NON-MIXING LAGRANGIAN SOLUTIONS TO THE MULTISPECIES POROUS MEDIA EQUATION

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Abstract. In this paper, we consider a multispecies generalization of the porous media equation used in cancer modelling. Here each species represents a different cell population e.g. healthy cells versus tumor cells. The density of each population solves a continuity equation where the velocity field is given by a pressure gradient that is induced by the total cell density. The resulting model is a challenging system of coupled hyperbolic and parabolic equations, indeed, the individual species do not regularize over time and discontinuities can both form and persist. As a result, the existence of weak solutions to this model has only been achieved recently and many important questions still remain. A particularly important open question is whether it is possible for the different populations to mix together if they were separated at initial time. The main result of this paper is the construction of solutions that do not mix. To do this, we show that it is possible to construct both the forward and backward Lagrangian flows along the pressure gradient — a result that may be of independent interest as the pressure gradient lacks sufficient regularity to apply the theory of regular Lagrangian flows. To overcome this difficulty, we combine ideas from [CdL08] and [GPŚG19] to show that the bad parts of the pressure gradient can be ignored. Once we have the flow maps, it is straightforward to show that the populations do not mix.

1. Introduction

The Porous Media Equation (PME) is a non-linear analogue of the heat equation that has various important physical applications [Váz07]. PME describes the evolution of a density ρ flowing down a pressure gradient ∇p, where the pressure function p is coupled to the density through a monotone relation. A particularly important application of PME is the modelling of living cells and tissues, particularly in the context of tumor growth [BKMP03, PT08, RBE$^+$10, PQV14]. In these models, the cells are treated as a viscous and nearly incompressible fluid. When cells proliferate and grow, there is a buildup of mechanical pressure, which both pushes cells down the pressure gradient and affects growth rates via the biological phenomenon of contact inhibition [PQV14]. This can be modeled by PME with a pressure dependent source term.

For realistic modelling, it is important to take into account multiple cell populations (e.g. healthy cells versus tumor cells) and nutrient availability. In this paper, we will be interested in studying a system of evolution equations for a finite number of cell populations with densities $\rho_1, \dots, \rho_\ell$, whose total density $\rho = \sum_{i=1}^{\ell} \rho_i$ evolves according to PME with a source term. Given a parameter $\gamma > 0$, each individual population evolves according to the continuity equation

$$\partial_t \rho_i - \nabla \cdot (\rho_i \nabla p) = \rho_i G_i, \quad p = \rho^\gamma, \tag{1.1}$$

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where G_i is a growth function that depends on the pressure and a nutrient variable n. The nutrient is coupled to the other variables through the diffusion equation

$$\partial_t n - \alpha \Delta n = -n \sum_{i=1}^{\ell} \beta_i \rho_i, \tag{1.2}$$

where α, β_i are scalars that determine the diffusion rate and consumption rate of the nutrients respectively. The connection between the system (1.1-1.2) and the classical PME can be seen by summing (1.1) over each of the populations. Doing so produces the equation

$$\partial_t \rho - \nabla \cdot (\rho \nabla p) = \rho G, \quad p = \rho^\gamma, \tag{1.3}$$

where $G = \sum_{i=1}^{\ell} \frac{\rho_i}{\rho} G_i$.

In the case of multiple cell populations, the model is a challenging system of coupled PDEs. Indeed, existence of solutions to these systems in dimensions $d > 1$ was only achieved recently (see [BHI$^+$20, CFSS18] for results in one dimension) in the series of papers [GPG19, BCP20, LX21, Jac21] for $d > 1$, while well-posedness remains open. The difficulty of these systems stems from the fact that although equation (1.3) is degenerate parabolic, the evolution equations for the individual populations (1.1) are hyperbolic. Hence, the equation does not have any regularizing effect on the ρ_i (for instance discontinuities at initial time will persist throughout the evolution). While [CFSS18] was able to obtain strong compactness of the ρ_i in one dimension, the situation in $d > 1$ is more complicated. Following the approach of [GPŚG19], all of the results for $d > 1$ have constructed solutions by obtaining strong compactness for the pressure variable instead. The advantage of working with the pressure is that one can focus on the good properties of equation (1.3), however, this approach cannot say much about the properties of the limiting ρ_i.

An important question that has remained open is whether the individual populations remain unmixed throughout the evolution if they were separated at initial time. More concretely, given initial data $\{\rho_i^0\}_{i \in \{1,\dots,\ell\}}$ such that $\min(\rho_i^0, \rho_j^0) = 0$ for all $i \neq j$ one wishes to know whether it is possible to construct solutions such that $\min(\rho_i, \rho_j) = 0$ almost everywhere for all $i \neq j$. The main result of this paper is an affirmative answer to this question: it is possible to construct nonmixing solutions. To establish the nonmixing property, we prove the existence of the Lagrangian flow along $-\nabla p$ — a result that may be of independent interest. Indeed, let us emphasize that $-\nabla p$ does not have sufficient regularity to apply the theory of regular Lagrangian flows [DL89, Amb04, Amb08], forcing us to develop a new approach based on the quantitative arguments in [CdL08]. Once we have the flow map, we show that the densities satisfy an explicit representation formula (c.f. Definition 1.1) that implies the non-mixing property. Finally, let us also note that our arguments are strong enough to pass to the incompressible limit $\gamma \to \infty$, which is an important special case for realistic modelling.

1.1. Lagrangian flows for PME.

As we mentioned above, constructing Lagrangian flows for PME is difficult to due the low regularity of the pressure variable p. Even in the case of a single population, it is well known that when ρ^0 has compact support Δp is a singular measure [Váz07]. Notably, this is not strong enough to bound ∇p in BV. Hence, the theory of regular Lagrangian flows cannot be applied to $-\nabla p$ even in the most classical case.

On the other hand, PME has much more structure than a flow along an arbitrary vector field with poor regularity. To understand this better, we can use the relation $p = \rho^\gamma$ to rewrite (1.3) in terms of p, leading to the equivalent formulation

$$\partial_t p - |\nabla p|^2 - \gamma p(\Delta p + G) = 0. \tag{1.4}$$

From (1.4), one can see that PME is a degenerate parabolic equation, whose second order irregularities must occur in the vicinity of the level set $\{p = 0\}$. Indeed, it is known that quantities of the form $\int_{Q_T} p|D^2 p|^2$ are finite under rather general assumptions on the structure of G

[MPQ17, GPG19, DP21]. Hence, there is hope to construct Lagrangian flows provided that one can show that most trajectories stay away from $\{p = 0\}$.

The question of whether Lagrangian trajectories avoid $\{p = 0\}$ is closely tied to the structure of equation (1.4). If one assumes formally that X is a Lagrangian flow map satisfying the equation $\partial_t X = -\nabla p \circ X$, then

$$\frac{d}{dt}(p \circ X) = \big(\partial_t p - |\nabla p|^2\big) \circ X = \big(\gamma p(\Delta p + G)\big) \circ X. \tag{1.5}$$

The singularity of Δp at $\{p = 0\}$ is known to be positive, but we could still hope to gain information from (1.5) by bounding the negative part of $\gamma(\Delta p+G)$. Indeed, a uniform bound on the negative part would imply that trajectories that are not near $\{p = 0\}$ at time t were never near $\{p = 0\}$ in the past. This would allow us to construct regular Lagrangian flows along such trajectories using existing theory.

Due to the complicated structure of our growth term G, uniform bounds seem extremely unlikely ($G = \sum_{i=1}^{\ell} \frac{\rho_i}{\rho} G_i$ can have discontinuities from the ρ_i) and there are cases where they are known to be false [DP21]. This is in contrast to the classical case $G = 0$, where bounds are known through the celebrated Aronson-Bénilan (AB) estimates [AB79] (see also [PQV14] for similar uniform estimates in the case of one population where G is a decreasing function of the pressure only).

Luckily, the situation is not entirely hopeless. In [GPŚG19], the authors developed new arguments to obtain L^2 and L^3 analogues of the AB estimates in the presence of multiple populations; however, these estimates require somewhat restrictive assumptions on the structure of the G_i's and they deteriorate in the $\gamma \to \infty$ limit. In this paper, we make a meaningful improvement to these estimates by instead considering the slightly weaker quantity $\int_{[0,T]\times\mathbb{R}^d} \gamma^2 \rho \log(1 + \frac{1}{p})^{2(1-\lambda')}(\Delta p + G)_-^2$ for some $\lambda' \in (0, 1]$. By working with this quantity instead, we are able to drop the restrictive assumptions on the G_i's when $\gamma < \infty$ and we can pass to the $\gamma \to \infty$ limit for $\lambda' \in (0, 1/2)$. Once we have a bound on the negative part of $\gamma(\Delta p + G)$, we can use the structure of (1.4) to conclude that $\int \gamma\rho|\Delta p+G| \log(1 + \frac{1}{p})^{(1-\lambda')}$ is bounded which gives us an L^1 control on the forward and backward in time behavior of (1.5).

Unsurprisingly, a price must be paid for downgrading L^∞ estimates to L^1 estimates. L^1 bounds applied to (1.5) can only be used to obtain the following logarithmic version of Gronwall's inequality

$$\sup_{t\in[0,T]} \int_E \rho^0(x) \log(1 + \frac{1}{p(t, X(t, x))})\, dx \lesssim_T 1 + \int_E \rho^0(x) \log(1 + \frac{1}{p(0, x)})\, dx \tag{1.6}$$

where $E \subset \mathbb{R}^d$ is chosen to make the right hand side finite. This is not strong enough to rule out the possibility that every Lagrangian trajectory spends some time near $\{p = 0\}$. Thus, we cannot just hope to apply the existing theory of regular Lagrangian flows.

To circumvent this problem, we instead adapt the logarithmic Gronwall estimates in [CdL08] to show that doubly logarithmic quantities measuring the stability of approximations to the forward and backward flow maps can be controlled. Although the doubly logarithmic bounds are extremely weak, they will provide sufficient compactness to deduce that the approximations converge to the correct limit. Once we have the flow maps, everything else is smooth sailing and the nonmixing results neatly follow.

1.2. **Preliminaries and main results.** We begin by giving a more concrete description of the growth terms and our important assumptions on them. Throughout the paper we shall place the following assumptions on the G_i.

(G1) Each $G_i := G_i(p, n)$ is a continuous and uniformly bounded function of the pressure p and nutrient n.

(G2) If the pressure is sufficiently high, no growth occurs regardless of nutrient availability, i.e. there exists some $p_h > 0$ such that $G_i(p, n) < 0$ for all $i \in \{1, \dots, \ell\}, n \in [0, \infty)$ and $p > p_h$ (the value p_h has been called the homeostatic pressure in the literature [PQV14]).

(G3) The following mild technical condition on the derivatives holds:

$$\Big(\sum_{i=1}^{\ell} p\partial_p G_i(p, n)\Big)_+ + \max_{i\in\{1,\dots,\ell\}} |\partial_n G_i(p, n)| \in L^\infty_{\text{loc}}([0, \infty)^2).$$

When we pass to the incompressible limit $\gamma \to \infty$ we will require the G_i to satisfy two additional condition:

(G4) $\min_{i\in\{1,\dots,\ell\}} \inf_{(p,n)\in[0,\infty)^2} \frac{1}{2} G_i(p, n) - \big(p\partial_p G_i(p, n)\big)_+ > 0.$

Unlike (G1-G3), this last condition is much more restrictive from a modelling perspective. (G4) forces $G_i(0, 0) > 0$, meaning the cells must grow even in the absence of nutrients. Let us note however that (G4) is not purely technical, some property related to (G4) is necessary to guarantee the nonmixing property in the incompressible case. Given two populations with growth functions satisfying $G_1(0, 0) > 0$ and $G_2(0, 0) < 0$, it is easy to cook up a scenario where population 1 instantaneously mixes into population 2. For instance, this will always happen in a scenario where one sets the initial nutrients to zero and chooses a starting condition where the populations are separated, share a codimension 1 boundary, and both saturate the incompressible constraint on their respective supports.

For the initial data, all of our conditions are on the total density $\rho^0 = \sum_{i=1}^{\ell} \rho_i^0$, the corresponding initial pressure p^0, and the starting nutrient level n^0. We shall require the following regularity conditions.

(ID1) $\rho^0 \in L^1(\mathbb{R}^d)$, $\rho^0 \in [0, p_h^{\frac{1}{\gamma}}]$, and $|x|^2\rho^0 \in L^1(\mathbb{R}^d)$.

(ID2) $\nabla p^0 \in L^2(\mathbb{R}^d)$, $p^0 \le p_h$ almost everywhere, $\rho^0 \log(1+\frac{1}{p^0}) \in L^1(\mathbb{R}^d)$, and $n^0 \in W^{1,\infty}(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$.

(ID3) $\gamma\rho^0(\Delta p^0 + \sum_{i=1}^{\ell} \frac{\rho_i^0}{\rho_i} G_i(p^0, n^0))_-^2 \in L^1(\mathbb{R}^d)$.

When we pass to the incompressible limit we shall require the two following additional conditions

(ID4) $\rho^0 \in \{0, 1\}$ almost everywhere.

(ID5) There exists a constant $\lambda > 0$ such that $\rho^0 \log(1 + \frac{1}{p^0})^\lambda \in L^1(\mathbb{R}^d)$.

Next, we give a concrete description of the solutions that we are interested in constructing.

Definition 1.1. We will say that a tuple $(\rho_1, \dots, \rho_k, p, n)$ is a *complete Lagrangian solution* to the system (1.1-1.2) with initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ if the following conditions are met.

(i) $(\rho_1, \dots, \rho_k, p, n)$ is a weak solution to (1.1-1.2) with initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ such that for any $T > 0$, $\rho = \sum_{i=1}^{\ell} \rho_i \in L^\infty([0, T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$.

(ii) For all $t, s \ge 0$ there exist unique forward and backward flow maps X, Y satisfying the Lagrangian Flow equations

$$X(t, s, x) = x - \int_s^{t+s} \nabla p(\tau, X(\tau, s, x))\, d\tau \quad \text{for almost all } x \in \text{spt}(\rho(s, \cdot)), \tag{1.7}$$

and

$$Y(t, s, x) = x + \int_{\min(s-t,0)}^{s} \nabla p(\tau, Y(\tau, s, x)) d\tau \quad \text{for almost all } x \in \text{spt}(\rho(s, \cdot)). \tag{1.8}$$

(iii) There exists a constant $B \geq 0$ such that for all $t, s \geq 0$

$$e^{-tB}\rho(s+t,\cdot) \leq X(t,s,\cdot)_{\#}\rho(s,\cdot) \leq e^{tB}\rho(s+t,\cdot) \quad \text{for almost all } x \in \operatorname{spt}(\rho(s+t,\cdot)), \tag{1.9}$$

and

$$e^{-tB}\rho(s,\cdot) \leq Y(t,s,\cdot)_{\#}\rho(s+t,\cdot) \leq e^{tB}\rho(s,\cdot) \quad \text{for almost all } x \in \operatorname{spt}(\rho(s,\cdot)). \tag{1.10}$$

(iv) The maps satisfy the semigroup property

$$X(t,s,x) = X(t-t', s+t', X(t',s,x)) \quad \text{for almost all } x \in \operatorname{spt}(\rho(s,\cdot)), \tag{1.11}$$

$$Y(t,s,x) = Y(t-t', s-t', Y(t',s,x)) \quad \text{for almost all } x \in \operatorname{spt}(\rho(s,\cdot)), \tag{1.12}$$

and the inversion formulas

$$X(t,s,Y(t,s+t,x)) = x \quad \text{for almost all } x \in \operatorname{spt}(\rho(t+s,\cdot)), \tag{1.13}$$

$$Y(t,s,X(t,s-t,x')) = x' \quad \text{for almost all } x' \in \operatorname{spt}(\rho(s,\cdot)). \tag{1.14}$$

(v) For any test function φ, each ρ_i satisfies the representation formula

$$\int_{\mathbb{R}^d} \rho_i(s+t,x)\varphi(x) = \int_{\mathbb{R}^d} \rho_i(s,x)\varphi(X(t,s,x)) \exp\Big(\int_s^{t+s} G_i \circ X(\tau,s,x)d\tau\Big)\, dx, \tag{1.15}$$

where $G_i \circ X(\tau,s,x)$ is shorthand for $G_i\Big(p\big(\tau, X(\tau,s,x)\big), n\big(\tau, X(\tau,s,x)\big)\Big)$.

