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2},$$

and satisfies the minimal polynomial $\Psi(C) = 0$ where

$$\Psi(x) = \prod_{j=1}^{p}(x-b_j), \qquad b_j = \frac{q^j + q^{-j}}{(q-q^{-1})^2}$$

Using the polynomials $\phi_j(x) = \prod_{b_s \neq b_j}(x-b_s)$ $j = 0, \ldots, h$, any polynomial $R(C)$ in C can be expressed in the forms of

$$R(C) = \sum_{j=0}^{h} R(b_j)P_j + \sum_{j=1}^{h} R'(b_j)N_j.$$

where

$$P_j = \frac{1}{\phi_j(b_j)}\phi_j(C) - \frac{\phi_j'(b_j)}{\phi_j(b_j)^2}(C-b_j)\phi_j(C), \quad j = 0, \dots, h$$

$$N_j = \frac{1}{\phi_j(b_j)}(C-b_j)\phi_j(C), \quad j = 1, \dots, h$$

This can be understood as some kind of Lagrange Interpolation.

In order to describe $\mathcal{Z}$, we need to introduce the projections

$$\pi_s^+ = \frac{1}{p}\sum_{n=0}^{s-1}\sum_{j=0}^{p-1} q^{(2n-s+1)j}K^j, \quad \pi_s^- = \frac{1}{p}\sum_{n=s}^{p-1}\sum_{j=0}^{p-1} q^{(2n-s+1)j}K^j, \quad s = 1, \dots, h$$

Let $N_j^+ = \pi_j^+ N_j$ and $N_j^- = \pi_j^- N_j$ for $j = 1, \dots, h$.

Proposition 1. *[Ker1] The center $\mathcal{Z}$ of $U_q sl(2)$ is a $3h+1$ dimensional commutative algebra with basis $\{P_i, N_j^+, N_j^- \mid i = 0, \dots, h; j = 1, \dots, h\}$ such that*

$$P_i P_j = \delta_{ij} P_j, \quad P_i N_j^{\pm} = \delta_{ij} N_j^{\pm}, \quad N_i^{\pm} N_j^{\pm} = N_i^{\pm} N_j^{\mp} = 0$$

Proposition 2. *[Ker1] The ribbon element is decomposed in terms of the canonical central elements as*

$$\theta = \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)} P_s + (q-q^{-1})\sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)}\left(\frac{p-s}{[p-s]}N_s^+ - \frac{s}{[s]}N_s^-\right)$$

4 Representation theory of $U_q sl(2)$

One important idea of [FGST] is to explore the representation theoretical meaning of P_i, N_j^+, N_j^-. Now we first review the representation theory of structure $U_q sl(2)$. First, there are p irreducible $U_q sl(2)$-modules V_s's for $s = 1, \dots, p$. V_s is linearly spanned by $v_n^{(s)}$, $0 \le n \le s-1$ with $v_0^{(s)}$ the highest weight vector. The $U_q sl(2)$-action is given by

$$K v_n^{(s)} = q^{s-1-2n} v_n^{(s)}$$

$$E v_n^{(s)} = [n][s-n] v_{n-1}^{(s)}$$

$$F v_n^{(s)} = v_{n+1}^{(s)}$$

where we set $v_0^{(s)} = v_s^{(s)} = 0$. In particular, V_1 is the trivial module.

Besides the irreducible modules, the projective modules play an important role in the representation theory of $U_q sl(2)$. There are p indecomposable projective $U_q sl(2)$-modules P_s's for $s = 1, \ldots, p$. P_s, $1 \leq s \leq p$ is spanned by $\{x_k^{(s)}, y_k^{(s)}\}_{0\leq k\leq p-s-1} \cup \{a_n^{(s)}, b_n^{(s)}\}_{0\leq n\leq s-1}$. The action of $U_q sl(2)$ on P_s is given by

$$Kx_k^{(s)} = q^{p-s-1-2k}x_k^{(s)}, \quad Ky_k^{(s)} = q^{p-s-1-2k}y_k^{(s)}, \quad 0 \leq k \leq p-s-1,$$
$$Ka_n^{(s)} = q^{s-1-2n}a_n^{(s)}, \quad Kb_n^{(s)} = q^{s-1-2n}b_n^{(s)}, \quad 0 \leq n \leq s-1,$$
$$Ex_k^{(s)} = [k][p-s-k]x_{k-1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad (with \;\; x_{-1}^{(s)} = 0)$$
$$Ey_k^{(s)} = [k][p-s-k]y_{k-1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad Ey_0^{(s)} = a_{s-1}^{(s)},$$
$$Ea_n^{(s)} = [n][s-n]a_{n-1}^{(s)}, \quad 0 \leq n \leq s-1, \quad (with \;\; a_{-1}^{(s)} = 0)$$
$$Eb_n^{(s)} = [n][s-n]b_{n-1}^{(s)} + a_{n-1}^{(s)}, \quad 0 \leq n \leq s-1, \quad Eb_0^{(s)} = x_{p-s-1}^{(s)},$$
$$Fx_k^{(s)} = x_{k+1}^{(s)}, \quad 0 \leq k \leq p-s-2, \quad Fx_{p-s-1}^{(s)} = a_0^{(s)}$$
$$Fy_k^{(s)} = y_{k+1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad (with \;\; y_{p-s}^{(s)} = 0)$$
$$Fa_n^{(s)} = a_{n+1}^{(s)}, \quad 0 \leq n \leq s-1, \quad (with \;\; a_s^{(s)} = 0)$$
$$Fb_n^{(s)} = b_{n+1}^{(s)}, \quad 0 \leq n \leq s-2, \quad Fb_{s-1}^{(s)} = y_0^{(s)}$$

In particular, P_p coincides with V_p.

Note that $\{x_k^{(s)}, y_k^{(s)}\}_{0\leq k\leq p-s-1} \cup \{a_n^{(s)}\}_{0\leq n\leq s-1}$ spans a submodule W_s of P_s, and $\{x_k^{(s)}\}_{0\leq k\leq p-s-1} \cup \{a_n^{(s)}\}_{0\leq n\leq s-1}$ spans a submodule M_s of W_s. We have a composition series:

$$P_s \supseteq W_s \supseteq M_s \supseteq V_s \supseteq 0 \tag{1}$$

with composition factors V_s, V_{p-s}, V_{p-s}, V_s.

The regular representation of $U_q sl(2)$ is decomposed as

$$U_q sl(2) \cong \sum_{s=1}^{p} dim(V_s) P_s$$

Moreover, its bimodule decomposition is

$$U_q sl(2) \cong \sum_{s=0}^{h} Q_s$$

where $Q_0 = dim(V_p)P_p, \quad Q_s = dim(V_s)P_s \oplus dim(V_{p-s})P_{p-s}, \quad s = 1, \ldots, h$

In general, for an algebra, the bimodule endomorphisms of the regular representation are in 1:1 correspondence with elements in the center. In deed, suppose f is a bimodule endomorphism of the regular representation, then $f(a) = af(1)$ by viewing f as a left module endomorphism. Similarly, when viewed as a right module endomorphism, $f(a) = f(1)a$. So, f is in 1:1 correspondence with $f(1)$ in $\mathcal{Z}$.

In particular, P_j corresponds the identity on Q_j for $j = 0, \ldots, h$; and $N_s^{\pm}$ act as

$$N_s^+ b_n^{(s)} = a_n^{(s)}, \quad N_s^- b_k^{(p-s)} = a_k^{(p-s)}, \quad s = 1, \ldots, h \tag{2}$$

The fusion rules among the irreducible modules are

$$V_i \otimes V_j = \sum_{k=|i-j|+1, step=2}^{\min(i+j-1, 2p-1-i-j)} V_k \oplus \sum_{r=2p+1-i-j, step=2}^{p} P_r$$

In particular,

$$\begin{aligned} V_2 \otimes V_1 &= V_2 \\ V_2 \otimes V_s &= V_{s-1} \oplus V_{s+1}, \quad s = 2, \ldots, p-1 \end{aligned}$$

5 Drinfeld maps

In order to study the center of $U_q sl(2)$ via the representations, we need to introduce the q-characters. For a Hopf algebra A, $Ch(A) = \{\beta \in A^* \mid \beta(xy) = \beta(S^2(y)x) \;\; \forall x, y \in A\}$ is the space of so called q-characters of A.