Remark 1.2. The uniqueness of the flow maps along $-\nabla p$ guarantees the uniqueness of the ρ_i and n when p is held fixed. However, we are not able to prove that the system itself has a unique solution. Indeed, we cannot rule out the possibility that there could be solutions with different pressure variables.

We are now ready to give our main results. For convenience we restrict our attention to values of $\gamma \geq 1$.

Theorem 1.3. *Given growth terms satisfying assumptions (G1-G3), initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ satisfying (ID1-ID3), and $\gamma \in [1, \infty)$, there exists a complete Lagrangian solution $(\rho_1, \dots, \rho_\ell, n, p)$ for (1.1-1.2). Furthermore if for some $i \neq j$ we have $\min(\rho_i^0(x), \rho_j^0(x)) = 0$ almost everywhere, then for every $t \geq 0$ we have $\min(\rho_i(t,x), \rho_j(t,x)) = 0$ almost everywhere in x.*

Theorem 1.4. *Given growth terms satisfying assumptions (G1-G4) and initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ satisfying (ID1-ID5) along with the additional condition $\rho^0 \in \{0, 1\}$ almost everywhere, there exists a complete Lagrangian solution $(\rho_1, \dots, \rho_\ell, n, p)$ to (1.1-1.2) with $\gamma = \infty$, i.e. the incompressible system*

$$\partial_t \rho_i - \nabla \cdot (\rho_i \nabla p) = \rho_i G_i, \quad p(1-\rho) = 0, \quad \rho \leq 1 \tag{1.16}$$

$$\partial_t n - \alpha \Delta n = -n \sum_{i=1}^{\ell} \beta_i \rho_i. \tag{1.17}$$

Furthermore if for some $i \neq j$ we have $\min(\rho_i^0(x), \rho_j^0(x)) = 0$ almost everywhere, then for every $t \geq 0$ we have $\min(\rho_i(t,x), \rho_j(t,x)) = 0$ almost everywhere in x.

Remark 1.5. In the incompressible case, one must take some extra care in verifying the assumption (ID2) on the initial data. Indeed, one cannot directly obtain the initial pressure p^0 from the initial total density ρ^0. Instead p^0 must be obtained by solving the equation

$$\Delta p^0 + \sum_{i=1}^{\ell} \frac{\rho_i^0}{\rho^0} G_i(p^0, n^0), \quad p^0(1-\rho^0) = 0. \tag{1.18}$$

While the condition $\nabla p^0 \in L^2(\mathbb{R}^d)$ is essentially automatic, the condition $\rho^0 \log(1+\frac{1}{p^0}) \in L^1(\mathbb{R}^d)$ forces p^0 to grow sufficiently quickly as one moves away from the boundary of $\mathrm{spt}(\rho^0)$. This forces some regularity on the geometry of the boundary $\mathrm{spt}(\rho^0)$. For instance, an interior ball condition is sufficient.

The rest of the paper is structured as follows. In Section 2, we assume that we have a smooth solution to the system and collect a number of important estimates, most crucially, the weighted AB type estimate on $\gamma^2(\Delta p+G)_-^2$ and the weighted L^1 estimate on $\gamma|\Delta p+G|$. In section 3, we show how these estimates can be used to establish equicontinuity properties for the associated Lagrangian flow maps. In the final Section, Section 4, we show how one can construct smooth approximations to the system and then take limits to prove the main results.

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2. PME estimates

Throughout this section, we will assume that we have a smooth solution $(\rho_1, \dots, \rho_\ell, p, n)$ to the system (1.1-1.2) where the initial data satisfies assumptions (ID1-ID3), the growth terms satisfy (G1-G3) and $\gamma \in [1, \infty)$. This will allow us to investigate properties of the system without having to worry about integrability or differentiability issues. Our main goal in this section will be to build towards bounds on $p|D^2 p|^2$ and $\rho(\Delta p+G)_-^2$ that only depend on the information (ID1-ID4). For notational convenience we shall use Q_T to denote the space time domain $Q_T := [0,T] \times \mathbb{R}^d$ for any $T > 0$.

Our analysis and estimates will be focused on the "nice" parabolic equations (1.3) and (1.4), rather than the hyperbolic equation (1.1). Nonetheless, we will still need to work with the individual densities ρ_i through their presence in the growth term $G = \sum_{i=1}^{\ell} \frac{\rho_i}{\rho} G_i$. A formal calculation shows that the ratios $c_i := \frac{\rho_i}{\rho}$ satisfy the transport equation

$$\partial_t c_i - \nabla c_i \cdot \nabla p = c_i(G_i - G). \tag{2.1}$$

Since we have already assumed we are working with smooth solutions, we can use this formula without issue. It will play an important role in some of the subsequent estimates.

We begin with some standard estimates for PME type equations.

Lemma 2.1. *Let $B = \sup_{(p,n)\in[0,\infty)^2} \max_i |G_i(p,n)|$ and fix some time $T \geq 0$. For any $m \geq 1$*

$$\int_{\mathbb{R}^d \times \{T\}} \frac{1}{m}\rho^m + \int_{Q_T} (m-1)\rho^{m-1}\nabla\rho \cdot \nabla p = e^{BmT}\int_{\mathbb{R}^d} \frac{1}{m}(\rho^0)^m, \tag{2.2}$$

and

$$\int_{\mathbb{R}^d \times \{T\}} \rho\log(\rho) + \int_{Q_T} \nabla\rho\cdot\nabla p \leq \int_{\mathbb{R}^d \times \{T\}} \rho + \int_{Q_T} B\rho\log(\rho) \tag{2.3}$$

$$\int_{\mathbb{R}^d \times \{T\}} |x|^2\rho \leq e^{T(B+1)}\Big(\int_{\mathbb{R}^d} |x|^2\rho^0 + \int_{Q_T} \rho|\nabla p|^2\Big). \tag{2.4}$$

Furthermore, $p \leq p_h$ almost everywhere.

Remark 2.2. Note that the negative part of $\rho\log(\rho)$ can be controlled by the second moments of ρ. Indeed, one has $-\rho\log(\rho) \lesssim_d \rho^{1-\frac{1}{d+1}} \lesssim_d \rho(1+|x|^2) + (1+|x|^2)^{-d}$.

Proof. The first relation follows from integrating equation (1.3) against ρ^{m-1} and using Gronwall's inequality. The second relation follows from integrating (1.3) against $\log(\rho+\delta)$, and then sending $\delta \to 0$. The third bound follows from integrating (1.3) against $|x|^2$, using Young's inequality (one can first integrate against $|x|^2 e^{-\delta|x|^2}$ and then send $\delta \to 0$ to check that $|x|^2$ is a valid test function), and then using Gronwall's inequality.

For the bound $p \leq p_h$, we can multiply equation (1.4) against $(p-p_h)_+$ to obtain

$$\int_{\mathbb{R}^d} \frac{d}{dt}\frac{1}{2}(p-p_h)_+^2 - (p-p_h)_+|\nabla p|^2 \leq \int_{\mathbb{R}^d} \gamma p(p-p_h)_+\Delta p$$

where we have used the fact that each $G_i \leq 0$ whenever $p \geq p_h$. After integrating by parts and dropping a good term, we see that

$$\int_{\mathbb{R}^d} \frac{d}{dt}\frac{1}{2}(p-p_h)_+^2 + (\gamma-1)(p-p_h)_+|\nabla p|^2 \leq 0,$$

and the result follows. □

The following Lemma will play a crucial role in our AB estimates.

Lemma 2.3. *If $\zeta : [0,\infty) \to \mathbb{R}$ is an increasing function such that ζ is C^1 on $(0,\infty)$ and $\eta(0) = 0$, then*

$$\int_{Q_T} \rho\zeta'(p)|\nabla p|^2 \leq \zeta(p_h)\big(\|\rho^0\|_{L^1(\mathbb{R}^d)} + B\|\rho\|_{L^1(Q_T)}\big).$$

Proof. Integrating (1.3) against $\zeta(p)$, we see that

$$\int_{\mathbb{R}^d} \zeta(p)\partial_t\rho + \rho\zeta'(p)|\nabla p|^2 = \int_{\mathbb{R}^d} \rho\eta(p)G.$$

Note that $\zeta(p)\partial_t\rho = \frac{\zeta(p)}{\gamma p^{1-\gamma}}\partial_t p$. If we set $\bar{\zeta}(a) = \int_0^a \frac{\zeta(a)}{\gamma a^{1-\gamma}}\,da$, then it follows that

$$\int_{\mathbb{R}^d} \rho\zeta'(p)|\nabla p|^2 + \frac{d}{dt}(\bar{\zeta}(p)) = \int_{\mathbb{R}^d} \rho\zeta(p)G.$$

Integrating with respect to time, we get

$$\int_{\mathbb{R}^d\times\{T\}} \bar{\zeta}(p) + \int_{Q_T} \rho\zeta'(p)|\nabla p|^2 = \int_{\mathbb{R}^d} \bar{\zeta}(p^0) + \int_{Q_T} \rho\zeta(p)G.$$

Since ζ is positive and increasing, we have $\zeta(p) \leq \zeta(p_h)$ and $\bar{\zeta}(p) \leq \zeta(p_h)p^{\frac{1}{\gamma}} = \zeta(p_h)\rho$. The result now follows. □

For the nutrient equation, we have the following estimates that are standard for the heat equation

Lemma 2.4. *For any time $T > 0$,*

$$\|n\|_{W^{1,\infty}(\{T\}\times\mathbb{R}^d)} \leq \|n^0\|_{W^{1,\infty}(\{T\}\times\mathbb{R}^d)} + \|\rho\|_{L^\infty(Q_T)}\|n\|_{L^3([0,T];L^\infty(\mathbb{R}^d))}T^{1/3}\sum_{i=1}^{\ell}\beta_i \tag{2.5}$$

$$\|\nabla n\|^2_{L^2(\{T\}\times\mathbb{R}^d)} + \|\partial_t n\|^2_{L^2(Q_T)} \lesssim \|\nabla n^0\|^2_{L^2(\mathbb{R}^d)} + \|n\|^2_{L^2(Q_T)}\|\rho\|^2_{L^2(Q_T)} \tag{2.6}$$

The next estimate is essentially taken directly from [DP21]. We reproduce the argument here since we are in the case of multiple populations, however, the differences are relatively minor. For notational convenience we will adopt the shorthand

$$u := -\gamma(\Delta p + G). \tag{2.7}$$

With this shorthand, equation (1.4) now reads

$$\partial_t p - |\nabla p|^2 + pu = 0. \tag{2.8}$$

Proposition 2.5. *For any time $T > 0$ there exists a constant $C(T)$ such that*

$$\int_{\mathbb{R}^d \times \{T\}} |\nabla p|^2 + \int_{Q_T} p|D^2 p|^2 \leq C(T). \tag{2.9}$$

For any increasing function $\eta : \mathbb{R} \to \mathbb{R}$ and $m \in [2, 4]$

$$\int_{Q_T} \eta'(p)|\nabla p|^m \lesssim 1 + \|\frac{\eta(p)^2}{p\eta'(p)^{\frac{2m-4}{4}}}\|_{L^{\frac{m}{4-m}}(Q_T)}\big(\int_{Q_T} p|\Delta p|^2 + p|D^2 p|^2\big). \tag{2.10}$$

Proof. Integrating equation (2.8) against $\frac{1}{\gamma}u$, we get

$$\frac{1}{\gamma}\int_{Q_T} u\partial_t p - u|\nabla p|^2 + pu^2 = 0.$$

Note that $-\frac{1}{\gamma}u|\nabla p|^2 = G|\nabla p|^2 + \Delta p|\nabla p|^2$. Integrating by parts, we see that

$$\int_{Q_T} \Delta p|\nabla p|^2 = \int_{Q_T} 2p|D^2 p|^2 + 2p\nabla\Delta p \cdot \nabla p = \int_{Q_T} 2p|D^2 p|^2 - 2\Delta p|\nabla p|^2 - 2p|\Delta p|^2.$$

Hence, we obtain the identity,

$$\int_{Q_T} \Delta p|\nabla p|^2 = \frac{2}{3}\int_{Q_T} p|D^2 p|^2 - p|\Delta p|^2.$$

Expanding $p|\Delta p|^2 = \frac{1}{\gamma^2}u^2 - \frac{2}{\gamma}puG + pG^2$, our combined work gives us

$$\frac{1}{\gamma}\int_{Q_T} u\partial_t p + \gamma G|\nabla p|^2 + (1 - \frac{2}{3\gamma})pu^2 - \frac{4}{3}puG + \frac{2\gamma}{3}pG^2 + \frac{2\gamma}{3}p|D^2 p|^2 \leq 0.$$

Applying Young's inequality to puG, we can conclude that for any $\gamma \geq 1$

$$\int_{Q_T} \frac{1}{\gamma}u\partial_t p + \frac{1}{\gamma}pu^2 + p|D^2 p|^2 \lesssim \int_{Q_T} pG^2 + G|\nabla p|^2 \tag{2.11}$$

Now we turn our attention to the time derivative term. We see that

$$\int_{Q_T} \frac{1}{\gamma}u\partial_t p = \int_{Q_T} -G\partial_t p - \Delta p\partial_t p = \|\nabla p(T, \cdot)\|^2_{L^2(\mathbb{R}^d)} - \|\nabla p^0\|^2_{L^2(\mathbb{R}^d)} - \int_{Q_T} G\partial_t p.$$

Recall that $G = \sum_{i=1}^{\ell} c_i G_i(p, n)$ where c_i satisfy (2.1). We then see that

$$G\partial_t p = \sum_{i=1}^{\ell} c_i\big(\frac{d}{dt}\bar{G}_i(p, n) - \partial_n \bar{G}_i(p, n)\partial_t n\big),$$

where $\bar{G}_i : \mathbb{R}^2 \to \mathbb{R}$ is defined as $\bar{G}_i(p, n) := \int_0^p G_i(a, n)\, da$. Hence,

$$\int_{Q_T} G\partial_t p = \sum_{i=1}^{\ell}\int_{Q_T} \frac{d}{dt}\big(c_i\bar{G}_i(p, n)\big) - c_i\partial_n\bar{G}_i(p, n)\partial_t n - \bar{G}_i(p, n)\partial_t c_i.$$

Using (2.1), we can now estimate

$$\Big|\int_{Q_T} G\partial_t p\Big| \leq \sum_{i=1}^{\ell}\|G_i\|_{L^\infty(\mathbb{R}^2)}\|p^0+p(T,\cdot)\|_{L^1(\mathbb{R}^d)} + \|\partial_n G_i\|_{L^\infty(\mathbb{R}^2)}\|p\|_{L^2(Q_T)}\|\partial_t n\|_{L^2(Q_T)} + \sum_{i=1}^{\ell}\Big|\int_{Q_T}\bar{G}_i(p,n)\big(\nabla c_i\cdot\nabla p + c_iG_i - c_i\sum_j c_jG_j\big)\Big|. \tag{2.12}$$

For the final integral we want to remove derivatives from c_i. Integrating by parts and then using Young's inequality, we get

$$\int_{Q_T}\bar{G}_i(p,n)\nabla c_i\cdot\nabla p = \int_{Q_T} c_i(\frac{1}{\gamma}u+G)\bar{G}_i(p,n) - c_iG_i(p,n)|\nabla p|^2$$
$$\leq \int_{Q_T} c_i\frac{1}{\gamma}pu^2 + \frac{1}{\gamma}\frac{\bar{G}_i(p,n)^2}{p} + G\bar{G}_i(p,n) - c_iG_i(p,n)|\nabla p|^2.$$

The first result now follows from combining the previous line with (2.11) and (2.12).