Theorem 2. *[Dr] Given an M-matrix $M = R_{21}R_{12} = \sum_I m_I \otimes n_I$ of a factorizable finite dimensional Hopf algebra A, the Drinfeld map $\chi : A^* \to A$ defined by*

$$\chi(\beta) = (\beta \otimes id)M = \sum_I \beta(m_I)n_I,$$

restricted on $Ch(A)$ is an isomorphism of commutative algebras between $Ch(A)$ and the center $\mathcal{Z}(A)$ of A

For a $U_q sl(2)$-module V, its q-character is defined to be $qCh_V = Tr_V(K^{-1}?)$. By $S^2(x) = KxK^{-1}$ for $x \in U_q sl(2)$, we know that $qCh_V \in Ch(U_q sl(2))$. For the irreducible $U_q sl(2)$-modules $V_1, \ldots, V_p$, the images of their q-characters in the center are

$$\chi(s) := \chi(qCh_{V_s}) = (Tr_{V_s} \otimes id)((K^{-1} \otimes 1)M), \quad 1 \le s \le p$$

We can calculate them explicitly.

Proposition 3.

$$\chi(s) = \sum_{n=0}^{s-1} \sum_{m=0}^{n} (q-q^{-1})^{2m} q^{-(m+1)(m+s-1-2n)} \begin{bmatrix} s-n+m-1 \\ m \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} E^m F^m K^{s-1-2n+m}.$$

In particular, $\chi(2) := \hat{C} = (q-q^{-1})^2 C$

Proof. Note that by induction

$$F^r E^r = \prod_{s=0}^{r-1} (C - \frac{q^{2s+1}K + q^{-2s-1}K^{-1}}{(q-q^{-1})^2}),$$

for $r < p$, Then

$$Tr_{V_s}(F^m E^m K^l) = ([m]!)^2 \sum_{n=0}^{s-1} q^{l(s-1-2n)} \begin{bmatrix} s-n+m-1 \\ m \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix}.$$

□

By observing the fusion rules, we find $\chi(s)$ can be calculated by the Chebyshev polynomials of the second kind

$$U_s(2\cos t) = \frac{\sin st}{\sin t}$$

which satisfy the recursive relation $xU_s(x) = U_{s-1}(x) + U_{s+1}(x)$ for $s \ge 2$. Then $\chi(s) = U_s(\hat{C})$ for $s = 1, \ldots, p$. The following proposition provides a formula expanding $\chi(s)$ in terms of the canonical basis of the center.

Proposition 4. *For* $s = 1, \ldots, p-1$,

$$\chi(s) = \sum_{j=0}^{h} \frac{[js]}{[j]} P_j(C) - \sum_{j=1}^{h} \frac{(s+1)[j(s-1)] - (s-1)[j(s+1)]}{[j]^3} N_j(C)$$

$$\chi(p) = pP_0(C) + 2p \sum_{j=1}^{h} \frac{1}{[j]^2} N_j(C)$$

Proof. Recall $\hat{C}$ satisfies

$$\prod_{j=1}^{p}(\hat{C}-\hat{b}_j)=0, \quad \hat{b}_j=q^j+q^{-j}$$

Similarly as the case of C, any polynomial $R(\hat{C})$ in $\hat{C}$ can be expressed in terms of $\hat{P}_j$ and $\hat{N}_j$ by

$$R(\hat{C})=\sum_{j=0}^{h}R(\hat{b}_j)\hat{P}_j+\sum_{j=1}^{h}\hat{R}'(\hat{b}_j)\hat{N}_j(\hat{C}).$$

Here $\hat{P}_j(\hat{C})=P_j(C), \quad \hat{N}_j(\hat{C})=(q-q^{-1})^2N_j(C)$. Then

$$\begin{aligned}\chi(s) &= U_s(\hat{C})=\sum_{j=0}^{h}U_s(\hat{b}_j)\hat{P}_j(\hat{C})+\sum_{j=1}^{h}U_s'(\hat{b}_j)\hat{N}_j(\hat{C}) \\ &= \sum_{j=0}^{h}U_s(\hat{b}_j)P_j(C)+(q-q^{-1})^2\sum_{j=1}^{h}U_s'(\hat{b}_j)N_j(C)\end{aligned}$$

By plug in

$$U_s(\hat{b}_j)=U_s(2\cos\frac{2\pi j}{p})=\sin\frac{2\pi js}{p}/\sin\frac{2\pi j}{p} = \frac{[js]}{[j]}$$

$$U_s'(\hat{b}_j) = -\frac{1}{(q-q^{-1})^2}\frac{(s+1)[j(s-1)]-(s-1)[j(s+1)]}{[j]^3}$$

We obtain the formulas for $\chi(s)$. Here the derivative is calculated by differentiating both sides of $U_s(2\cos t)\sin t=\sin st$ with respect to t, then evaluating at $2\cos t=\hat{b}_j$. □

Define $\nu(s)=\chi(s)+\chi(p-s)$ for $s=1,\dots,p-1$ and $\nu(0)=\nu(p)=\chi(p)$. By the composition (1), we know that these are the images of the q-characters of indecomposable projective modules under the Drinfeld map.

Corollary 1. *Then for $s=0,\dots,p-1$*

$$\nu(s) \quad = \quad pP_0(C)+p\sum_{j=1}^{h}\frac{q^{js}+q^{-js}}{[j]^2}N_j(C)$$

6 Radford maps

For a finite dimensional Hopf algebra A with cointegral c, the Radford map $\phi : A^* \to A$ given by

$$\phi(\alpha) = (\alpha \otimes id)\Delta(c) = \sum_{(c)} \alpha(c')c''$$

is an isomorphism between left A-modules A^* and A (see [Ra1]). The left action of A on itself is given by left multiplication and the left action of A on A^* is given by $a(\beta) = \beta(S(a)?)$. For $U_q sl(2)$, it is shown in [La] that the image of the set of q-characters $Ch(A)$ under ϕ coincides with the center $\mathcal{Z}(A)$.

For the irreducible $U_q sl(2)$-modules $V_1, \ldots, V_p$, the images of their q-characters in the center are

$$\phi(s) := \phi(qCh_{V_s}) = \sum_{(c)} Tr_{V_s}(K^{-1}c')c'', \quad 1 \le s \le p$$

Using the Casimir element C, we have

Proposition 5.

$$\begin{aligned}
\phi(s) &= \frac{p\sqrt{p}}{[s]^2} N_s^+, \quad 1 \le s \le h \\
\phi(s) &= \frac{p\sqrt{p}}{[s]^2} N_{p-s}^-, \quad h+1 \le s \le 2h \\
\phi(p) &= p\sqrt{p} P_0
\end{aligned}$$

Proof. Plugging the formula of cointegral c, we have

$$\phi(s) = \zeta \sum_{n=0}^{s-1} \sum_{0 \le i,j \le p} ([i]!)^2 q^{j(s-1-2n)} \begin{bmatrix} s-n+i-1 \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} F^{p-1-i} E^{p-1-i} K^j$$

By the same calculation for the lemma 4.5.1 in [FGST], one can see that $\phi(s)$ acts by zero on $P(s')$ and it is proportional to $N^{\pm}$. The proportionality coefficients are determined by the action of $\phi(s)$ on $b_0^{(s)}$ by (2). □

Let us denote the images of qCh_{V_s}'s under Drinfeld and Radford maps by $\mathcal{D}_p$ and $\mathcal{R}_p$ respectively. Then $\mathcal{D}_p$ is spanned by $\{\chi(1), \ldots, \chi(p)\}$ and $\mathcal{R}_p$ is spanned by $\{\phi(1),\ldots,\phi(p)\}$. It is shown in [La] that $\mathcal{D}_p \bigcup \mathcal{R}_p = \mathcal{Z}$. And they have a subspace P_{h+1} in common spanned by $\chi(p), \chi(1)+\chi(p-1), \ldots, \chi(h)+\chi(h+1)$. This space is also spanned by $\phi(p), \phi(1)+\phi(p-1), \ldots, \phi(h)+\phi(h+1)$. $\mathcal{P}_{h+1}$ has a categorical meaning that it is the image of the q-characters of projective indecomposable modules under both Drinfeld's and Radford's maps. The following proposition shows that $\mathcal{P}_{h+1}$ is actually spanned by $\nu(0),\ldots,\nu(h)$.