For the second result, we integrate by parts and then use Young's inequality to get

$$\int_{Q_T}\eta'(p)|\nabla p|^m = -\int_{Q_T}\eta(p)(\Delta p|\nabla p|^{m-2} + (m-2)|\nabla p|^{m-4}D^2p:\nabla p\otimes\nabla p)$$
$$\leq \frac{1}{2}\int_{Q_T} ap|\Delta p|^2 + a(m-2)p|D^2p|^2 + a^{-1}(m-1)\frac{\eta(p)^2}{p}|\nabla p|^{2m-4}$$

for some constant a. After applying Holder's inequality we obtain

$$\int_{Q_T}\eta'(p)|\nabla p|^m \lesssim a\|\frac{\eta(p)^2}{p\eta'(p)^{\frac{2m-4}{4}}}\|_{L^{\frac{m}{4-m}}(Q_T)}\|\eta'(p)|\nabla p|^m\|_{L^1(Q_T)}^{\frac{2m-4}{m}} + a^{-1}\big(\int_{Q_T} p|\Delta p|^2 + p|D^2p|^2\big).$$

Since $\frac{2m-4}{m}\leq 1$ for $m\in[2,4]$, it follows that

$$\int_{Q_T}\eta'(p)|\nabla p|^m \lesssim 1 + \|\frac{\eta(p)^2}{p\eta'(p)^{\frac{2m-4}{4}}}\|_{L^{\frac{m}{4-m}}(Q_T)}\big(\int_{Q_T} p|\Delta p|^2 + p|D^2p|^2\big).$$

as desired.

□

The rest of this section will be building towards the weighted L^1 bounds on $\gamma(\Delta p+G)$ (c.f. Proposition 2.10). Most of the effort will be in establishing weighted L^2 AB type estimates on $\gamma(\Delta p+G)_-$ which is equivalent to estimating u_+^2. Our estimate of u_+^2 is a modification of the estimate from [GPŚG19]. Instead of directly estimating u_+^2, we consider the weighted quantity $\omega(p)u_+^2$. We shall require that our weight $\omega:\mathbb{R}\to\mathbb{R}$ satisfies the following properties

(W1) ω is nonnegative, increasing, and concave.
(W2) $\omega(a)\leq\gamma a\omega'(a)$ for all $a\in[0,p_h]$.
(W3) $\int_0^{p_h}\frac{\omega(a)}{a}\,da<\infty$.

We will keep the weights abstract until our L^1 estimate, Proposition 2.10, where we will finally fix a choice.

Let us note that the main advantage of working with these weaker weighted quantity is that we can have far less restrictive structural assumptions on the growth terms and our estimates will hold in the incompressible limit $\gamma\to\infty$. In addition, the calculation itself will be a bit simpler since we do not need to include a localizing function (more precisely, one can think of $\omega(p)$ as a special choice of a localizing function). Nonetheless, the calculation is still quite complicated and will be separated into a few different steps. Readers who are just interested in the bound itself can skip to the statements of Propositions 2.9 and more importantly 2.10. Readers who are interested in the argument itself will be "rewarded" with many "fun" (tedious) applications of integration by parts and Young's inequality.

Lemma 2.6. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^2 convex increasing function such that $f(a)=0$ for all $a\leq 0$. If we let f^* denote the convex conjugate of f, then*

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)f^*(f'(u))(\frac{1}{\gamma}u+G)+p\omega(p)f''(u)|\nabla u|^2 \\ \leq \int_{\mathbb{R}^d}Gf(u)(\frac{2}{\gamma}\omega(p)-p\omega'(p))+\omega(p)f'(u)(2\nabla G\cdot\nabla p-\partial_t G) \tag{2.13}$$

Remark 2.7. Rather than directly work with $f(u)=u_+^2$ we instead consider a more generic function f, which makes it easier to see when an integration by parts will be useful and helps us see why we will eventually be forced into the choice $f(u)=u_+^2$.

Proof. Differentiating in time and using (1.4), we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)=\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega'(p)[|\nabla p|^2-pu]f(u)+\omega(p)f'(u)\frac{1}{\gamma}\partial_t u.$$

Using (1.4) again, we see that

$$\frac{1}{\gamma}\partial_t u=\Delta(pu)-\Delta|\nabla p|^2-\partial_t G.$$

Expanding the terms with the Laplacian, we get

$$\frac{1}{\gamma}\partial_t u=-(\frac{1}{\gamma}u+G)u+2\nabla u\cdot\nabla p+p\Delta u-2|D^2p|^2+2\nabla(\frac{1}{\gamma}u+G)\cdot\nabla p-\partial_t G$$

Hence, after some rearranging, we have shown that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)uf'(u)(\frac{1}{\gamma}u+G)+2\omega(p)f'(u)|D^2p|^2 \\ =\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega'(p)\big(|\nabla p|^2-pu\big)f(u)+\omega(p)f'(u)\Big(2\nabla u\cdot\nabla p+p\Delta u+2\nabla(\frac{1}{\gamma}u+G)\cdot\nabla p\Big)-\omega(p)f'(u)\partial_t G \tag{2.14}$$

Now we want to move spatial derivatives off of u. Moving $f'(u)$ inside the parentheses, we see that the second term on the right hand side of (2.14) is equal to

$$\int_{\mathbb{R}^d}\omega(p)\Big(2(1+\frac{1}{\gamma})\nabla f(u)\cdot\nabla p+p\Delta f(u)-pf''(u)|\nabla u|^2++2f'(u)\nabla G\cdot\nabla p\Big)$$

Integrating by parts, the previous line is equal to

$$\int_{\mathbb{R}^d}f(u)\Delta(p\omega(p))-2(1+\frac{1}{\gamma})f(u)\nabla\cdot(\omega(p)\nabla p)-p\omega(p)f''(u)|\nabla u|^2+2\omega(p)f'(u)\nabla G\cdot\nabla p \tag{2.15}$$

Now we expand $\Delta(p\omega(p))=\nabla\cdot(\omega(p)\nabla p)+\nabla\cdot(p\omega'(p)\nabla p)$ to see that (2.15) is equal to

$$\int_{\mathbb{R}^d}-(1+\frac{2}{\gamma})f(u)\nabla\cdot(\omega(p)\nabla p)+f(u)\nabla\cdot(p\omega'(p)\nabla p)-p\omega(p)f''(u)|\nabla u|^2+2\omega(p)f'(u)\nabla G\cdot\nabla p \tag{2.16}$$

Plugging this back into (2.14) and rearranging, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)uf'(u)(\frac{1}{\gamma}u+G)+2\omega(p)f'(u)|D^2p|^2+\omega(p)f'(u)\partial_t G$$
$$=\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega'(p)\big(|\nabla p|^2-pu\big)f(u)-(1+\frac{2}{\gamma})f(u)\nabla\cdot(\omega(p)\nabla p)$$
$$+\int_{\mathbb{R}^d}f(u)\nabla\cdot(p\omega'(p)\nabla p)-p\omega(p)f''(u)|\nabla u|^2+2\omega(p)f'(u)\nabla G\cdot\nabla p \tag{2.17}$$

Expanding the terms with the divergence operator, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)uf'(u)(\frac{1}{\gamma}u+G)+2\omega(p)f'(u)|D^2p|^2+\omega(p)f'(u)\partial_t G$$
$$=\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega'(p)\big(|\nabla p|^2-pu\big)f(u)+(1+\frac{2}{\gamma})f(u)\omega(p)(\frac{1}{\gamma}u+G)-\frac{2}{\gamma}f(u)\omega'(p)|\nabla p|^2$$
$$+\int_{\mathbb{R}^d}f(u)(p\omega''(p)|\nabla p|^2-p\omega'(p)(\frac{1}{\gamma}u+G))-p\omega(p)f''(u)|\nabla u|^2+2\omega(p)f'(u)\nabla G\cdot\nabla p \tag{2.18}$$

Combining similar terms and rearranging, we get

(2.19)
$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)(uf'(u)-f(u))(\frac{1}{\gamma}u+G)+2\omega(p)f'(u)|D^2p|^2+p\omega(p)f''(u)|\nabla u|^2$$
$$=\int_{\mathbb{R}^d}\frac{2}{\gamma}\big(\frac{1}{\gamma}\omega(p)-p\omega'(p)\big)uf(u)+Gf(u)(\frac{2}{\gamma}\omega(p)-p\omega'(p))+f(u)|\nabla p|^2\big(p\omega''(p)-\frac{1}{\gamma}\omega'(p)\big)$$
$$+\int_{\mathbb{R}^d}\omega(p)f'(u)(2\nabla G\cdot\nabla p-\partial_t G)$$

Thanks to our assumptions on f and ω the terms $2\omega(p)f'(u)|D^2p|^2$, $\frac{2}{\gamma}\big(\frac{1}{\gamma}\omega(p)-p\omega'(p)\big)uf(u)$ and $f(u)|\nabla p|^2\big(p\omega''(p)-\frac{1}{\gamma}\omega'(p)\big)$ are all terms with favorable signs. Dropping these terms, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)f(u)+\int_{\mathbb{R}^d}\omega(p)\big(uf'(u)-f(u)\big)(\frac{1}{\gamma}u+G)+p\omega(p)f''(u)|\nabla u|^2$$
$$\leq\int_{\mathbb{R}^d}Gf(u)(\frac{2}{\gamma}\omega(p)-p\omega'(p))+\omega(p)f'(u)(2\nabla G\cdot\nabla p-\partial_t G) \tag{2.20}$$

The result now follows from the identity $uf'(u)-f(u)=f^*(f'(u))$.

□

In the next Lemma, we tackle the estimate of the term $(\partial_t G-2\nabla G\cdot p)$. This term is quite annoying since it has the form of a transport equation along $-2\nabla p$ instead of $-\nabla p$. Ultimately, we would like to estimate this term in such a way that there are no derivatives on the ratio variables c_i.

Lemma 2.8. *There exists a constant C depending only on T and the initial data such that for any $\theta>0$*

$$\int_{Q_T}\omega(p)f'(u)(2\nabla G\cdot\nabla p-\partial_t G)\leq C(1+\theta^{-1})+\int_{Q_T}\theta\omega(p)f'(u)^2$$
$$+\int_{Q_T}\frac{1}{2}p\omega(p)f''(u)^2|\nabla u|^2+\omega(p)f'(u)G(\frac{1}{\gamma}u+G)+uf'(u)p\omega(p)\partial_p G. \tag{2.21}$$

Proof. We recall that $G = \sum_{i=1}^{\ell} c_i G_i(p,n)$. Hence,

$$2\nabla G \cdot \nabla p = \sum_{i=1}^{\ell} 2\nabla c_i \cdot \nabla p G_i(p,n) + 2c_i \partial_p G_i(p,n)|\nabla p|^2 + 2c_i \partial_n G_i(p,n) \nabla p \cdot \nabla n$$

and

$$\partial_t G = \sum_{i=1}^{\ell} \partial_t c_i G_i(p,n) + c_i \partial_p G_i(p,n) \partial_t p + c_i \partial_n G_i(p,n) \partial_t n.$$

Using equation (2.1), we have

$$2\nabla G \cdot \nabla p - \partial_t G = \sum_{i=1}^{\ell} \nabla c_i \cdot \nabla p G_i(p,n) + 2c_i \partial_p G_i(p,n)|\nabla p|^2 + 2c_i \partial_n G_i(p,n) \nabla p \cdot \nabla n \\ + c_i\big(G - G_i(p,n)\big) + c_i \partial_p G_i(p,n)\partial_t p - c_i \partial_n G_i(p,n) \partial_t n \tag{2.22}$$

Using equation (1.4) and grouping similar terms, we get

$$2\nabla G \cdot \nabla p - \partial_t G = \sum_{i=1}^{\ell} \nabla c_i \cdot \nabla p G_i(p,n) + c_i \partial_p G_i(p,n)\big(|\nabla p|^2 + pu\big) \\ + c_i \partial_n G_i(p,n)(2\nabla p \cdot \nabla n - \partial_t n) + c_i(G - G_i(p,n))G_i(p,n) \tag{2.23}$$

Now we are ready to begin estimating. Note that

$$\sum_{i=1}^{\ell} c_i (G - G_i) G_i = (\sum_{i=1}^{\ell} c_i G_i)^2 - \sum_{i=1}^{\ell} c_i G_i^2.$$

Since $\sum_{i=1}^{\ell} c_i = 1$, we can use Jensen's inequality to conclude that $\sum_{i=1}^{\ell} c_i(G - G_i(p,n))G_i(p,n) \leq 0$. Hence, returning to our integral, we have

$$\int_{Q_T} \omega(p) f'(u)(2\nabla G \cdot \nabla p - \partial_t G) \leq \tag{2.24}$$

$$\sum_{i=1}^{\ell} \int_{Q_T} \omega(p) f'(u)\Big(c_i \partial_n G_i(p,n)(2\nabla p \cdot \nabla n - \partial_t n) + c_i \partial_p G_i(p,n)(|\nabla p|^2 + pu) + G_i(p,n) \nabla c_i \cdot \nabla p\Big).$$

Now we want to integrate by parts in the final term to eliminate the bad quantity ∇c_i. After doing so, the second line of (2.25) becomes

$$\sum_{i=1}^{\ell} \int_{Q_T} c_i \omega(p) f'(u)\Big(\partial_n G_i(p,n)(\nabla p \cdot \nabla n - \partial_t n) + \partial_p G_i(p,n) pu - G_i(p,n)\Delta p\Big) - c_i G_i(p,n) \nabla p \cdot \nabla(\omega(p) f'(u))$$

Using $\partial_n G$ and $\partial_p G$ as shorthands for $\sum_{i=1}^{\ell} c_i \partial_n G_i(p,n)$ and $\sum_{i=1}^{\ell} c_i \partial_p G_i(p,n)$ respectively, we can rewrite the previous line as

$$\int_{Q_T} \omega(p) f'(u)\Big((\nabla p \cdot \nabla n - \partial_t n)\partial_n G + pu\partial_p G - G\Delta p\Big) - G\nabla p \cdot \nabla(\omega(p) f'(u))$$

After replacing $-\Delta p$ by $\frac{1}{\gamma}u + G$ and expanding $\nabla(\omega(p) f'(u))$, our combined work gives us

$$\int_{Q_T} \omega(p) f'(u)(2\nabla G \cdot \nabla p - \partial_t G) \leq \tag{2.25}$$

$$\int_{Q_T} \omega(p) f'(u)\Big((\nabla p \cdot \nabla n - \partial_t n)\partial_n G + \gamma pu \partial_p G + (\frac{1}{\gamma}u + G)G\Big) - G\omega'(p) f'(u)|\nabla p|^2 - G\omega(p) f''(u) \nabla p \cdot \nabla u.$$

Although $-G\omega'(p)f'(u)|\nabla p|^2$ appears to be a good term, we also want to handle the case where G can be negative. Let $B = \max_{i\in\{1,\dots,\ell\}} \sup_{(p,n)\in[0,\infty)^2} |G_i(p,n)|$. Using the inequality $-G\omega'(p)f'(u)|\nabla p|^2 \leq B\omega'(p)f'(u)|\nabla p|^2$ and then integrating by parts, we get

$$(2.26)\quad \int_{Q_T} \omega(p)f'(u)(2\nabla G\cdot\nabla p - \partial_t G) \leq$$

$$\int_{Q_T} \omega(p)f'(u)\Big((\nabla p\cdot\nabla n - \partial_t n)\partial_n G + pu\partial_p G + (\frac{1}{\gamma}u + G)G\Big) - (G+B)\omega(p)f''(u)\nabla p\cdot\nabla u.$$

Now we can use Young's inequality to obtain

$$(2.27)$$

$$\int_{Q_T} \omega(p)f'(u)(2\nabla G\cdot\nabla p - \partial_t G) \leq \int_{Q_T} \theta^{-1}\omega(p)\partial_n G(|\nabla p|^4 + |\nabla n|^4 + |\partial_t n|^2) + |\nabla p|^2\frac{(G+B)^2\omega(p)}{2p}$$

$$+ \int_{Q_T} \frac{1}{2}p\omega(p)f''(u)^2|\nabla u|^2 + \theta\omega(p)f'(u)^2 + \omega(p)f'(u)G(\frac{1}{\gamma}u + G) + uf'(u)p\omega(p)\partial_p G.$$

Our estimates in Lemmas 2.3-2.4 and Proposition 2.5 imply that all of the terms in the first line are bounded and only depend on the initial data and T. Hence, the result follows. □

At last we obtain the following AB type estimate.