Proposition 6. $\mathcal{D}_p \bigcap \mathcal{R}_p = \mathcal{P}_{h+1}$

Proof.

$$
\begin{aligned}
& \nu(0) + \sum_{s'=1}^{h} (q^{ss'} + q^{-ss'})\nu(s') \\
= \; & \nu(0) + \frac{1}{2}\sum_{s'=1}^{p-1} (q^{ss'} + q^{-ss'})\nu(s') \\
= \; & \nu(0) + \frac{p}{2}\sum_{s'=1}^{p-1} (q^{ss'} + q^{-ss'})P_0(C) + \frac{p}{2}\sum_{s'=1}^{p-1}\sum_{j=1}^{h} (q^{ss'} + q^{-ss'})\frac{q^{js'} + q^{-js'}}{[j]^2} N_j(C) \\
= \; & \nu(0) - pP_0(C) + \frac{p}{2}\sum_{j=1}^{h}\sum_{s'=1}^{p-1} \frac{q^{(s+j)s'} + q^{(s-j)s'} + q^{(-s+j)s'} + q^{(-s-j)s'}}{[j]^2} N_j(C) \\
= \; & 2p\sum_{j=1}^{h} \frac{1}{[j]^2}N_j(C) + \frac{p}{2}\left(\frac{2p}{[s]^2}N_s(C) - 4\sum_{j=1}^{h}\frac{1}{[j]^2}N_j(C)\right) \\
= \; & \frac{p^2}{[s]^2}N_s(C) \\
= \; & \frac{1}{\sqrt{p}}(\phi(s) + \phi(p-s))
\end{aligned}
$$

$$\nu(0) + 2\sum_{s=1}^{h} \nu(s) = \nu(0) + \sum_{s=1}^{p-1} \nu(s)$$

$$\begin{aligned}
&= pP_0(C) + 2p\sum_{j=1}^{h} \frac{1}{[j]^2} N_j(C) + (p-1)pP_0(C) + p\sum_{s=1}^{p-1}\sum_{j=1}^{h} \frac{q^{js}+q^{-js}}{[j]^2} N_j(C) \\
&= p^2 P_0(C) \\
&= \frac{1}{p\sqrt{p}}\phi(p)
\end{aligned}$$

□

7 $SL(2,\mathbb{Z})$-representations on the center of $U_q sl(2)$

Let A be any factorizable finite-dimensional ribbon Hopf algebra, $\mu \in A^*$ be a left integral on A, suitably normalized. Then there are $\mathcal{S}, \mathcal{T} : A \to A$ obeying the modular identities

$$(\mathcal{S}\mathcal{T})^3 = \lambda \mathcal{S}^2, \quad \mathcal{S}^2 = S^{-1}$$

where λ is some constant and S is the antipode of A. $\mathcal{S}$ and $\mathcal{T}$ are given for all $x \in A$ by

$$\mathcal{S}(x) = (id \otimes \mu)(R^{-1}(1 \otimes x)R_{21}^{-1}), \quad \mathcal{T}x = \theta x$$

where R is the R-matrix of A and v is the ribbon element. Restricted to the center of A, $\mathcal{S}^4 = S^{-2} = id_{\mathcal{Z}(A)}$ since $S^{-2}(x) = G^{-1}xG$ for all $x \in A$. Here G is the balancing element of A. Thus we actually have a representation of $SL(2,\mathbb{Z})$ on $\mathcal{Z}(A)$.

As [La], when restricted to the center $\mathcal{Z}(A)$ of A, we may slightly modify the definition of this representation: for $a \in \mathcal{Z}(A)$

$$\mathcal{S}(a) = (\mu(S(a)) \otimes 1)R_{21}R = \phi(\chi^{-1}(a)),$$

$$\mathcal{T}(a) = \lambda \mathcal{S}^{-1}(\theta^{-1}(\mathcal{S}(a))).$$

The following theorem was conjectured by Kerler in [Ker1].