Proposition 2.9. *There exists a constant $C_\gamma(T)$ depending only on T, γ, and the initial data, such that*

$$(2.28)\quad \int_{Q_T} \omega(p)u_+^2 \leq C_\gamma(T).$$

If in addition G satisfies assumption (G4), then $C_\gamma(T) = C(T)$ can be taken independently of γ.

Proof. Combining Lemmas 2.6 and 2.8, there exists a constant $C > 0$ depending only on the initial data and T such that for any $\theta > 0$

$$(2.29)\quad \frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{\gamma}\omega(p)f(u) + \int_{\mathbb{R}^d} \omega(p)f^*(f'(u))(\frac{1}{\gamma}u + G) + p\omega(p)f''(u)(1 - \frac{1}{2}f''(u))|\nabla u|^2$$

$$\leq C(1+\theta^{-1}) + \int_{\mathbb{R}^d} Gf(u)(\frac{2}{\gamma}\omega(p) - p\omega'(p)) + \theta\omega(p)f'(u)^2 + \omega(p)f'(u)G(\frac{1}{\gamma}u + G) + uf'(u)p\omega(p)\partial_p G.$$

Since we need $1 - \frac{1}{2}f''(u) > 0$, the fastest growing choice for f is to take $f(u) = u_+^2$. Plugging in this choice, we get

$$(2.30)\quad \frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{\gamma}\omega(p)u_+^2 + \int_{\mathbb{R}^d} \omega(p)u_+^2(\frac{1}{\gamma}u + G) - 2u_+^2 p\omega(p)\partial_p G$$

$$\leq C(1+\theta^{-1}) + \int_{\mathbb{R}^d} Gu_+^2(\frac{4}{\gamma}\omega(p) - p\omega'(p)) + 4\theta\omega(p)u_+^2 + 2G^2\omega(p)u_+.$$

We use Young's inequality to get $2G^2\omega(p)u_+ \leq \theta^{-1}G^4\omega(p) + \theta\omega(p)u_+^2$ and $Gu_+^2\frac{4}{\gamma} \leq \frac{2}{3\gamma}u_+^3 + \frac{64}{3}|G|^3$. Relying on the fact that $\omega(p) \in L^1(Q_T)$ and G is bounded, we can conclude that

$$(2.31)\quad \frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{\gamma}\omega(p)u_+^2 + \int_{\mathbb{R}^d} \omega(p)u_+^2\big(\frac{1}{3\gamma}u + G\big) - 2u_+^2 p\omega(p)\partial_p G + Gu_+^2 p\omega'(p)$$

$$\leq C'(1+\theta^{-1}) + \int_{\mathbb{R}^d} 5\theta\omega(p)u_+^2$$

for some constant C'. Since concavity implies that $p\omega'(p) \leq \omega(p)$, the first result now follows from Gronwall's inequality and our assumptions on the initial data.

For the second result, if assumption (G4) holds, then we can see from (2.31) that there exists some $\epsilon > 0$ independent of γ such that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)u_+^2 + \int_{\mathbb{R}^d}\epsilon\omega(p)u_+^2 \leq C'(1+\theta^{-1}) + \int_{\mathbb{R}^d}5\theta\omega(p)u_+^2. \tag{2.32}$$

By choosing $\theta \leq \epsilon/10$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{\gamma}\omega(p)u_+^2 + \int_{\mathbb{R}^d}\epsilon\omega(p)u_+^2 \leq C'(1+\epsilon^{-1}) \tag{2.33}$$

and the second result now follows.

□

We have at last reached the final estimate of this section where we provide a weighted L^1 bound on $|u|$. Crucially, this bound controls both the positive and negative part of u, which will allow us to construct both the forward and backward Lagrangian flows along $-\nabla p$ in the next section.

Proposition 2.10. *There exists a constant $C_\gamma(T)$ depending only on T, γ and the initial data such that*

$$\int_{Q_T}\rho|u| + \int_{\mathbb{R}^d\times\{T\}}\rho\log(1+\frac{1}{p}) \leq C_\gamma(T). \tag{2.34}$$

Furthermore, if G satisfies condition (G4) and the initial data satisfies (ID5), then for any $\lambda' \in (0,1/2)\cap(0,\lambda]$ there exists a constant $C(T)$ that is independent of $\gamma \in [1,\infty)$ such that

$$\int_{Q_T}\rho\log(1+\frac{1}{p})^{1-\lambda'}|u| + \int_{\mathbb{R}^d\times\{T\}}\rho\log(1+\frac{1}{p})^{\lambda'} \leq C(T) \tag{2.35}$$

where λ is the constant in (ID5).

Proof. Let $u_- = \min(u,0)$ and let $\eta : \mathbb{R}\to\mathbb{R}$ be a nonnegative increasing function that we will choose at the end. Expanding $|u| = u_+ + |u_-|$ and using Young's inequality, we see that

$$\int_{Q_T}\rho\eta(p)^{1/2}|u| \leq \int_{Q_T}\frac{1}{2}\rho + \frac{1}{2}\rho\eta(p)u_+^2 + \rho\eta(p)^{1/2}|u_-|.$$

We can also write

$$\int_{Q_T}\rho\eta(p)^{1/2}|u_-| = \int_{Q_T}\eta(p)^{1/2}\rho u_+ - \eta(p)^{1/2}\rho u \leq \int_{Q_T}\frac{1}{2}\rho\eta(p)u_+^2 + \frac{1}{2}\rho - \rho\eta(p)^{1/2}u,$$

hence,

$$\int_{Q_T}\rho\eta(p)^{1/2}|u| \leq \int_{Q_T}\frac{1}{2}\rho + \frac{1}{2}\rho\eta(p)u_+^2 - \rho\eta(p)^{1/2}u. \tag{2.36}$$

It remains to estimate $-\int_{Q_T}\rho u\eta(p)^{1/2}$.

Using equation (1.4), it follows that

$$-\int_{Q_T}\rho u\eta(p)^{1/2} = \int_{Q_T}\rho\eta(p)^{1/2}\frac{\partial_t p - |\nabla p|^2}{p} = \int_{Q_T}-\rho(\frac{d}{dt}\zeta(p) - \nabla\zeta(p)\cdot\nabla p)$$

where $\zeta : [0,\infty)\to\mathbb{R}$ is the antiderivative $\zeta(a) = -\int_1^a \frac{\eta(b)^{1/2}}{b}\,db$. Integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^d\times\{T\}}\rho\zeta(p) - \int_{Q_T}\rho u\eta(p)^{1/2} &= \int_{\mathbb{R}^d}\rho^0\zeta(p^0) - \int_{Q_T}\zeta(p)\big(\nabla\cdot(\rho\nabla p) - \partial_t\rho\big)\\ &= \int_{\mathbb{R}^d}\rho^0\zeta(p^0) + \int_{Q_T}\rho\zeta(p)G\end{aligned}$$

Combining this with (2.36), we have

$$\int_{\mathbb{R}^d\times\{T\}} \rho\zeta(p) + \int_{Q_T} \rho\eta(p)^{1/2}|u| \leq \int_{\mathbb{R}^d} \rho^0\zeta(p^0) + \int_{Q_T} \rho\zeta(p)G + \frac{1}{2}\rho + \frac{1}{2}\rho\eta(p)u_+^2 \tag{2.37}$$

Now we are ready to make choices for η. For $\gamma \in [1,\infty)$ the weight $\omega(p) = p^{\frac{1}{\gamma}} = \rho$ satisfies the conditions (W1-W3), hence we can choose $\eta(a) = 1$ to obtain

$$\int_{\mathbb{R}^d\times\{T\}} \rho\log(1/p) + \int_{Q_T} \rho|u| \leq \int_{\mathbb{R}^d} \rho^0\log(1/p^0) + \int_{Q_T} \rho\log(1/p)G + \frac{1}{2}\rho + \frac{1}{2}\rho u_+^2 \tag{2.38}$$

Hence, Gronwall's inequality, Proposition 2.9, and the identity $\log(1/p) = \log(1+1/p) - \log(1+p)$ imply the existence of a constant $C_\gamma(T)$ such that

$$\int_{Q_T} \rho|u| + \int_{\mathbb{R}^d\times\{T\}} \rho\log(1+\frac{1}{p}) \leq C_\gamma(T). \tag{2.39}$$

To get a bound that also is valid in the limit $\gamma\to\infty$, let us choose some $\lambda' \in (0,\lambda]\cap(0,1/2)$ where $\lambda > 0$ is the constant in assumption (ID5). We then choose $\eta(a) = \log(1+\frac{1}{p})^{2(\lambda'-1)}$, which gives $\zeta(a) = -\int_1^a \frac{\log(1+\frac{1}{b})^{\lambda'-1}}{b}\,db$. Integrating by parts, we see that

$$\zeta(a) = -\log(a)\log(1+\frac{1}{a})^{\lambda'-1} + (1-\lambda')\int_1^a \frac{\log(b)}{b+b^2}\log(1+\frac{1}{b})^{\lambda'-2}$$

$$= \log(\frac{1}{a})\log(1+\frac{1}{a})^{\lambda'-1} + (1-\lambda')\zeta(a) + (1-\lambda')\int_1^a \frac{\log(1+\frac{1}{b})^{\lambda'-1}}{b}(\frac{\log(b)}{(1+b)\log(1+\frac{1}{b})} + 1)$$

Thus,

$$\lambda'\zeta(a) = \log(1/a)\log(1+\frac{1}{a})^{\lambda'-1} + (1-\lambda')\int_1^a \frac{\log(1+\frac{1}{b})^{\lambda'-1}}{b}(\frac{\log(b)+(1+b)\log(1+1/b)}{(1+b)\log(1+\frac{1}{b})})$$

$$= \log(1+\frac{1}{a})^{\lambda'} - \log(1+a)\log(1+\frac{1}{a})^{\lambda'-1} + (1-\lambda')\int_1^a \frac{\log(1+\frac{1}{b})^{\lambda'-1}}{1+b} + \frac{\log(1+\frac{1}{b})^{\lambda'-2}\log(1+b)}{b(1+b)}$$

Hence,

$$\zeta(a) = \frac{1}{\lambda'}\log(1+\frac{1}{a})^{\lambda'} + h(a)$$

where h is a function that is bounded on $[0,p_h]$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d\times\{T\}} \rho\log(1+\frac{1}{p})^{\lambda'} + \int_{Q_T} \rho|u| \lesssim \|\rho\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} + \int_{\mathbb{R}^d} \rho^0\log(1+\frac{1}{p^0})^{\lambda'} + \\ \int_{Q_T} \rho\log(1+\frac{1}{p})|G| + \frac{1}{2}\rho\log(1+\frac{1}{p})^{2(\lambda'-1)}u_+^2\end{aligned} \tag{2.40}$$

In order to use Proposition 2.9 to bound $\rho\log(1+\frac{1}{p})^{2(\lambda'-1)}u_+^2$, we need to check to see if there exists a weight $\omega(p)$ satisfying (W1-W3) such that $C\omega(p) \geq \rho\log(1+\frac{1}{p})^{2(\lambda'-1)}$ for some constant C. We shall choose $\omega(p) = p^{\frac{1}{\gamma}}z(p) = \rho z(p)$ where z is a nonnegative increasing function. To ensure that ω is concave we need

$$\omega''(a) = \frac{1}{\gamma}a^{\frac{1}{\gamma}-2}\big((\frac{1}{\gamma}-1)z(a) + 2az'(a) + \gamma a^2 z''(a)\big) \leq 0 \tag{2.41}$$

for all $a \in (0, p_h]$. We now consider the choice $z(a) = \log(\xi(a)^{-1})^{2(\lambda'-1)}$ where $\xi : [0, \infty) \to [0, \infty)$ is an increasing concave function that is bounded above by e^{-6} such that $\xi(a) = a$ on $[0, \frac{e^{-6}}{2}]$. Testing this choice, we get

$$(2.42)\quad (\frac{1}{\gamma} - 1)z(a) + 2az'(a) + \gamma a^2 z''(a) = (\frac{1}{\gamma} - 1)\log(\xi(a)^{-1})^{2(\lambda'-1)} + 4(1-\lambda')\frac{a\xi'(a)}{\xi(a)}\log(\xi(a)^{-1})^{2\lambda'-3}$$
$$-2\gamma(1-\lambda')(\frac{a\xi'(a)}{\xi(a)})^2\log(\xi(a)^{-1})^{2\lambda'-3}\big(1-(3-2\lambda')\log(\xi(a)^{-1})^{-1}\big)+2\gamma(1-\lambda')\frac{a^2\xi''(a)}{\xi(a)}\log(\xi(a)^{-1})^{2\lambda'-3}.$$

Exploiting the concavity of ξ and the upper bound of e^{-6}, it follows that

$$(2.43)\quad (\frac{1}{\gamma} - 1)z(a) + 2az'(a) + \gamma a^2 z''(a) \leq (\frac{1}{\gamma} - 1)\log(\xi(a)^{-1})^{2(\lambda'-1)} + \frac{2}{3}(1-\lambda')\log(\xi(a)^{-1})^{2(\lambda'-1)}$$
$$- \gamma(1-\lambda')(\frac{a\xi'(a)}{\xi(a)})^2\log(\xi(a)^{-1})^{2\lambda'-3},$$

which is nonpositive for all $\gamma \geq 3$.

It is now easy to check that the remaining properties (W1-W3) are satisfied by $\omega(p) = p^{\frac{1}{\gamma}}\log(\xi(a)^{-1})^{2(\lambda'-1)} = \rho\log(\xi(a)^{-1})^{2(\lambda'-1)}$. Since ξ is increasing and $\xi(a) = a$ on $[0, \frac{e^{-6}}{2}]$ it also follows that there exists a constant $C > 0$ such that $\rho\log(1+\frac{1}{p})^{2(\lambda'-1)} \leq C\omega(p)$. Thus, (2.35) now follows from (2.40), Gronwall's inequality, and Proposition 2.9 (note that (2.35) also holds for $\gamma \in [1, 3]$ since (2.34) is a strictly stronger bound and $C_\gamma(T)$ only blows up as $\gamma \to \infty$). □

3. Stability of Lagrangian flows

Once again, in this Section, we will assume that we are working with smooth solutions $(\rho_1, \ldots, \rho_\ell, p, n)$ to the system (1.1-1.2). Thanks to the smoothness of p, the regular Lagrangian flow along $-\nabla p$ must exist by classic Cauchy-Lipschitz theory. Thus, $(\rho_1, \ldots, \rho_\ell, p, n)$ is already a complete Lagrangian solution in the sense of Definition 1.1. Hence, we can freely assume the existence of the forward and backward flow maps X, Y satisfying equations (1.7) and (1.8) respectively. The main purpose of this Section is to use our bounds from Section 2 to show that X and Y satisfy certain quantitative stability bounds (c.f. Proposition 3.5).

Our stability bounds will compare X and Y to the forward and backward flows S, Z along some vector field $V \in L^\infty_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$ with an associated nonnegative density $\mu \in C_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. Specifically, we shall assume that S and Z satisfy the flow equations

$$(3.1)\qquad S(t, s, x) = x + \int_s^{s+t} V(\tau, S(\tau, s, x))\, d\tau \quad \text{for a.e } x \in \mathrm{spt}(\mu(s, x)),$$

$$(3.2)\qquad Z(t, s, x) = x - \int_{s-t}^{s} V(\tau, Z(\tau, s, x))\, d\tau \quad \text{for a.e } x \in \mathrm{spt}(\mu(s, x)),$$

and there exists a constant $C > 0$ such that

$$(3.3)\qquad e^{-tC}\mu(s+t, \cdot) \leq S(t, s, \cdot)_{\#}\mu(s, \cdot) \leq e^{tC}\mu(s+t, \cdot)$$

and

$$(3.4)\qquad e^{-tC}\mu(s-t, \cdot) \leq Z(t, s, \cdot)_{\#}\mu(s, \cdot) \leq e^{tC}\mu(s-t, \cdot).$$

We will then show that the difference between X and S (respectively Y and Z) on $p\min(\mu, \rho)$ can be controlled in terms of the difference between V and $-\nabla p$.

Remark 3.1. Since we only assume that $V \in L^\infty_{\text{loc}}([0,\infty); L^2(\mathbb{R}^d))$ there is no guarantee that one can find S, Z, and μ satisfying the properties (3.1-3.4). Nonetheless, when we use the results of this Section, it will always end up being in cases where we already know that the analogues of S, Z, μ exist and satisfy the desired properties.

Let us emphasize that the estimates in this section are heavily inspired by the quantitative estimates on Lagrangian flows from [CdL08]. The insight in [CdL08] was that certain logarithmic quantities related to the flow maps could be controlled with just Sobolev regularity on the flow field. Here we introduce doubly logarithmic quantities that can be controlled without needing to bound D^2p in any L^r space. Specifically, our quantities take the form

(3.5)
$$I_T(t,s) := \int_{\mathbb{R}^d} \bar{\rho}(s,x) \log\Big(1 + p(s+t, X(t,s,x)) \log\big(1 + \frac{\min(|X(t,s,x) - S(t,s,x)|, 1)}{\delta_{2T}}\big)\Big)\, dx,$$

and

(3.6)
$$J_T(t,s) := \int_{\mathbb{R}^d} \bar{\rho}(s,x) \log\Big(1 + p(s-t, Y(t,s,x)) \log\big(1 + \frac{\min(|Y(t,s,x) - Z(t,s,x)|, 1)}{\delta_{T}}\big)\Big)\, dx,$$

where

$$\bar{\rho} := \min(\rho, \mu), \quad \delta_T = \Big(\int_{Q_T} \mu |\nabla p + V|^2\Big)^{1/2}. \tag{3.7}$$

In what follows, we will show that our bounds on $p|D^2p|^2$ and $\gamma\rho|u|$ from Section 2 are sufficient to control the above integrals. In particular, our control on $p|D^2p|^2$ will replace the usual need for Sobolev regularity on ∇p, while our control on $\gamma\rho|u|$ will help us make sure that we can keep the factor of p attached to $p|D^2p|^2$ in our calculations. We will then show that bounds on I and J can be used to bound the differences $\sup_{s\leq T}\sup_{t\leq(T-s)} \int \bar{\rho}(s,x)|X(t,s,x) - S(t,s,x)|\, dx$ and $\sup_{s\leq T}\sup_{t\leq s} \int \bar{\rho}(s,x)|Y(t,s,x) - Z(t,s,x)|\, dx$ in terms of δ_T.

Before we get into the main results of this section, we review some important properties of maximal functions.

3.1. **Maximal functions.** The maximal functions

$$f(t,x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |\nabla p(t,y)|^2 + p(t,y)|D^2p(t,y)|\, dy, \tag{3.8}$$

and

$$g(t,x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |\nabla p(t,y)|\, dy. \tag{3.9}$$

will play an important role in our calculations. It is a classical fact [Ste16] that for any $r \in (1,\infty]$

$$\|f\|_{L^r(\{t\}\times\mathbb{R}^d)} \lesssim_{r,d} \|\nabla p\|^2_{L^{2r}(\{t\}\times\mathbb{R}^d)} + \|pD^2p\|_{L^r(\{t\}\times\mathbb{R}^d)} \tag{3.10}$$

and

$$\|g\|_{L^r(\{t\}\times\mathbb{R}^d)} \lesssim_{r,d} \|\nabla p\|_{L^r(\{t\}\times\mathbb{R}^d)}. \tag{3.11}$$

f and g will show up in our estimates through the following crucial bound.

Lemma 3.2. *Given any two points $x_1, x_2 \in \mathbb{R}^d$ and any time $t \geq 0$, we have*

$$p(t,x_1)|\nabla p(t,x_1) - \nabla p(t,x_2)| \leq |x_1 - x_2|\Big(f(t,x_1) + f(t,x_2) + g(t,x_1)^2 + g(t,x_2)^2\Big) \tag{3.12}$$

Proof. By the triangle inequality

$$p(t,x_1)|\nabla p(t,x_1)-\nabla p(t,x_2)| \leq |p(t,x_1)\nabla p(t,x_1)-p(t,x_2)\nabla p(t,x_2)|+|p(t,x_1)-p(t,x_2)||\nabla p(t,x_2)|.$$

Noting that $|D(p\nabla p)| \leq |\nabla p|^2 + p|D^2 p|$, one can use standard maximal function theory [Ste16] to obtain the bounds

$$|p(t,x_1)\nabla p(t,x_1) - p(t,x_2)\nabla p(t,x_2)| \leq |x_1 - x_2|(f(t,x_1) + f(t,x_2))$$

and

$$|p(t,x_1) - p(t,x_2)| \leq |x_1 - x_2|(g(t,x_1) + g(t,x_2)|.$$

The result now follows from Young's inequality and the fact that $|\nabla p| \leq g$ pointwise everywhere. □

3.2. **Quantitative stability.** We are now ready to prove the main results of this section. We begin with some basic estimates on the flow maps and their pushforwards.

Lemma 3.3. *Let* $B' = \max_{i\in\{1,\dots,\ell\}} \sup_{(p,n)\in[0,\infty)^2} |G_i(p,n)|$ *and* $B = \max(B', C)$. *where* C *is the constant from (3.3). For any time* $s \geq 0$ *and* $t \leq s$ *we have*

$$e^{-Bt}\rho(s+t,x) \leq X(t,s,\cdot)_{\#}\rho(s,x) \leq e^{Bt}\rho(s+t,x), \tag{3.13}$$

$$e^{-Bt}\rho(s-t,x) \leq Y(t,s,\cdot)_{\#}\rho(s,x) \leq e^{Bt}\rho(s-t,x). \tag{3.14}$$

Furthermore, for any $s,t \geq 0$

$$\Big(\int_{\mathbb{R}^d} \bar{\rho}(s,x)|X(t,s,x) - S(t,s,x)|^2\Big)^{1/2} \leq (te^{tB})^{1/2}\big(\delta_{s+t} + \|\rho^{1/2}\nabla p\|_{L^2([s,s+t]\times\mathbb{R}^d)}\big), \tag{3.15}$$

and for any $s \geq 0$ *and* $t \leq s$

$$\Big(\int_{\mathbb{R}^d} \bar{\rho}(s,x)|Y(t,s,x) - Z(t,s,x)|^2\Big)^{1/2} \leq (te^{tB})^{1/2}\big(\delta_{s} + \|\rho^{1/2}\nabla p\|_{L^2([s-t,s]\times\mathbb{R}^d)}\big). \tag{3.16}$$

Proof. Since $Y(t,s,\cdot)$ is the inverse of $X(t,s-t,\cdot)$ (3.14) will follow from (3.13). Using the representation formula (1.15), it is clear that $X(t,s,\cdot)_{\#}\rho(s,x) \leq e^{Bt}\rho(s+t,x)$ and $e^{-Bt}\rho(s+t,x) \leq X(t,s,\cdot)_{\#}\rho(s,x)$. The last two bounds follow from Jensen's inequality and the pushforward bounds. □

Most of the action in this section occurs in the following Lemma where we provide bounds on I and J.

Lemma 3.4. *For any* $T > 0$ *and any* $\lambda' \in (0,1]$ *define*

$$\mathcal{B}_{\lambda'}(T) = \|\rho\log(1+\frac{1}{p})^{\lambda'-1}u\|_{L^1(Q_T)}+p_h^2+\|\rho+\mu\|_{L^2(Q_T)}\Big(1+\|\nabla p\|^2_{L^4(Q_T)}+\|pD^2p\|_{L^2(Q_T)}\Big), \tag{3.17}$$

we then have the estimates

$$\sup_{s,t\leq T} I_T(t,s) \leq \mathcal{B}_{\lambda'}(2T)e^{BT}\log(1+\log(1+\delta_{2T}^{-1}))^{1-\lambda'}, \tag{3.18}$$

and

$$\sup_{s,t\leq T} J_T(t,s) \leq \mathcal{B}_{\lambda'}(T)e^{BT}\log(1+\log(1+\delta_{T}^{-1}))^{1-\lambda'}. \tag{3.19}$$

Proof. We will provide the argument for the bound on I, the bound on J has a nearly identical proof. To bound I, we will proceed by estimating its time derivative with respect to t. Since the expressions are complicated, we will break down the calculation into smaller pieces first by defining the inner logarithm term $L_X(t,s,x) := \log(1 + \frac{\min(|X(t,s,x)-S(t,s,x)|,1)}{\delta_{2T}})$.

Differentiating L_X with respect to t, we see that

$$\partial_t L_X(t,s,x) \leq \frac{|\nabla p(s+t, X(t,s,x)) + V(s+t, S(t,s,x))|}{\delta_{2T} + |X(t,s,x) - S(t,s,x)|}.$$

After an application of the triangle inequality, we can bound the previous line by

$$\frac{|\nabla p(s+t, X(t,s,x)) - \nabla p(s+t, S(t,s,x))|}{|X(t,s,x) - S(t,s,x)|} + \frac{|\nabla p(s+t, S(t,s,x)) + V(s+t, S(t,s,x))|}{\delta_{2T}}.$$

After combining these bounds with (3.12), we can conclude that

$$\text{(3.20)}\quad p(s+t, X(t,s,x))\partial_t L_X(t,s,x) \leq f(s+t, X(t,s,x)) + f(s+t, S(t,s,x))$$
$$+g(s+t, X(t,s,x))^2 + g(s+t, S(t,s,x))^2 + p(t+s, X(t,s,x))|\nabla p(s+t, S(t,s,x)) + V(s+t, S(t,s,x))|\delta_{2T}^{-1}.$$

Next, we calculate

$$\frac{d}{dt}p(s+t, X(t,s,x)) = \partial_t p(s+t, X(t,s,x)) - |\nabla p(s+t, X(t,s,x))|^2 = u(s+t, X(t,s,x))p(s+t, X(t,s,x)),$$

and

$$\frac{d}{dt}\log(1+p(s+t, X(t,s,x)L_X(t,s,x)) = \frac{p(s+t, X(t,s,x))\partial_t L_X(t,s,x) + L_X(t,s,x)\frac{d}{dt}p(s+t, X(t,s,x))}{1 + p(s+t, X(t,s,x))L_X(t,s,x)}$$
$$= \frac{p(s+t, X(t,s,x))\partial_t L_X(t,s,x)}{1 + p(s+t, X(t,s,x))L_X(t,s,x)} + \frac{L_X(t,s,x)p(s+t, X(t,s,x))u(s+t, X(t,s,x))}{1 + p(s+t, X(t,s,x))L_X(t,s,x)}$$

Thus,

$$\text{(3.21)}\quad \frac{d}{dt}\log(1 + p(t+s, X(t,s,x)L_X(t,s,x)) \leq$$
$$f(s+t, X(t,s,x)) + f(s+t, S(t,s,x)) + g(s+t, X(t,s,x))^2 + g(s+t, S(t,s,x))^2$$
$$+p(t+s, X(t,s,x))|\nabla p(s+t, S(t,s,x)) + V(s+t, S(t,s,x))|\delta_{2T}^{-1} + \frac{L_X(t,s,x)p(s+t, X(t,s,x))u(s+t, X(t,s,x))}{1 + p(s+t, X(t,s,x))L_X(t,s,x)}$$

Now we note that $I(0,s) = 0$. Using the above bounds on the t derivative of the integrand of I, we can conclude that for any $t \geq 0$

(3.22)

$$I_T(t,s) \leq \int_{Q_t} \bar{\rho}(s,x)\Big(f(s+t, X(t,s,x)) + f(s+t, S(t,s,x)) + g(s+t, X(t,s,x))^2 + g(s+t, S(t,s,x))^2$$
$$+ p(t+s, X(t,s,x))|\nabla p(s+t, S(t,s,x)) + V(s+t, S(t,s,x))|\delta_{2T}^{-1}$$
$$+ \frac{L_X(t,s,x)p(s+t, X(t,s,x))u(s+t, X(t,s,x))}{1 + p(s+t, X(t,s,x))L_X(t,s,x)}\Big)\, dx\, dt,$$

Using the pushforward bounds from Lemma 3.3 and changing variables in time, it follows that

(3.23)

$$I_T(t,s) \leq e^{tB}\int_{[s,s+t]\times\mathbb{R}^d} p_h\delta_{2T}^{-1}\mu(\tau,x)|\nabla p(\tau,x) + V(\tau,x)| + \big(\rho(\tau,x) + \mu(\tau,x)\big)\big(f(\tau,x) + g(\tau,x)^2\big)\, dx\, d\tau,$$
$$+ e^{tB}\int_{[s,s+t]} \rho(\tau,x)\frac{u(\tau,x)p(\tau,x)L_X(t,s,Y(t,s+t,x))}{1 + p(\tau,x)L_X(t,s,Y(t,s+t,x))},$$

where we have also used the fact that $Y(t,s+t,x)$ is the inverse of $X(t,s,x)$. Since $L_X \leq \log(1+\delta_{2T}^{-1})$ and $a \mapsto \frac{a}{1+a}$ is an increasing function, it follows that

(3.24)
$$I_T(t,s) \leq e^{tB}\int_{[s,s+t]\times\mathbb{R}^d} p_h\delta_{2T}^{-1}\mu(\tau,x)|\nabla p(\tau,x)+V(\tau,x)|+\big(\rho(\tau,x)+\mu(\tau,x)\big)\big(f(\tau,x)+g(\tau,x)^2\big)\,dx\,d\tau,$$
$$+e^{tB}\int_{[s,s+t]}\rho(\tau,x)\frac{|u(\tau,x)|p(\tau,x)\log(1+\delta_{2T}^{-1})}{1+p(\tau,x)\log(1+\delta_{2T}^{-1})},$$

Using the bounds (3.10) and (3.11) and the definition of δ_T, we see that

(3.25)
$$I_T(t,s) \lesssim e^{tB}\|\rho u\log(1+\frac{1}{p})^{\lambda'-1}\|_{L^1([s,s+t]\times\mathbb{R}^d)}\|\log(1+\frac{1}{p})^{1-\lambda'}\frac{p\log(1+\delta_{2T}^{-1})}{1+p\log(1+\delta_{2T}^{-1})}\|_{L^\infty([s,s+t]\times\mathbb{R}^d)}$$
$$+e^{tB}p_h\|\mu\|^{1/2}_{L^2([s,s+t]\times\mathbb{R}^d)}+e^{tB}\|\rho+\mu\|_{L^2([s,s+t]\times\mathbb{R}^d)}\Big(\|\nabla p\|^2_{L^4([s,s+t]\times\mathbb{R}^d)}+\|pD^2p\|_{L^2([s,s+t]\times\mathbb{R}^d)}\Big).$$

For $b>0$ large, the function $\log(1+\frac{1}{a})^{1-\lambda'}\frac{ab}{1+ab}$ is roughly maximized at $a=1/b$, thus,

$$\|\log(1+\frac{1}{p})^{1-\lambda'}\frac{p\log(1+\delta_{2T}^{-1})}{1+p\log(1+\delta_{2T}^{-1})}\|_{L^\infty([s,s+t]\times\mathbb{R}^d)} \lesssim \log(1+\log(1+\delta_{2T}^{-1}))^{1-\lambda'}.$$

The result now follows from (3.25) and the above bound.