Theorem 3 (Kerler)**.** *Let $p = 2h+1$ be an odd number and q be a p-th primitive root of unity. The $SL(2,\mathbb{Z})$ representation on the center $\mathcal{Z}$ of $U_q sl(2)$ decomposes as*

$$\mathcal{Z} = \mathcal{P}_{h+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_h$$

$\mathcal{P}_{h+1}$ is an $(h+1)$ dimensional representation and $\mathbb{C}^2$ is the standard representation of $SL(2,\mathbb{Z})$. $\mathcal{V}_h$ is an h dimensional representation when restricted on which the matrices $\mathcal{S}$ and $\mathcal{T}$ are the same as those obtained by RT TQFT.

Proof. We choose a basis for $\mathcal{Z}$ as

$$\begin{aligned}
\nu(s) &= \chi(s) + \chi(p-s), \quad s = 1, \dots, h; \qquad \nu(0) = \chi(p) \\
\rho(s) &= \frac{p-s}{p}\chi(s) - \frac{s}{p}\chi(p-s), \quad s = 1, \dots, h \\
\varphi(s) &= \frac{1}{\sqrt{p}} \sum_{r=1}^{h} (q^{rs} - q^{-rs})(\frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s)), \quad s = 1, \dots, h
\end{aligned}$$

First,

$$\begin{aligned}
\mathcal{S}(\nu(s)) &= \phi\chi^{-1}(\chi(s) + \chi(p-s)) = \phi(s) + \phi(p-s) \\
&= \frac{1}{\sqrt{p}}(\nu(0) + \sum_{s'=1}^{h}(q^{ss'} + q^{-ss'})\nu(s')) \\
\mathcal{S}(\nu(0)) &= \frac{1}{\sqrt{p}}(\nu(0) + 2\sum_{s=1}^{h}\nu(s))
\end{aligned}$$

So $\mathcal{P}_{h+1}$ is invariant under the action of $\mathcal{S}$. Further,

$$\begin{aligned}
\mathcal{S}(\rho(s)) &= \phi\chi^{-1}(\frac{p-s}{p}\chi(s) - \frac{s}{p}\chi(p-s)) = \frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s) \\
&= -\frac{1}{\sqrt{p}}\sum_{r=1}^{h}(q^{rs} - q^{-rs})\varphi(s)
\end{aligned}$$

Note the facts that $\mathcal{S}^2 = S^{-1}$ and the antipode S acts identically on the center $\mathcal{Z}$. Then $\mathcal{S}(\frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s)) = \rho(s)$. And

$$\begin{aligned}
\mathcal{S}(\varphi(s)) &= \frac{1}{\sqrt{p}}\sum_{r=1}^{h}(q^{rs} - q^{-rs})\mathcal{S}(\frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s)) \\
&= \frac{1}{\sqrt{p}}\sum_{r=1}^{h}(q^{rs} - q^{-rs})\rho(s)
\end{aligned}$$

$\{\rho(s),\varphi(s)\}_{s=1,\ldots,h}$ span the $\mathbb{C}^2\otimes\mathcal{V}_h$ that is invariant under the action of $\mathcal{S}$, and the matrix of $\mathcal{S}$ is

$$\begin{pmatrix} 0_{h\times h} & -\frac{q-q^{-1}}{\sqrt{p}}S_{semi} \\ \frac{q-q^{-1}}{\sqrt{p}}S_{semi} & 0_{h\times h}\end{pmatrix} = -\frac{q-q^{-1}}{\sqrt{p}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\otimes S_{semi}$$

where S_{semi} is the semisimple S-matrix $([q^{rs}])_{h\times h}$.

Next, θ acts on $\phi(s)$ as

$$\theta\phi(s) = q^{-\frac{1}{2}(s^2-1)}\phi(s), \quad s=1,\ldots,h$$

Then we have

$$\mathcal{T}(\chi(s)) = \lambda\mathcal{S}^{-1}(\theta^{-1}\phi(s)) = q^{\frac{1}{2}(s^2-1)}\lambda\chi(s)$$

and so $\mathcal{P}_{h+1}$ is invariant under $\mathcal{T}$ and actually a $(h+1)$ dimensional representation of $SL(2,\mathbb{Z})$. Moreover, $\mathcal{T}(\rho(s)) = q^{\frac{1}{2}(s^2-1)}\lambda\rho(s)$.