□

Now we are ready to establish the stability property.

Proposition 3.5. *If the initial data satisfies (ID1-ID3) and the growth terms satisfy (G1-G3), then for any $T\geq 0$ and $\lambda'\in(0,1]$ there exists a constant $\mathcal{C}_{\gamma,\lambda'}(T)$ depending only on the initial data, λ',γ,T and d such that*

$$\sup_{s\leq T}\sup_{t\leq T}\int_{\mathbb{R}^d}\bar\rho(s,x)|X(t,s,x)-S(t,s,x)| \leq \mathcal{C}_{\gamma,\lambda'}(2T)\log(1+\log(1+\delta_T^{-1}))^{-\lambda'/2}, \tag{3.26}$$

$$\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d}\bar\rho(s,x)|Y(t,s,x)-Z(t,s,x)| \leq \mathcal{C}_{\gamma,\lambda'}(T)\log(1+\log(1+\delta_T^{-1}))^{-\lambda'/2}. \tag{3.27}$$

Additionally, if the growth terms satisfy (G4) the initial data satisfies (ID5), and $\lambda'<\min(1/2,\lambda)$ where λ is the constant in (ID5), then $\mathcal{C}_{\gamma,\lambda'}(T)$ is independent of γ.

Proof. We provide the proof for (3.27), the argument for (3.26) is essentially identical.

Given $r>0$ let $D_r(t,s) := \{x\in\mathbb{R}^d : |Y(t,s,x)-Z(t,s,x)|>r\}$. We can then estimate

$$\int_{\mathbb{R}^d}\bar\rho(s,x)|Y(t,s,x)-Z(t,s,x)| \leq rp_h\|\rho\|_{L^\infty([0,s];L^1(\mathbb{R}^d))}+\int_{D_r(t,s)}\bar\rho(s,x)p(s,x)|Y(t,s,x)-Z(t,s,x)|\,dx$$
$$\leq rp_h\|\rho\|_{L^\infty([0,s];L^1(\mathbb{R}^d))}+\Big(\int_{D_\lambda(t,s)}\bar\rho(s,x)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^d}\bar\rho(s,x)|Y(t,s,x)-Z(t,s,x)|^2\,dx\Big)^{1/2}$$

From Lemma 3.3, we already have a bound for $\Big(\int_{\mathbb{R}^d}\bar\rho(s,x)|Y(t,s,x)-Z(t,s,x)|^2\,dx\Big)^{1/2}$. Thus we focus on the other integral.

Fix some $\epsilon>0$ and note that D_r is contained in the union $D_{r,\epsilon}(t,s)\cup A_{Y,\epsilon}(t,s)$ where

$$D_{r,\epsilon}(t,s) := \{x\in D_r(t,s) : p(s-t,Y(t,s,x))>\epsilon\}$$

and

$$A_{Y,\epsilon}(t,s) := \{x \in \mathbb{R}^d : p(s-t, Y(t,s,x)) < \epsilon\}$$

Using these sets, we see that

$$\int_{D_r(t,s)} \bar{\rho}(s,x)\,dx \leq \log(1+\epsilon^{-1})^{-\lambda'} \int_{A_{Y,\epsilon}(t,s)} \bar{\rho}(s,x) \log(1+\frac{1}{p(s-t,Y(t,s,x))})^{\lambda'}\,dx + \int_{D_{r,\epsilon}(t,s)} \bar{\rho}(s,x)\,dx.$$

Pushing forward by Y in the first integral, we get

$$\int_{D_r(t,s)} \bar{\rho}(s,x)\,dx \leq e^{tB} \log(1+\epsilon^{-1})^{-\lambda'} \int_{\mathbb{R}^d} \rho(s-t,x) \log(1+\frac{1}{p(s-t,x)})^{\lambda'}\,dx + \int_{D_{r,\epsilon}(t,s)} \bar{\rho}(s,x)\,dx.$$

To estimate the final integral, we write

$$\int_{D_{r,\epsilon}(t,s)} \bar{\rho}(s,x)\,dx \leq \log(1+\epsilon\log(1+\frac{\min(r,1)}{\delta_T}))^{-1} \int_{D_{r,\epsilon}(t,s)} \bar{\rho}(s,x) \log\Big(1+\epsilon\log\big(1+\frac{\min(r,1)}{\delta_T}\big)\Big)$$

$$\leq \log(1+\epsilon\log(1+\frac{\min(r,1)}{\delta_T}))^{-1} \int_{D_{r,\epsilon}(t,s)} \bar{\rho}(s,x) \log\Big(1+p(s-t,Y(t,s,x))\log\big(1+\frac{|Y(t,s,x)-Z(t,s,x)|}{\delta_T}\big)\Big),$$

where we have taken advantage of the definition of $D_{r,\epsilon}(t,s)$ to obtain the last inequality. Recognizing that the final integral is bounded above by J_T, it follows that

$$\int_{D_{\lambda,\epsilon}(t,s)} \bar{\rho}(s,x)p(s,x)^2\,dx \leq \log(1+\epsilon\log(1+\frac{\min(r,1)}{\delta_T}))^{-1} J_T(t,s).$$

Thus, after combining our work, we see that

$$\int_{D_r(t,s)} \bar{\rho}(s,x)\,dx \leq$$

$$e^{tB}\log(1+\epsilon^{-1})^{-\lambda'} \int_{\mathbb{R}^d} \rho(s-t,x)\log(1+\frac{1}{p(s-t,x)})^{\lambda'}\,dx + \log(1+\epsilon\log(1+\frac{\min(r,1)}{\delta_T}))^{-1} J_T(t,s).$$

Using Proposition 2.10 and Lemma 3.4, it follows that

$$\int_{D_r(t,s)} \bar{\rho}(s,x)\,dx \lesssim \log(1+\epsilon^{-1})^{-\lambda'} + \log(1+\epsilon\log(1+\frac{\min(r,1)}{\delta_T}))^{-1}\log(1+\log(1+\delta_T^{-1}))^{1-\lambda'}$$

Now we make the choices $r = \delta_T^{1/2}$ and $\epsilon = \log(1+\delta_T^{-1/2})^{-1/2}$. Up to constants, the previous line becomes

$$\log(1+\log(1+\delta_T^{-1}))^{-\lambda'}$$

Combining our work, the result follows. □

4. Compactness

In this final section, we will at last construct complete Lagrangian solutions to the system (1.1-1.2) under our various assumptions on the initial data and structure of the growth terms (c.f. Section 1.2). To construct these solutions, we will take a sequence of smooth solutions to (1.1-1.2) and use our results from Sections 2 and 3 to prove that strong limit points exist and satisfy Definition 1.1. We will first construct solutions in the case $\gamma \in [1,\infty)$ and then consider the incompressible limit $\gamma \to \infty$.

4.1. **Compactness for γ fixed.** We begin with the following Proposition which guarantees the existence of smooth solutions under certain assumptions on the initial data and growth terms. Here the crucial property will be that the initial data is not compactly supported. Let us emphasize that the existence of smooth solutions for PME equations with data bounded away from zero is very well-known in the literature [Váz07]. We take an approach similar to [GPŚG19].

Proposition 4.1. *Let $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ be initial data satisfying (ID1-ID3) and let $G_1, \dots, G_\ell$ be growth terms satisfying (G1-G3). If $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ and $G_1, \dots, G_\ell$ are smooth and there exists some $\delta > 0$ such that $\sum_{i=1}^\ell \rho_i^0 \geq \delta e^{-|x|^2}$ then there exists a smooth complete Lagrangian solution to (1.1-1.2) with initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ and growth terms G_i.*

Proof. We can construct solutions through the following iteration scheme. To initialize the scheme we first set $\rho_{i,0}(t,x) = \rho_i^0(x)$ and $n_0(t,x) = n^0(x)$ for all (t,x), then we set $\rho_0 = (\sum_{i=1}^\ell \rho_{i,0})$, $p_0 = \rho_0^\gamma$, We then iterate by solving the following equations

$$c_{i,m} = \frac{\rho_{i,m}}{\rho_m}, \quad G^m = \sum_{i=1}^{\ell} c_{i,m} G_i(p_m, n_m), \tag{4.1}$$

$$\partial_t p_{m+1} - \gamma(p_{m+1} + \frac{1}{m})\Delta p_{m+1} - |\nabla p_{m+1}|^2 = \gamma p_{m+1} G^m, \tag{4.2}$$

$$\partial_t \rho_{i,m+1} - \nabla \cdot (\rho_{i,m+1} \nabla p_{m+1}) = \rho_{i,m+1} G_i(p_{m+1}, n_m) \tag{4.3}$$

$$\partial_t n_{m+1} - \alpha \Delta n_{m+1} = -n_{m+1} \sum_{i=1}^{\ell} \beta_i \rho_{i,m+1} \tag{4.4}$$

By construction, each step of the scheme produces a smooth solution (this is clear for (4.2) and (4.4) and we then note that (4.3) is a continuity equation with smooth initial data, smooth vector field, and smooth source). We can also check that $\delta^\gamma \exp(-\gamma|x|^2 - \theta t)$ is a subsolution to (4.2) once θ is chosen to be sufficiently large. Thus, $p_m(t,x) \geq \delta^\gamma \exp(-\gamma|x|^2 - \theta t)$ for all m. This guarantees that on balls of finite radius equation (4.2) is uniformly parabolic independently of m. From here, the convergence of the scheme to a smooth solution is well-known in the parabolic literature folklore. □

Given initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$ satisfying (ID1-ID3) and growth terms G_i satisfying (G1-G3), we want to use the previous Proposition to construct a sequence of smooth solutions that will converge to a complete Lagrangian solution with the desired initial data. Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a smooth compactly supported mollifier. For each $k \in \mathbb{Z}_+$ we define

$$\rho_{i,k}^0 := \frac{1}{k} e^{-|x|^2} + \eta_{\frac{1}{k}} * \rho_i^0, \tag{4.5}$$

$$n_k^0 = \eta_{\frac{1}{k}} * n^0, \tag{4.6}$$

and we choose $G_{i,k}$ to be a sequence of smooth approximations to G_i. By Proposition 4.1, for each $k \in \mathbb{Z}_+$, there exists a smooth solution $(\rho_{1,k}, \dots, \rho_{\ell,k}, p_k, n_k)$ to (1.1-1.2) with initial data $(\rho_{1,k}^0, \dots, \rho_{\ell,k}^0, n_k^0)$. Note that the smoothness of the variables implies that $(\rho_{1,k}, \dots, \rho_{\ell,k}, p_k, n_k)$ is a complete Lagrangian solution to the system. Hence, we are assured the existence of the forward and backward flow maps X_k, Y_k along $-\nabla p_k$ satisfying all of the properties in Definition 1.1. It remains to verify that these sequences have sufficient compactness to extract limit points and prove that the limit points are the desired complete Lagrangian solutions.

Lemma 4.2. *Fix some $T \geq 0$. Both p_k and n_k are $L^2([0,T]; H^1(\mathbb{R}^d)) \cap C([0,T]; L^2(\mathbb{R}^d))$ strongly precompact. For each $i \in \{1, \dots, \ell\}$ the family $\rho_{i,k}$ is $L^1(Q_T) \cap W^{1,2}([0,T]; H^{-1}(\mathbb{R}^d))$ weakly precompact.*

Proof. Thanks to Proposition 2.10, we have

$$\sup_k \int_{Q_T} \rho_k |\partial_t p_k| + \rho_k |\Delta p_k| = \sup_k \int_{Q_T} \rho_k |\nabla p|^2 + \rho_k p_k |u_k| + \frac{1}{\gamma} \rho_k |u_k| + \rho_k |G_k| < \infty$$

We can also compute

$$\sup_k \int_{Q_T} |x|^{1/2} |\nabla p_k(t,x)|^2 \leq \sup_k \int_{Q_T} \rho_k |x|^2 + \frac{|\nabla p_k(t,x)|^{8/3}}{\rho_k^{1/3}} < \infty.$$

If we define $q_k = p_k^{1+\frac{1}{\gamma}}$, then $|\partial_t q_k| = \rho_k |\partial_t p_k|$, $(1 + |x|^{1/2}) |\nabla q_k|^2 = (1 + |x|^{1/2}) \rho_k^2 |\nabla p_k|^2$ and $|\Delta q_k| \leq |\nabla \rho_k \cdot \nabla p_k| + \rho_k |\Delta p_k|$. Hence, q_k is $L^2([0,T]; H^1(\mathbb{R}^d))$ precompact.

Now we want to transfer these precompactness properties to p_k. We need to be a little careful since the transformation $a \mapsto a^{(1+\frac{1}{\gamma})^{-1}}$ is not C^1. Let k_j be a subsequence such that q_{k_j} is $L^2([0,T]; H^1(\mathbb{R}^d))$ Cauchy. Fix some $\epsilon > 0$ and let $\chi_\epsilon : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[0, \epsilon]$. We can then compute

$$\|\nabla p_{k_j} - \nabla p_{k_m}\|_{L^2(Q_T)} \leq \|\nabla p_{k_j} \chi_\epsilon(p_{k_j})\|_{L^2(Q_T)} + \|\nabla p_{k_m} \chi_\epsilon(p_{k_m})\|_{L^2(Q_T)}$$
$$+ \|\nabla p_{k_j}(1 - \chi_\epsilon(p_{k_j})) - \nabla p_{k_m}(1 - \chi_\epsilon(p_{k_m}))\|_{L^2(Q_T)}$$

Hence,

$$\lim_{j,m \to \infty} \|\nabla p_{k_j} - \nabla p_{k_m}\|_{L^2(Q_T)} \leq 2 \sup_j \|\nabla p_{k_j} \chi_\epsilon(p_{k_j})\|_{L^2(Q_T)}$$
$$\leq 2\epsilon^{1/4} \sup_j \|p_{k_j}^{-1/4} \nabla p_{k_j}\|_{L^2(Q_T)} \lesssim \epsilon^{1/4}.$$

Hence, the p_k are $L^2([0,T]; H^1(\mathbb{R}^d))$ precompact.

To get precompactness in $C([0,T]; L^2(\mathbb{R}^d))$ we note that $|\partial_t p_k|^2 \leq 2|\nabla p_k|^4 + 2p_k^2 u_k^2$, hence $\partial_t p_k$ is uniformly bounded in $L^2(Q_T)$ thanks to Proposition 2.5. Now the precompactness in $C([0,T]; L^2(\mathbb{R}^d))$ follows from the precompactness in $L^2([0,T]; H^1(\mathbb{R}^d))$.

The precompactness of the nutrients is clear from the uniform L^2 bounds on $\partial_t n_k$ and Δn_k. The weak precompactness of the $\rho_{i,k}$ in $L^1(Q_T) \cap W^{1,2}([0,T]; H^{-1}(\mathbb{R}^d))$ follows from the bound $0 \leq \rho_{i,k} \leq \rho_k$ and the equation (1.1). □

Now that we have established the precompactness of the family of smooth solutions, we can deduce the existence of a limit point $(\rho_1, \dots, \rho_\ell, p, n)$. In what follows, we shall assume that we have extracted a subsequence (that we do not relabel) such that $(\rho_{1,k}, \dots, \rho_{\ell,k}, p_k, n_k)$ converges to $(\rho_1, \dots, \rho_\ell, p, n)$ with the various notions of convergence identified in Lemma 4.2. We will then show that this limit point is a complete Lagrangian solution and prove Theorem 1.3.

First we show that the maps X_k, Y_k are Cauchy on the support of p.

Lemma 4.3. *For any $T \geq 0$,*

$$\lim_{j,k \to \infty} \sup_{s \leq T} \sup_{t \leq s} \int_{\mathbb{R}^d} \rho(s,x) |X_j(t,s,x) - X_k(t,s,x)| \, dx = 0.$$

$$\lim_{j,k \to \infty} \sup_{s \leq T} \sup_{t \leq s} \int_{\mathbb{R}^d} \rho(s,x) |Y_j(t,s,x) - Y_k(t,s,x)| \, dx = 0.$$

Proof. Thanks to the strong convergence of p_k in $C([0,T];L^r(\mathbb{R}^d))$ and hence the strong convergence of ρ_k we can replace $\rho(s,x)$ in the above integrals with $\min(\rho_k(s,x),\rho_j(s,x))$. If we allow $(-\nabla p_k,\rho_k)$ and $(-\nabla p_j,\rho_j)$ to respectively play the roles of $(-\nabla p,\rho)$ and (V,μ) in Proposition 3.5, the result follows from the vanishing of $\|\nabla p_k - \nabla p_j\|_{L^2(Q_T))}$ as $j,k\to\infty$. □

The strong convergence of the flow maps in Lemma 4.3 implies the existence of the forward and backward Lagrangian flow maps X,Y along $-\nabla p$. The strong convergence guarantees that these maps satisfy all of the properties in requirement (ii) of the definition of complete Lagrangian solutions (i.e. the flow equations, semigroup property, and inversion formulas). Hence, we have almost succeeded in constructing our desired solution. Before we prove Theorem 1.3, we establish two uniqueness properties for the flow along $-\nabla p$. First we show that the flow maps X,Y have a stability property a lá Proposition 3.5 and then we show that solutions to the continuity equation along $-\nabla p$ are unique provided that the density stays within the support of p.

Proposition 4.4. *Let X and Y be the $L^1_{\text{loc}}([0,\infty);L^1(p^2))$ limits of X_k and Y_k. Let V,μ,S,Z be as in Section 3. We have the estimates*

$$\sup_{s\le T}\sup_{t\le T}\int_{\mathbb{R}^d}\min(\mu(s,x),\rho(s,x))|X(t,s,x)-S(t,s,x)|\le \mathcal{C}_\gamma(2T)\log(1+\log(1+\delta_T^{-1}))^{-1/2}, \tag{4.7}$$

$$\sup_{s\le T}\sup_{t\le s}\int_{\mathbb{R}^d}\min(\mu(s,x),\rho(s,x))|Y(t,s,x)-Z(t,s,x)|\le \mathcal{C}_\gamma(T)\log(1+\log(1+\delta_T^{-1}))^{-1/2}, \tag{4.8}$$

where $\delta_T=\|\nabla p+V\|_{L^2(Q_T)}$ and $\mathcal{C}_\gamma(T)$ is the same constant as in Proposition 3.5.

Proof. Arguing as in Lemma 4.3, it follows that

$$\int_{\mathbb{R}^d}\min(\mu(s,x),\rho(s,x))|X(t,s,x)-S(t,s,x)|\,dx=$$

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\min(\mu(s,x),\rho_k(s,x))|X_k(t,s,x)-S(t,s,x)|\,dx$$

Choosing $\lambda'=1$ in Proposition 3.5, we know that

$$\sup_{s\le T}\sup_{t\le T}\int_{\mathbb{R}^d}\bar\rho_k(s,x)|X_k(t,s,x)-S(t,s,x)|\le \mathcal{C}_\gamma(2T)\log(1+\log(1+\delta_{T,k}^{-1}))^{-1/2}, \tag{4.9}$$

where $\bar\rho_k=\min(\rho_k,\mu)$ and $\delta_{T,k}=\|\mu^{1/2}(\nabla p_k+V)\|_{L^2(Q_T)}$. Hence the first result follows from the convergence $\lim_{k\to\infty}\delta_{T,k}=\delta_T$. The second result has the same argument. □

Proposition 4.5. *Let X and Y be the $L^1_{\text{loc}}([0,\infty);L^1(p^2))$ limits of X_k and Y_k. Suppose that $\nu\in L^\infty_{\text{loc}}([0,\infty);L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ is a weak solution to the continuity equation*

$$\partial_t\nu-\nabla\cdot(\nu\nabla p)=0, \tag{4.10}$$

with initial data ν^0. If ν is everywhere nonnegative and

$$\int_{\{x\in\mathbb{R}^d:\rho(s,x)=0\}}\nu(s,x)=0 \tag{4.11}$$

for all $s\ge 0$, then $X(t,s,\cdot)_\#\nu(s,\cdot)=\nu(s+t,\cdot)$ for all $s,t\ge 0$ almost everywhere in space.

Remark 4.6. This proposition gives another way to argue that X and Y are the unique forward and backward flow maps along $-\nabla p$ when restricted to the support of ρ (see Ambrosio's superposition principle [Amb08]).

Proof. Fix a time step $\tau > 0$ and for each $k \in \mathbb{Z}_+$, we construct the optimal transport interpolants between $\nu(\tau k, \cdot)$ and $\nu(\tau(k+1), \cdot)$. Doing so, we obtain $\nu^\tau, \varphi^\tau, S^\tau$ where ν^τ is a density such that $\nu^\tau(k\tau, \cdot) = \nu(k\tau, \cdot)$ for all k, ν^τ, φ^τ are weak solutions to the continuity equation

$$\partial_t \nu^\tau + \nabla \cdot (\nu^\tau \nabla \varphi^\tau) = 0, \tag{4.12}$$

and S^τ satisfies

$$\partial_t S^\tau(t,s,x) = \nabla \varphi^\tau(s+t, S^\tau(t,s,x)), \quad S^\tau(t,s,\cdot)_\# \nu(s,\cdot) = \nu(s+t,\cdot),$$

see for instance [San15]. Furthermore, if we define $m^\tau = \nu^\tau \nabla \varphi^\tau$, then for any $j \in \mathbb{Z}_+$ we have

$$\int_0^{j\tau} \int_{\mathbb{R}^d} \frac{|m^\tau|^2}{2\nu^\tau} = \inf_{(\mu,b)\in \mathcal{C}_\tau} \int_0^{\tau j} \int_{\mathbb{R}^d} \frac{|b|^2}{2\mu}$$

where C_τ is the space of all density-flux pairs $(\mu, b) \in L^1_{\mathrm{loc}}(Q_\infty) \times L^2_{\mathrm{loc}}(Q_\infty)$ that are weak solutions to the continuity equation $\partial_t \mu + \nabla \cdot b = 0$ such that $\mu(\tau k, \cdot) = \nu(\tau k, \cdot)$ for all $k \in \mathbb{Z}_+$. Note that $(\nu, \nu\nabla p) \in C_\tau$ for any choice of τ, hence,

$$\int_0^{j\tau} \int_{\mathbb{R}^d} \frac{|m^\tau|^2}{2\nu^\tau} \leq \int_0^{j\tau} \int_{\mathbb{R}^d} \frac{\nu}{2} |\nabla p|^2.$$

Given any $\psi \in H^1_c(Q_\infty)$ and $j \in \mathbb{Z}_+$, we have

$$\int_0^{j\tau} \int_{\mathbb{R}^d} (\tilde{\nu}^\tau - \nu^\tau)\psi = \int_{\mathbb{R}^d} \sum_{k=0}^{j-1} \int_{k\tau}^{(k+1)\tau} \int_{k\tau}^{s} \Big(\nu^\tau(\theta,x)\nabla\varphi^\tau(\theta,x) - \nu(\theta,x)\nabla p(\theta,x)\Big) \cdot \nabla\psi(\theta,x)\, d\theta\, ds\, dx,$$

thus it follows that ν^τ converges to ν in $\dot{H}^{-1}_{\mathrm{loc}}(Q_\infty)$ as $\tau \to 0$. Hence, for any $\psi \in H^1_c(Q_\infty)$ it follows from (4.10) and (4.12) that

$$\lim_{\tau\to 0} \int_{Q_\infty} (m^\tau - \nu \nabla p) \cdot \nabla \psi = 0$$

so m^τ converges weakly to $\nu \nabla p + w$ where w is some divergence free vector field. Given some $T > 0$ let $j_\tau = \lceil \frac{T}{\tau} \rceil$. We can then compute

$$\int_0^{j_\tau \tau} \int_{\mathbb{R}^d} \frac{1}{2} \nu^\tau |\nabla \varphi^\tau - \nabla p|^2 = \int_0^{j_\tau \tau} \int_{\mathbb{R}^d} \frac{|m^\tau|^2}{2\nu^\tau} - m^\tau \cdot \nabla p + \frac{1}{2}\nu^\tau |\nabla p|^2.$$

$$\leq \int_0^{j_\tau \tau} \int_{\mathbb{R}^d} \frac{\nu}{2}|\nabla p|^2 - m^\tau \cdot \nabla p + \frac{1}{2}\nu^\tau |\nabla p|^2.$$

ν^τ must converge weakly to ν in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^d))$, therefore

$$\lim_{\tau \to 0} \int_0^{j_\tau \tau} \int_{\mathbb{R}^d} \frac{1}{2} \nu^\tau |\nabla \varphi^\tau - \nabla p|^2 = 0.$$

Now we can use Proposition 4.4 to deduce that for any $s, t \geq 0$

$$\lim_{\tau \to 0} \int_{\mathbb{R}^d} \min(\nu^\tau(s,x), \rho(s,x)) |S^\tau(t,s,x) - X(t,s,x)|\, dx = 0. \tag{4.13}$$

Finally, we can establish the pushforward formulas for ν. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a smooth compactly supported test function. We can compute

$$|\int_{\mathbb{R}^d} \big(\nu^\tau(s+t,x) - X(t,s,\cdot)_\# \nu(s,\cdot)\big)\varphi(x)\, dx| =$$

$$|\int_{\mathbb{R}^d} \nu^\tau(s,x)\varphi(S^\tau(t,s,x)) - \nu(s,x)\varphi(X(t,s,x))\, dx| =$$

$$|\int_{\mathbb{R}^d} \nu^\tau(s,x)\Big(\varphi(S^\tau(t,s,x)) - \varphi(X(t,s,x))\Big)\,dx|$$

Fix some $\epsilon > 0$ and let $\Omega_\epsilon(s) = \{x \in \mathbb{R}^d : \rho(s,x) < \epsilon\}$. The previous line is then bounded from above by

$$\epsilon^{-1}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \nu^\tau(s,x)\rho(s,x)|S^\tau(t,s,x)) - X(t,s,x)|\,dx \\ + 2\|\varphi\|_{L^\infty(\mathbb{R}^d)} \int_{\Omega_\epsilon(s)} \nu^\tau(s,x)\,dx. \tag{4.14}$$

Sending $\tau \to 0$ we see that (4.14) is equal to

$$2\|\varphi\|_{L^\infty(\mathbb{R}^d)} \int_{\Omega_\epsilon(s)} \nu(s,x)\,dx.$$

Thanks to our assumption (4.11), this last integral vanishes as $\epsilon \to 0$. Thus it follows that $X(t,s,\cdot)_{\#}\nu(s,\cdot) = \nu(s+t,\cdot)$ almost everywhere in space and for every $s,t \geq 0$. Since $Y(t,s,x)$ is the inverse of $X(t,s-t,x)$ we also have $Y(t,s,\cdot)_{\#}\nu(s,\cdot) = \nu(s-t,\cdot)$ almost everywhere in space and for every $s \geq 0$ and $t \leq s$.

□

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. We have already verified that our solution satisfies property (ii) of a complete Lagrangian solution. From our convergence properties, it follows that $(\rho_1,\dots,\rho_\ell,p,n)$ is a weak solution to the tumor growth system with initial data $(\rho_1^0,\dots,\rho_\ell^0,n^0)$. Finally, since we know that $\rho_i \leq p^{\frac{1}{\gamma}}$ we can deduce property (iii) from Lemma 4.3. Thus, $(\rho_1,\dots,\rho_\ell,p,n)$ is the desired complete Lagrangian solution.

It remains to prove the nonmixing property. Let $\rho_{i,j} = \min(\rho_i,\rho_j)$. From the pushforward representation formula (1.15), it follows that $Y(t,t,\cdot)_{\#}\rho_{i,j}(t,\cdot) \leq e^{tB}\min(\rho_i^0,\rho_j^0)$. If $\min(\rho_i^0,\rho_j^0) = 0$, then it follows that $\rho_{i,j} = 0$. Hence the nonmixing property holds. □

4.2. **The incompressible limit.** Now we want to construct solutions in the case $\gamma = \infty$. Given growth terms $G_1,\dots,G_\ell$ satisfying (G1-G4) and initial data $(\rho_1^0,\dots,\rho_\ell^0,n^0)$ satisfying (ID1-ID5), we create a modified sequence of initial data as follows. Recall that the initial pressure p^0 must solve the equation

$$\Delta p^0 + \sum_{i=1}^{\ell} \frac{\rho_i^0}{\rho^0} G_i(p^0,n^0) = 0, \quad p^0(1-\rho^0) = 0. \tag{4.15}$$

Using p^0, we create the sequence by setting $\rho_\gamma^0 = (p^0)^{\frac{1}{\gamma}}$ and $\rho_{i,\gamma}^0 = \rho_\gamma^0 \frac{\rho_i^0}{\rho_0}$. The key properties of this sequence are summarized below.

Lemma 4.7. $(\rho_{1,\gamma}^0,\dots,\rho_{\ell,\gamma},n^0)$ *satisfies (ID1-ID3),*

$$\sup_\gamma \int_{\mathbb{R}^d} \gamma\rho_\gamma^0(\Delta p_0 + \sum_{i=1}^{\ell} \frac{\rho_{i,\gamma}^0}{\rho_\gamma^0} G_i(p^0,n^0))_-^2 = 0, \tag{4.16}$$

and $\lim_{\gamma\to\infty}\|\rho_i^0 - \rho_{i,\gamma}^0\|_{L^1(\mathbb{R}^d)} = 0$.

Proof. The first two claims are clear from our construction. For the last property, we note that (ID5) implies the existence of some $\lambda > 0$ such that $\int_{\mathbb{R}^d} \rho^0 \log(1 + 1/p^0)^\lambda < \infty$, thus, for any $\epsilon > 0$ the set $\{x \in \mathbb{R}^d : p^0 < \epsilon\}$ has ρ^0 measure at most $\frac{1}{|\log(\epsilon)|^\lambda}$. Thus $\lim_{\gamma\to\infty}(p^0(x))^{\frac{1}{\gamma}} = 1$

almost everywhere on the support of ρ^0. Since $p^0 = 0$ on the complement of the support of ρ^0, we can deduce $\lim_{\gamma\to\infty}\|\rho_i^0 - \rho_{i,\gamma}^0\|_{L^1(\mathbb{R}^d)} = 0$ from dominated convergence. □

Now that we have a sequence of initial data satisfying (ID1-ID3), for each $\gamma \geq 1$ we can use Theorem 1.3 to construct complete Lagrangian solutions $(\rho_{1,\gamma}, \dots, \rho_{\ell,\gamma}, p_\gamma, n_\gamma)$ to (1.1-1.2) with initial data $(\rho_1^0, \dots, \rho_\ell^0, n^0)$. Our goal is now to show that these solutions converge to a complete Lagrangian solution to the incompressible system as we send $\gamma \to \infty$. Due to the fact that we only have uniform L^1 regularity for the time derivative of the pressure along the sequence, we will need to proceed more carefully than we did in the case $\gamma < \infty$.