Finally, we want to evaluate $\mathcal{T}(\varphi(s))$. Recall that

$$\begin{aligned}\theta &= \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)}P_s + (q-q^{-1})\sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)}(\frac{p-s}{[p-s]}N_s^+ - \frac{s}{[s]}N_s^-)\\ &= \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)}P_s + \frac{q-q^{-1}}{\sqrt{p}}\sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)}[s](\frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s))\end{aligned}$$

It is easy to check

$$\theta^{-1} = \sum_{t=0}^{h} q^{\frac{1}{2}(t^2-1)}P_t - \frac{q-q^{-1}}{\sqrt{p}}\sum_{t=1}^{h} q^{\frac{1}{2}(t^2-1)}[t](\frac{p-t}{p}\phi(t) - \frac{t}{p}\phi(p-t))$$

Then

$$\begin{aligned}\mathcal{T}(\varphi(s)) &= \lambda\mathcal{S}^{-1}(\theta^{-1}\frac{1}{\sqrt{p}}\sum_{r=1}^{h}(q^{rs}-q^{-rs})(\frac{p-r}{p}\chi(r) - \frac{r}{p}\chi(p-r)))\\ &= \frac{\sqrt{p}}{q^s-q^{-s}}\lambda\mathcal{S}^{-1}(\theta^{-1}(P_s - \frac{q^s+q^{-s}}{[s]^2}N_s))\\ &= \frac{\sqrt{p}q^{\frac{1}{2}(s^2-1)}}{q^s-q^{-s}}\lambda\mathcal{S}^{-1}(P_s - \frac{q^s+q^{-s}}{[s]^2}N_s + \frac{q-q^{-1}}{\sqrt{p}}[s](\frac{p-s}{p}\phi(s) - \frac{s}{p}\phi(p-s)))\\ &= q^{\frac{1}{2}(s^2-1)}\lambda\varphi(s) + q^{\frac{1}{2}(s^2-1)}\lambda\rho(s)\end{aligned}$$

Here we used that $\mathcal{S}(\varphi(s)) = \frac{1}{\sqrt{p}} \sum_{r=1}^{h} (q^{rs} - q^{-rs})\rho(r) = \frac{\sqrt{p}}{q^s - q^{-s}}(P_s - \frac{q^s+q^{-s}}{[s]^2} N_s))$.

$\mathbb{C}^2 \otimes \mathcal{V}_h$ is also invariant under the action of $\mathcal{T}$. The matrix the matrix of $\mathcal{T}$ is

$$\begin{pmatrix} \lambda T_{semi} & \lambda T_{semi} \\ 0_{h\times h} & \lambda T_{semi} \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes T_{semi}$$

Here $T_{semi} = diag(\ldots, q^{\frac{1}{2}(s^2-1)}, \ldots)_{h\times h}$ is the semisimple T-matrix. □

8 Verlinde Formula

In the semisimple case, the S-matrix diagonalize the fusion matrices. Now we want to generalize the classical Verlinde formula to the case of projective modules. The fusion rule among indecomposable projective modules is

$$\nu(s)\nu(s') = \nu(0) + 2\sum_{r=1}^{h} \nu(r), \quad 0 \le s, s' \le h$$

The matrix of $\mathcal{S}$ acting on $\mathcal{P}_{h+1} = span(\nu_0, \ldots, \nu_h)$ is given as

$$S_N = S_N^{-1} = \frac{1}{\sqrt{p}} \begin{pmatrix} 1 & 1 & & \cdots & & 1 \\ 2 & q+q^{-1} & & \cdots & & q^h + q^{-h} \\ & & \ddots & & & \\ \vdots & \vdots & & q^{ss'} + q^{-ss'} & & \vdots \\ & & & & \ddots & \\ 2 & q^h + q^{-h} & & \cdots & & q^{h^2} + q^{-h^2} \end{pmatrix}_{(h+1)\times(h+1)}$$

The fusion matrix is

$$N_{\nu(s)}^{U_{h+1}} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 2 & \cdots & \cdots & 2 \\ \vdots & \cdots & \cdots & \vdots \\ 2 & \cdots & \cdots & 2 \end{pmatrix}_{(h+1)\times(h+1)} \qquad s = 0\ldots, h$$

Then

$$S_N N_{\nu(s)}^{U_{h+1}} S_N^{-1} = \begin{pmatrix} p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(h+1)\times(h+1)}$$

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