Lemma 4.8. *If γ_k is a sequence such that $\lim_{k\to\infty}\gamma_k = \infty$ then p_{γ_k} is precompact in $L^2([0,T]; H^1(\mathbb{R}^d))$, n_{γ_k} is precompact in $L^2([0,T]; H^1(\mathbb{R}^d)) \cap C([0,T]; L^2(\mathbb{R}^d))$, and ρ_{i,γ_k} is weakly precompact in $L^1(Q_T)$ for each $i \in \{1, \dots, \ell\}$.*

Proof. We argue as in Lemma 4.2, except that we can no longer establish that p_{γ_k} is $C([0,T]; L^2(\mathbb{R}^d))$ precompact. □

Now that we have established precompactness, in the rest of this subsection we will assume (without loss of generality) that γ_k is a subsequence such that $(\rho_{1,\gamma_k}, \dots, \rho_{\ell,\gamma_k}, p_{\gamma_k}, n_{\gamma_k})$ converges to a point $(\rho_1, \dots, \rho_\ell, p, n)$ where the convergence holds in the spaces that we identified in Lemma 4.8.

We now establish some properties of the limit point.

Lemma 4.9. *$(\rho_1, \dots, \rho_\ell, p, n)$ is a weak solution to the incompressible system (1.16-1.17). Furthermore, ρ is nondecreasing in time almost everywhere, and for any $T \geq 0$, the set $\{(t,x) \in Q_T : \rho(t,x) > 0, p(t,x) = 0\}$ has measure zero, and p satisfies the complementarity formula*

$$p = \operatorname*{argmin}_{\varphi(1-\rho)=0} \int_{Q_T} \frac{1}{2}|\nabla\varphi|^2 - \varphi G, \tag{4.17}$$

where $G = \sum_{i=1}^{\ell} \frac{\rho_i}{\rho} G_i(p,n)$.

Proof. The convergence properties that we have are strong enough to guarantee that $(\rho_1, \dots, \rho_\ell, p, n)$ is a weak solution to the equations

$$\partial_t \rho_i - \nabla\cdot(\rho_i \nabla p) = \rho_i G_i(p,n)$$

$$\partial_t n - \alpha\Delta n = \sum_{i=1}^{\ell} \beta_i \rho_i.$$

To prove that $(\rho_1, \dots, \rho_\ell, p, n)$ is a solution to the incompressible system (1.16-1.17) we still need to show that $\rho \leq 1$ and $p(1-\rho) = 0$ almost everywhere. Since $p_{\gamma_k} \leq p_h$ almost everywhere, it follows that $\rho_{\gamma_k} \leq p_h^{\frac{1}{\gamma_k}}$ almost everywhere. Therefore, $\rho \leq 1$ almost everywhere. Fix some $\epsilon > 0$ and some set $E \subset Q_T$ with finite measure. We can then compute

$$\int_E p(t,x)(1-\rho(t,x))\,dx\,dt = \lim_{k\to\infty}\int_E p_{\gamma_k}(t,x)(1-\rho_{\gamma_k}(t,x))$$

$$\leq \lim_{k\to\infty}\int_E \epsilon p_{\gamma_k} + (1-\epsilon)^{\gamma_k} \leq \epsilon\|p\|_{L^1(E)},$$

where the first inequality follows from splitting E into the sets $\{(t,x) \in E : \rho_{\gamma_k} < 1-\epsilon\}$ and $\{(t,x) \in E : \rho_{\gamma_k} \geq 1-\epsilon\}$. Sending $\epsilon \to 0$ we can conclude that $p(1-\rho) = 0$ almost everywhere.

Now that we know that $(\rho_1, \dots, \rho_\ell, p, n)$ satisfies (1.16-1.17) we can glean some more information. Summing (1.16) over the populations, we see that ρ, p are weak solutions of the equation

$$\partial_t \rho - \nabla\cdot(\rho\nabla p) = \rho G \quad p(1-\rho) = 0, \quad \rho \leq 1, \tag{4.18}$$

which is the Hele-Shaw equation with a source term. Condition (G4) guarantees that $G_i \geq 0$ everywhere for each i, thus, standard theory for the Hele-Shaw equation implies that ρ must be nondecreasing in time, the support of p must be nondecreasing in time, and $\rho(t,x) \leq e^{Bt}\rho^0(x)$ for almost every (t,x) where $p(t,x) = 0$ (see for instance [PQV14, JKT21]). Since $\{x \in \mathbb{R}^d : \rho^0(x) > 0,\, p^0(x) = 0\}$ has measure zero, it follows that $\{x \in \mathbb{R}^d : \rho(t,x) > 0,\, p(t,x) = 0\}$ has measure zero. Finally, the complementarity condition (4.17) is a consequence of the weak equation (4.18) when the pressure has $L^2([0,T]; H^1(\mathbb{R}^d))$ regularity (see for instance [PQV14, DP21, GKM22, Jac21]).

□

In the next Lemma, we consider properties of the time integrated pressure $w(s,x) := \int_0^t p(s,x)\,dx$ and $w_{\gamma_k} := \int_0^t p_{\gamma_k}(s,x)\,ds$. The main advantage of working with this quantity is that w has better time regularity than p while still having the same support.

Lemma 4.10. *There exists constants $C_1, C_2 > 0$ such that for every $t, \tau \geq 0$ and any $\epsilon > 0$*

$$|\{(x \in \mathbb{R}^d : w(t,x) > \epsilon + C_1 t p(t,x)\}| + \limsup_{k\to\infty} |\{(x \in \mathbb{R}^d : w_{\gamma_k}(t,x) > \epsilon + C_1 t p_{\gamma_k}(t,x)\}| = 0,$$

and

$$|\{(x \in \mathbb{R}^d : w(t+\tau,x) < C_2 \tau p(t,x) - \epsilon\}| + \limsup_{k\to\infty} |\{(x \in \mathbb{R}^d : w_{\gamma_k}(t+\tau,x) < C_2 \tau p_{\gamma_k}(t,x) - \epsilon\}| = 0.$$

Furthermore, for any $T \geq 0$, the set $\{(t,x) \in Q_T : \rho(t,x) > 0, w(t,x) = 0\}$ has measure zero.

Proof. To prove the first result, we note that (G1) and (G4) guarantee the existence of constants $0 < b_1 \leq b_2$ such that $b_1 \leq G \leq b_2$ almost everywhere. Define

$$\tilde{p}_i = \operatorname*{argmin}_{\varphi(1-\rho)=0} \int_{Q_T} \frac{1}{2}|\nabla \varphi|^2 - \varphi b_i.$$

Since ρ is nondecreasing in time almost everywhere, it follows that both $\tilde{p}_1, \tilde{p}_2$ are nondecreasing in time almost everywhere. The comparison principle also implies that $\tilde{p}_1 \leq p \leq \tilde{p}_2$. Thus, we have the string of inequalities

$$w(t,x) \leq \int_0^t \tilde{p}_2(s,x)\,ds \leq t\tilde{p}_2(t,x) \leq t\frac{b_2}{b_1}\tilde{p}_1(t,x) \leq t\frac{b_2}{b_1}p(t,x),$$

and

$$w(t+\tau,x) \geq \int_0^{t+\tau} \tilde{p}_1(s,x)\,ds \geq \tau\tilde{p}_1(t,x) \geq \tau\frac{b_1}{b_2}\tilde{p}_2(t,x) \geq \tau\frac{b_1}{b_2}p(t,x),$$

for almost every (t,x). Now using the strong convergence of p_{γ_k} to p in $L^2([0,T]; H^1(\mathbb{R}^d))$ we can conclude that

$$|\{(x \in \mathbb{R}^d : w(t,x) > \epsilon + Ctp(t,x)\}| + \limsup_{k\to\infty} |\{(x \in \mathbb{R}^d : w_{\gamma_k}(t,x) > \epsilon + Ctp_{\gamma_k}(t,x)\}| = 0,$$

and

$$|\{(x \in \mathbb{R}^d : w(t+\tau,x) < C_2 \tau p(t,x) - \epsilon\}| + \limsup_{k\to\infty} |\{(x \in \mathbb{R}^d : w_{\gamma_k}(t+\tau,x) < C_2 \tau p_{\gamma_k}(t,x) - \epsilon\}| = 0,$$

for almost every $t, \tau \geq 0$. To upgrade this to every $t, \tau \geq 0$ we simply note that $\partial_t w_{\gamma_k}$ and $(\partial_t p_{\gamma_k})_-$ are uniformly bounded in $L^2(Q_T)$ for any $T \geq 0$

Now we turn our attention to proving the second result. For each $\delta > 0$ and $t \geq 0$, define $E_\delta(t) := \{x \in \mathbb{R}^d : p(t,x) > \delta, w(t,x) = 0\}$ and note that $\int_{E_\delta(t)} \rho(t,x)\,dx = |E_\delta(t)|$ for almost every $t \geq 0$. From our work above, it follows that $\int_0^T |E_\delta(t) \cap E_\delta(t+\tau)|\, dt = 0$ for all $\tau > 0$. Hence, for any $t, \tau \geq 0$ and $m \in \mathbb{Z}_+$ we can deduce that

$$\frac{1}{\tau}\int_t^{t+\tau}\int_{\mathbb{R}^d} \rho(s,x)\, dx\, ds \geq \frac{1}{\tau}\int_t^{t+\tau}\sum_{j=1}^{2^m}\int_{E_\delta(\frac{js}{2^m})} \rho(s,x)\, dx\, ds \geq \frac{1}{\tau}\int_t^{t+\tau}\sum_{j=1}^{2^m} |E_\delta(\frac{js}{2^m})|\, ds.$$

Diving both sides by 2^m and then sending $m \to \infty$, we can conclude that

$$\frac{1}{\tau}\int_t^{t+\tau}\int_0^s |E_\delta(\theta)|\, d\theta\, ds = 0.$$

Hence, it follows that $\{(t,x) \in Q_T : p(t,x) > \delta, w(t,x) = 0\}$ has measure zero for all $\delta > 0$. Recalling that $\{(t,x) \in Q_T : \rho(t,x) > 0, p(t,x) = 0\}$ has measure zero, we can conclude that $\{(t,x) \in Q_T : \rho(t,x) > 0, w(t,x) = 0\}$ also has measure zero.

□

At last we can prove the analogues of Lemma 4.3 and Propositions 4.4 and 4.5.

Lemma 4.11. *Let X_k and Y_k be the forward and backward Lagrangian flows along $-\nabla p_{\gamma_k}$. For any $T \geq 0$,*

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} w(s,x)|X_j(t,s,x) - X_k(t,s,x)|\, dx = 0.$$

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} w(s,x)|Y_j(t,s,x) - Y_k(t,s,x)|\, dx = 0.$$

Proof. Clearly w_{γ_k} converges to w in $C([0,T]; L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$, therefore

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} w(s,x)|X_j(t,s,x) - X_k(t,s,x)|\, dx =$$

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} \min(w_{\gamma_k}(s,x), w_{\gamma_j}(s,x))|X_j(t,s,x) - X_k(t,s,x)|\, dx \lesssim$$

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} sp_h^{1-\frac{1}{\gamma}}\min(\rho_{\gamma_k}, \rho_{\gamma_j})(s,x)|X_j(t,s,x) - X_k(t,s,x)|\, dx$$

where the second inequality follows from our work in Lemma 4.10 and the bound $p_{\gamma_k} \leq \rho_{\gamma_k} p_h^{1-\frac{1}{\gamma}}$. Since the initial data satisfies (ID5) and G satisfies (G4), we can use Proposition 3.5 to get

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} \min(\rho_{\gamma_k}(s,x), \rho_{\gamma_j}(s,x))|X_j(t,s,x) - X_k(t,s,x)|\, dx = 0.$$

An identical argument proves that

$$\lim_{j,k\to\infty}\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} w(s,x)|Y_j(t,s,x) - Y_k(t,s,x)|\, dx = 0.$$

□

Proposition 4.12. *Let X and Y be the $L^1_{\text{loc}}([0,\infty); L^1(p^2))$ limits of X_k and Y_k. Let V, μ, S, Z be as in Section 3. For any $\lambda' \in [0, 1/2) \cap [0, \lambda]$ we have the estimates*

(4.19)

$$\sup_{s\leq T}\sup_{t\leq T}\int_{\mathbb{R}^d} \min(\mu(s,x), w(s,x))|X(t,s,x) - S(t,s,x)| \leq \min(T,1)\mathcal{C}_\gamma(2T)\log(1+\log(1+\delta_T^{-1}))^{-\lambda'/2},$$

(4.20)

$$\sup_{s\leq T}\sup_{t\leq s}\int_{\mathbb{R}^d} \min(\mu(s,x), w(s,x))|Y(t,s,x) - Z(t,s,x)| \leq \min(T,1)\mathcal{C}_\gamma(T)\log(1+\log(1+\delta_T^{-1}))^{-\lambda'/2},$$

where $\delta_T = \|\nabla p + V\|_{L^2(Q_T)}$ *and* $\mathcal{C}_\gamma(T)$ *is a multiple of the constant in Proposition 3.5 and* λ *is the constant in condition (ID5).*

Proof. See the arguments of Proposition 4.4 and Lemma 4.11. □

Proposition 4.13. *Let* X *and* Y *be the* $L^1_{\text{loc}}([0,\infty); L^1(w^2))$ *limits of* X_k *and* Y_k*. Suppose that* $\nu \in L^\infty_{\text{loc}}([0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ *is a weak solution to the continuity equation*

$$\partial_t \nu - \nabla \cdot (\nu \nabla p) = 0, \tag{4.21}$$

with initial data ν^0*. If* ν *is everywhere nonnegative and*

$$\int_{\{x\in\mathbb{R}^d : w(s,x)=0\}} \nu(s,x) = 0 \tag{4.22}$$

for all $s > 0$*, then* $X(t,s,\cdot)_\# \nu(s,\cdot) = \nu(s+t,\cdot)$ *for all* $s, t \geq 0$ *almost everywhere in space.*

Proof. See the arguments of Proposition 4.5 and Lemma 4.11. □

Finally we can prove Theorem 1.4.

Proof of Theorem 1.4. We have already established that $(\rho_1,\dots,\rho_\ell,p,n)$ is a solution to the incompressible system (1.16-1.17). The strong convergence of the X_k, Y_k to X, Y on the support of w implies that X and Y satisfy all the properties in Definition 1.1 when restricted to the support of w. Noting that $t \mapsto \int_{\mathbb{R}^d} \rho(t,x)\,dx$ is a Lipschitz function and $\text{spt}(\rho) = \text{spt}(w)$ almost everywhere in spacetime, it follows that X, Y satisfy all of the properties in Definition 1.1 for all $s \geq 0$ and almost every $x \in \text{spt}(\rho(s,x))$. Thus, $(\rho_1,\dots,\rho_\ell,p,n)$ is a complete Lagrangian solution to the incompressible system (1.16-1.17).

The proof of the nonmixing property is identical to the proof of the nonmixing property in Theorem 1.3.

□

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